1. **Representation of Integers**

    (a) Prove that any positive integer can be written as the sum of distinct *nonconsecutive* Fibonacci numbers–if F_n appears in the sum, then neither F_{n+1} nor F_{n-1} will. For example: $42 = F_9 + F_6$, $25 = F_8 + F_4 + F_2$, $17 = F_7 + F_4 + F_2$.

    (b) Prove that any integer (positive, negative, or zero) can be written in the form $\sum_i \pm 3^i$, where the exponents i are distinct non-negative integers. For example $42 = 3^4 - 3^3 - 3^2 - 3^1$, $25 = 3^3 - 3^1 + 3^0$, $17 = 3^3 - 3^2 - 3^0$.

2. **Minimal Dominating Set**

    Suppose you are given a rooted tree T (not necessarily binary). You want to label each node in T with an integer 0 or 1, such that every node either has the label 1 or is adjacent to a node with the label 1 (or both). The *cost* of a labeling is the number of nodes with label 1. Describe and analyze an algorithm to compute the minimum cost of any labeling of the given tree T.

3. **Names in Boxes**

    The names of 100 prisoners are placed in 100 wooden boxes, one name to a box, and the boxes are lined up on a table in a room. One by one, the prisoners are led into the room; each may look in at most 50 boxes, but must leave the room exactly as he found it and is permitted no further communication with the others.

    The prisoners have a chance to plot their strategy in advance, and they are going to need it, because unless *every single prisoner finds his own name* all will subsequently be executed. Find a strategy for them which has probability of success exceeding 30%. You may assume that the names are distributed in the boxes uniformly at random.

    (a) Calculate the probability of success if each prisoner picks 50 boxes uniformly at random.

    $\star$(b) Consider the following strategy.

    The prisoners number themselves 1 to 100. Prisoner i begins by looking in box i. There he finds the name of prisoner j. If $j \neq i$, he continues by looking in box j. As long as prisoner i has not found his name, he continues by looking in the box corresponding to the last name he found.

    Describe the set of permutations of names in boxes for which this strategy will succeed.

    $\star$(c) Count the number of permutations for which the strategy above succeeds. Use this sum to calculate the probability of success. You may find it useful to do this calculation for general n, then set $n = 100$ at the end.

    (d) We assumed that the names were distributed in the boxes uniformly at random. Explain how the prisoners could augment their strategy to make this assumption unnecessary.