CS 473: Undergraduate Algorithms, Spring 2009
Homework 3

Written solutions due Tuesday, March 2, 2009 at 11:59:59pm.

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1. A *meldable priority queue* stores a set of keys from some totally-ordered universe (such as the integers) and supports the following operations:

   - MAKEQUEUE: Return a new priority queue containing the empty set.
   - FINDMIN(Q): Return the smallest element of Q (if any).
   - DELETEMIN(Q): Remove the smallest element in Q (if any).
   - INSERT(Q, x): Insert element x into Q, if it is not already there.
   - DECREASEKEY(Q, x, y): Replace an element $x \in Q$ with a smaller key y. (If $y > x$, the operation fails.) The input is a pointer directly to the node in Q containing x.
   - DELETE(Q, x): Delete the element $x \in Q$. The input is a pointer directly to the node in Q containing x.
   - MELD(Q_1, Q_2): Return a new priority queue containing all the elements of Q_1 and Q_2; this operation destroys Q_1 and Q_2.

   A simple way to implement such a data structure is to use a heap-ordered binary tree, where each node stores a key, along with pointers to its parent and two children. MELD can be implemented using the following randomized algorithm:

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MELD(Q1, Q2):
  if Q1 is empty return Q2
  if Q2 is empty return Q1
  if key(Q1) > key(Q2)
      swap Q1 ↔ Q2
  with probability 1/2
      left(Q1) ← MELD(left(Q1), Q2)
  else
      right(Q1) ← MELD(right(Q1), Q2)
  return Q1
```

   (a) Prove that for *any* heap-ordered binary trees Q_1 and Q_2 (*not* just those constructed by the operations listed above), the expected running time of MELD(Q_1, Q_2) is $O(\log n)$, where n is the total number of nodes in both trees. *[Hint: How long is a random root-to-leaf path in an n-node binary tree if each left/right choice is made with equal probability?]*

   (b) Show that each of the other meldable priority queue operations can be implemented with at most one call to MELD and $O(1)$ additional time. (This implies that every operation takes $O(\log n)$ expected time.)

2. Recall that a *priority search tree* is a binary tree in which every node has both a *search key* and a *priority*, arranged so that the tree is simultaneously a binary search tree for the keys and a min-heap for the priorities. A *heater* is a priority search tree in which the priorities are given by the user, and the search keys are distributed uniformly and independently at random in the real interval $[0,1]$. Intuitively, a heater is the 'opposite' of a treap.

   The following problems consider an n-node heater T whose node priorities are the integers from 1 to n. We identify nodes in T by their priorities; thus, 'node 5' means the node in T with priority 5. The min-heap property implies that node 1 is the root of T. Finally, let i and j be integers with $1 \le i < j \le n$.

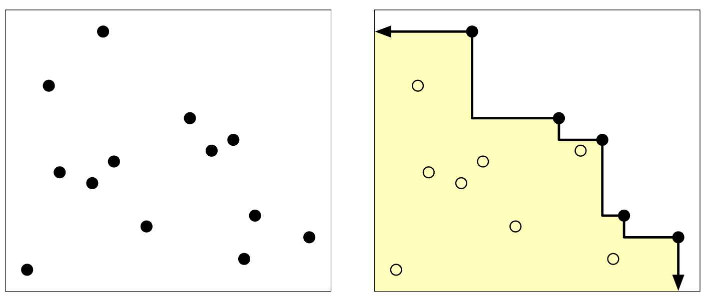
   (a) ***Prove*** that in a random permutation of the $(i+1)$-element set $\{1, 2, \dots, i, j\}$, elements i and j are adjacent with probability $2/(i+1)$.

   (b) ***Prove*** that node i is an ancestor of node j with probability $2/(i+1)$. *[Hint: Use part (a)!]*

   (c) What is the probability that node i is a *descendant* of node j? *[Hint:* ***Don't*** *use part (a)!]*

   (d) What is the *exact* expected depth of node j?

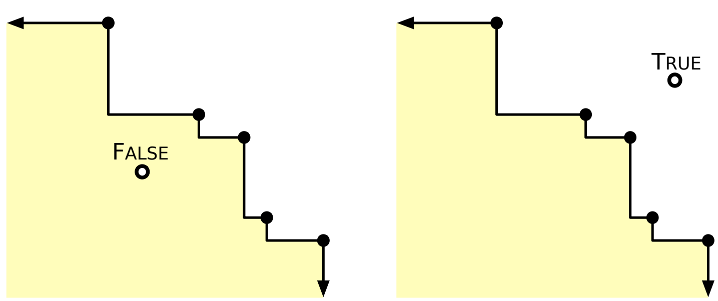
3. Let P be a set of n points in the plane. The *staircase* of P is the set of all points in the plane that have at least one point in P both above and to the right.



A set of points in the plane and its staircase (shaded).

   (a) Describe an algorithm to compute the staircase of a set of n points in $O(n \log n)$ time.

   (b) Describe and analyze a data structure that stores the staircase of a set of points, and an algorithm ABOVE?(x, y) that returns TRUE if the point (x, y) is above the staircase, or FALSE otherwise. Your data structure should use $O(n)$ space, and your ABOVE? algorithm should run in $O(\log n)$ time.



Two staircase queries.