CS/ECE 374 ✧ Fall 2016

Homework 6

Due Tuesday, October 16, 2016 at 8pm

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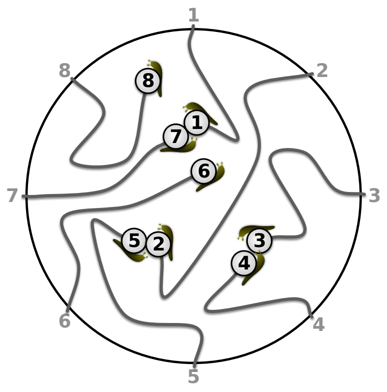
1. Every year, as part of its annual meeting, the Antarctican Snail Lovers of Upper Glacierville hold a Round Table Mating Race. Several high-quality breeding snails are placed at the edge of a round table. The snails are numbered in order around the table from 1 to n. During the race, each snail wanders around the table, leaving a trail of slime behind it. The snails have been specially trained never to fall off the edge of the table or to cross a slime trail, even their own. If two snails meet, they are declared a breeding pair, removed from the table, and whisked away to a romantic hole in the ground to make little baby snails. Note that some snails may never find a mate, even if the race goes on forever.



The end of a typical Antarctican SLUG race. Snails 6 and 8 never find mates.
The organizers must pay $M[3,4] + M[2,5] + M[1,7]$.

For every pair of snails, the Antarctican SLUG race organizers have posted a monetary reward, to be paid to the owners if that pair of snails meets during the Mating Race. Specifically, there is a two-dimensional array $M[1..n, 1..n]$ posted on the wall behind the Round Table, where $M[i,j] = M[j,i]$ is the reward to be paid if snails i and j meet. Rewards may be positive, negative, or zero.

Describe and analyze an algorithm to compute the maximum total reward that the organizers could be forced to pay, given the array M as input.

2. You and your eight-year-old nephew Elmo decide to play a simple card game. At the beginning of the game, the cards are dealt face up in a long row. Each card is worth a different number of points. After all the cards are dealt, you and Elmo take turns removing either the leftmost or rightmost card from the row, until all the cards are gone. At each turn, you can decide which of the two cards to take. The winner of the game is the player that has collected the most points when the game ends.

   Having never taken an algorithms class, Elmo follows the obvious greedy strategy—when it's his turn, Elmo *always* takes the card with the higher point value. Your task is to find a strategy that will beat Elmo whenever possible. (It might seem mean to beat up on a little kid like this, but Elmo absolutely *hates* it when grown-ups let him win.)

   (a) Prove that you should not also use the greedy strategy. That is, show that there is a game that you can win, but only if you do *not* follow the same greedy strategy as Elmo.

   (b) Describe and analyze an algorithm to determine, given the initial sequence of cards, the maximum number of points that you can collect playing against Elmo.

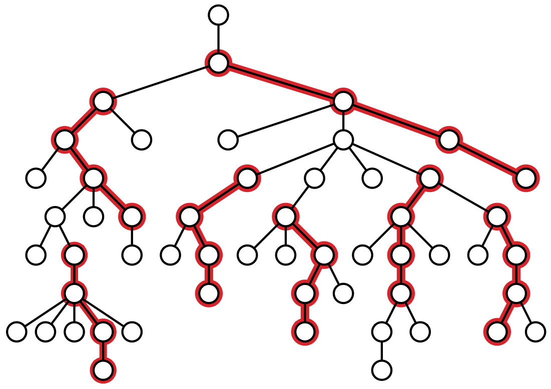
   (c) Five years later, Elmo has become a *significantly* stronger player. Describe and analyze an algorithm to determine, given the initial sequence of cards, the maximum number of points that you can collect playing against a *perfect* opponent. *[Hint: What is a perfect opponent?]*

3. One day, Alex got tired of climbing in a gym and decided to take a very large group of climber friends outside to climb. The climbing area where they went, had a huge wide boulder, not very tall, with various marked hand and foot holds. Alex quickly determined an "allowed" set of moves that her group of friends can perform to get from one hold to another.

