1. A *double-Hamiltonian* circuit in an undirected graph G is a closed walk that visits every vertex in G exactly *twice*, possibly by traversing some edges more than once. ***Prove*** that it is NP-hard to determine whether a given undirected graph contains a double-Hamiltonian circuit.

2. Suppose you are running a web site that is visited by the same set of people every day. Each visitor claims membership in one or more *demographic groups*; for example, a visitor might describe himself as male, 31-40 years old, a resident of Illinois, an academic, a blogger, a Joss Whedon fan[1], and a Sports Racer.[2] Your site is supported by advertisers. Each advertiser has told you which demographic groups should see its ads and how many of its ads you must show each day. Altogether, there are n visitors, k demographic groups, and m advertisers.

   Describe an efficient algorithm to determine, given all the data described in the previous paragraph, whether you can show each visitor exactly *one* ad per day, so that every advertiser has its desired number of ads displayed, and every ad is seen by someone in an appropriate demographic group.

3. Describe and analyze a data structure to support the following operations on an array $X[1\,..\,n]$ as quickly as possible. Initially, $X[i] = 0$ for all i.

   - Given an index i such that $X[i] = 0$, set $X[i]$ to 1.
   - Given an index i, return $X[i]$.
   - Given an index i, return the smallest index $j \geq i$ such that $X[j] = 0$, or report that no such index exists.

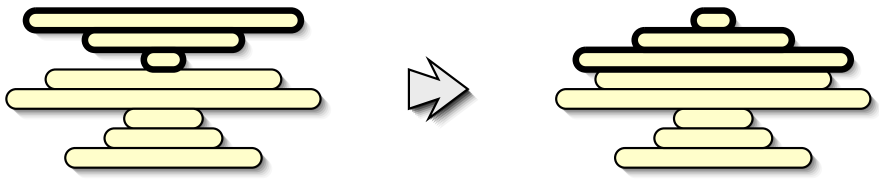
   For full credit, the first two operations should run in *worst-case constant* time, and the amortized cost of the third operation should be as small as possible. *[Hint: Use a modified union-find data structure.]*

4. The next time you are at a party, one of the guests will suggest everyone play a round of Three-Way Mumbledypeg, a game of skill and dexterity that requires three teams and a knife. The official Rules of Three-Way Mumbledypeg (fixed during the Holy Roman Three-Way Mumbledypeg Council in 1625) require that (1) each team *must* have at least one person, (2) any two people on the same team *must* know each other, and (3) everyone watching the game *must* be on one of the three teams. Of course, it will be a really *fun* party; nobody will want to leave. There will be several pairs of people at the party who don't know each other. The host of the party, having heard thrilling tales of your prowess in all things algorithmic, will hand you a list of which pairs of partygoers know each other and ask you to choose the teams, while he sharpens the knife.

   Either describe and analyze a polynomial time algorithm to determine whether the partygoers can be split into three legal Three-Way Mumbledypeg teams, or prove that the problem is NP-hard.

[1]Har har har! Mine is an evil laugh! Now *die!*
[2]It's Ride the Fire Eagle Danger Day!

5. Suppose you are given a stack of n pancakes of different sizes. You want to sort the pancakes so that smaller pancakes are on top of larger pancakes. The only operation you can perform is a *flip*—insert a spatula under the top k pancakes, for some integer k between 1 and n, and flip them all over.



Flipping the top three pancakes.

   (a) Describe an efficient algorithm to sort an arbitrary stack of n pancakes. *Exactly* how many flips does your algorithm perform in the worst case? (For full credit, your algorithm should perform as few flips as possible; an optimal $\Theta()$ bound is worth three points.)

   (b) Now suppose one side of each pancake is burned. *Exactly* how many flips do you need to sort the pancakes *and* have the burned side of every pancake on the bottom? (For full credit, your algorithm should perform as few flips as possible; an optimal $\Theta()$ bound is worth three points.)

6. Describe and analyze an efficient algorithm to find the length of the longest substring that appears both forward and backward in an input string $T[1..n]$. The forward and backward substrings must not overlap. Here are several examples:

   - Given the input string ALGORITHM, your algorithm should return 0.
   - Given the input string RECURSION, your algorithm should return 1, for the substring R.
   - Given the input string REDIVIDE, your algorithm should return 3, for the substring EDI. (The forward and backward substrings must not overlap!)
   - Given the input string DYNAMICPROGRAMMINGMANYTIMES, your algorithm should return 4, for the substring YNAM.

   For full credit, your algorithm should run in $O(n^2)$ time.

7. A *double-Eulerian* circuit in an undirected graph G is a closed walk that traverses every edge in G exactly twice. Describe and analyze a *polynomial-time* algorithm to determine whether a given undirected graph contains a double-Eulerian circuit.