You have 90 minutes to answer four of these questions.

Write your answers in the separate answer booklet.

You may take the question sheet with you when you leave.

1. You and your eight-year-old nephew Elmo decide to play a simple card game. At the beginning of the game, the cards are dealt face up in a long row. Each card is worth a different number of points. After all the cards are dealt, you and Elmo take turns removing either the leftmost or rightmost card from the row, until all the cards are gone. At each turn, you can decide which of the two cards to take. The winner of the game is the player that has collected the most points when the game ends.

   Having never taken an algorithms class, Elmo follows the obvious greedy strategy—when it's his turn, Elmo *always* takes the card with the higher point value. Your task is to find a strategy that will beat Elmo whenever possible. (It might seem mean to beat up on a little kid like this, but Elmo absolutely *hates* it when grown-ups let him win.)

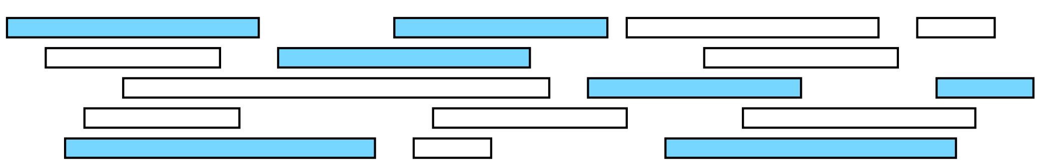
   (a) Prove that you should not also use the greedy strategy. That is, show that there is a game that you can win, but only if you do *not* follow the same greedy strategy as Elmo.

   (b) Describe and analyze an algorithm to determine, given the initial sequence of cards, the maximum number of points that you can collect playing against Elmo.

2. Suppose you are given a magical black box that can tell you in constant time whether or not a given graph has a Hamiltonian cycle. Using this magic black box as a subroutine, describe and analyze a *polynomial-time* algorithm to actually compute a Hamiltonian cycle in a given graph, if one exists.

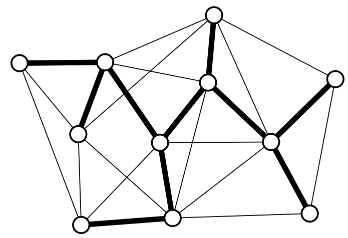
3. Let X be a set of n intervals on the real line. A subset of intervals $Y \subseteq X$ is called a *tiling path* if the intervals in Y cover the intervals in X, that is, any real value that is contained in some interval in X is also contained in some interval in Y. The *size* of a tiling cover is just the number of intervals.

   Describe and analyze an algorithm to compute the smallest tiling path of X as quickly as possible. Assume that your input consists of two arrays $X_L[1\,..\,n]$ and $X_R[1\,..\,n]$, representing the left and right endpoints of the intervals in X.



A set of intervals. The seven shaded intervals form a tiling path.

4. Prove that the following problem is NP-complete: Given an undirected graph, does it have a spanning tree in which every node has degree at most 3?

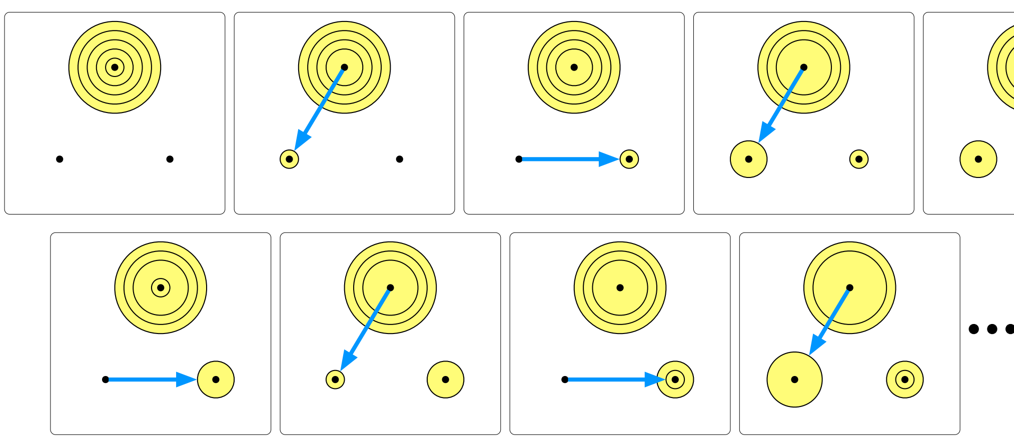


A graph with a spanning tree of maximum degree 3.

5. The *Tower of Hanoi* puzzle, invented by Edouard Lucas in 1883, consists of three pegs and n disks of different sizes. Initially, all n disks are on the same peg, stacked in order by size, with the largest disk on the bottom and the smallest disk on top. In a single move, you can move the topmost disk on any peg to another peg; however, you are never allowed to place a larger disk on top of a smaller one. Your goal is to move all n disks to a different peg.

   (a) Prove that the Tower of Hanoi puzzle can be solved in exactly $2^n - 1$ moves. *[Hint: You've probably seen this before.]*

   (b) Now suppose the pegs are arranged in a circle and you are *only* allowed to move disks *counterclockwise*. How many moves do you need to solve this restricted version of the puzzle? Give a upper bound in the form $O(f(n))$ for some function $f(n)$. Prove your upper bound is correct.



A top view of the first eight moves in a counterclockwise Towers of Hanoi solution