   The overall system of holds can be described by a rooted tree T with n vertices, where each vertex corresponds to a hold and each edge corresponds to an allowed move between holds. The climbing paths converge as they go up the boulder, leading to a unique hold at the summit, represented by the root of T.[1]

   Alex and her friends (who are all excellent climbers) decided to play a game, where as many climbers as possible are simultaneously on the boulder and each climber needs to perform a sequence of *exactly* k moves. Each climber can choose an arbitrary hold to start from, and all moves must move away from the ground. Thus, each climber traces out a path of k edges in the tree T, all directed toward the root. However, no two climbers are allowed to touch the same hold; the paths followed by different climbers cannot intersect at all.

   Describe and analyze an efficient algorithm to compute the maximum number of climbers that can play this game. More formally, you are given a rooted tree T and an integer k, and you want to find the largest possible number of disjoint paths in T, where each path has length k. For full credit, do ***not*** assume that T is a binary tree. For example, given the tree T below and $k = 3$ as input, your algorithm should return the integer 8.



Seven disjoint paths of length k=3 in a rooted tree.
This is *not* the largest such set of paths in this tree.

[1]Q: Why do computer science professors think trees have their roots at the top?
A: Because they've never been outside!

Solved Problems

4. A string w of parentheses **(** and **)** and brackets **[** and **]** is ***balanced*** if it is generated by the following context-free grammar:

$$S \to \varepsilon \mid \mathbf{(}S\mathbf{)} \mid \mathbf{[}S\mathbf{]} \mid SS$$

For example, the string $w =$ **([()][]())[()()]()** is balanced, because $w = xy$, where

$$x = \mathbf{(\ [()]\ []\ ()\ )} \qquad \text{and} \qquad y = \mathbf{[\ ()\ ()\ ]\ ()}.$$

Describe and analyze an algorithm to compute the length of a longest balanced subsequence of a given string of parentheses and brackets. Your input is an array $A[1..n]$, where $A[i] \in \{\mathbf{(}, \mathbf{)}, \mathbf{[}, \mathbf{]}\}$ for every index i.

Solution: Suppose $A[1..n]$ is the input string. For all indices i and j, we write $\boldsymbol{A[i] \sim A[j]}$ to indicate that $A[i]$ and $A[j]$ are matching delimiters: Either $A[i] =$ **(** and $A[j] =$ **)** or $A[i] =$ **[** and $A[j] =$ **]**.

For all indices i and j, let $\boldsymbol{LBS(i,j)}$ denote the length of the longest balanced subsequence of the substring $A[i..j]$. We need to compute $LBS(1,n)$. This function obeys the following recurrence:

$$LBS(i,j) = \begin{cases} 0 & \text{if } i \ge j \\ \max\left\{ \begin{array}{c} 2 + LBS(i+1, j-1) \\ \max\limits_{k=1}^{j-1} \big(LBS(i,k) + LBS(k+1,j)\big) \end{array} \right\} & \text{if } A[i] \sim A[j] \\ \max\limits_{k=1}^{j-1} \big(LBS(i,k) + LBS(k+1,j)\big) & \text{otherwise} \end{cases}$$

We can memoize this function into a two-dimensional array $LBS[1..n, 1..n]$. Since every entry $LBS[i,j]$ depends only on entries in later rows or earlier columns (or both), we can evaluate this array row-by-row from bottom up in the outer loop, scanning each row from left to right in the inner loop. The resulting algorithm runs in $\boldsymbol{O(n^3)}$ ***time***.

```
LongestBalancedSubsequence($A[1..n]$):
  for $i \leftarrow n$ down to 1
      $LBS[i,i] \leftarrow 0$
      for $j \leftarrow i+1$ to $n$
          if $A[i] \sim A[j]$
              $LBS[i,j] \leftarrow LBS[i+1,j-1] + 2$
          else
              $LBS[i,j] \leftarrow 0$
          for $k \leftarrow i$ to $j-1$
              $LBS[i,j] \leftarrow \max\{LBS[i,j],\ LBS[i,k] + LBS[k+1,j]\}$
  return $LBS[1,n]$
```

■

Rubric: 10 points, standard dynamic programming rubric

5. Oh, no! You've just been appointed as the new organizer of Giggle, Inc.'s annual mandatory holiday party! The employees at Giggle are organized into a strict hierarchy, that is, a tree with the company president at the root. The all-knowing oracles in Human Resources have assigned a real number to each employee measuring how "fun" the employee is. In order to keep things social, there is one restriction on the guest list: An employee cannot attend the party if their immediate supervisor is also present. On the other hand, the president of the company *must* attend the party, even though she has a negative fun rating; it's her company, after all.

   Describe an algorithm that makes a guest list for the party that maximizes the sum of the "fun" ratings of the guests. The input to your algorithm is a rooted tree T describing the company hierarchy, where each node v has a field $v.fun$ storing the "fun" rating of the corresponding employee.

Solution (two functions): We define two functions over the nodes of T.

- $\mathit{MaxFunYes}(v)$ is the maximum total "fun" of a legal party among the descendants of v, where v is definitely invited.
- $\mathit{MaxFunNo}(v)$ is the maximum total "fun" of a legal party among the descendants of v, where v is definitely not invited.

We need to compute $\mathit{MaxFunYes}(\mathit{root})$. These two functions obey the following mutual recurrences:

$$\mathit{MaxFunYes}(v) = v.\mathit{fun} + \sum_{\text{children } w \text{ of } v} \mathit{MaxFunNo}(w)$$

$$\mathit{MaxFunNo}(v) = \sum_{\text{children } w \text{ of } v} \max\{\mathit{MaxFunYes}(w), \mathit{MaxFunNo}(w)\}$$

(These recurrences do not require separate base cases, because $\sum\varnothing = 0$.) We can memoize these functions by adding two additional fields $v.\mathit{yes}$ and $v.\mathit{no}$ to each node v in the tree. The values at each node depend only on the vales at its children, so we can compute all $2n$ values using a postorder traversal of T.

```
BestParty(T):
    ComputeMaxFun(T.root)
    return T.root.yes
```

```
ComputeMaxFun(v):
    $v.yes \leftarrow v.fun$
    $v.no \leftarrow 0$
    for all children $w$ of $v$
        ComputeMaxFun($w$)
        $v.yes \leftarrow v.yes + w.no$
        $v.no \leftarrow v.no + \max\{w.yes, w.no\}$
```

(Yes, this is still dynamic programming; we're only traversing the tree recursively because that's the most natural way to traverse trees![2]) The algorithm spends $O(1)$ time at each node, and therefore runs in ***$O(n)$ time*** altogether. ■

[2]A naïve recursive implementation would run in $O(\phi^n)$ time in the worst case, where $\phi = (1+\sqrt{5})/2 \approx 1.618$ is the golden ratio. The worst-case tree is a path—every non-leaf node has exactly one child.

Solution (one function): For each node v in the input tree T, let $\mathit{MaxFun}(v)$ denote the maximum total "fun" of a legal party among the descendants of v, where v may or may not be invited.

The president of the company must be invited, so none of the president's "children" in T can be invited. Thus, the value we need to compute is

$$\mathit{root.fun} + \sum_{\text{grandchildren } w \text{ of } \mathit{root}} \mathit{MaxFun}(w).$$

The function *MaxFun* obeys the following recurrence:

$$\mathit{MaxFun}(v) = \max \left\{ \begin{array}{c} \displaystyle v.\mathit{fun} + \sum_{\text{grandchildren } x \text{ of } v} \mathit{MaxFun}(x) \\ \displaystyle \sum_{\text{children } w \text{ of } v} \mathit{MaxFun}(w) \end{array} \right\}$$

(This recurrence does not require a separate base case, because $\sum \varnothing = 0$.) We can memoize this function by adding an additional field $v.\mathit{maxFun}$ to each node v in the tree. The value at each node depends only on the values at its children and grandchildren, so we can compute all values using a postorder traversal of T.

```
BestParty(T):
    ComputeMaxFun(T.root)
    party ← T.root.fun
    for all children w of T.root
        for all children x of w
            party ← party + x.maxFun
    return party
```

```
ComputeMaxFun(v):
    yes ← v.fun
    no ← 0
    for all children w of v
        ComputeMaxFun(w)
        no ← no + w.maxFun
        for all children x of w
            yes ← yes + x.maxFun
    v.maxFun ← max{yes, no}
```

(Yes, this is still dynamic programming; we're only traversing the tree recursively because that's the most natural way to traverse trees![3])

The algorithm spends $O(1)$ time at each node (because each node has exactly one parent and one grandparent) and therefore runs in $\boldsymbol{O(n)}$ ***time*** altogether. ■

Rubric: 10 points: standard dynamic programming rubric. These are not the only correct solutions.

[3]Like the previous solution, a direct recursive implementation would run in $O(\phi^n)$ time in the worst case, where $\phi = (1+\sqrt{5})/2 \approx 1.618$ is the golden ratio.