DIVIDED POWER ALGEBRA

09PD

Contents

1. Introduction

09PE In this chapter we talk about divided power algebras and what you can do with them. A reference is the book [Ber74].

2. Divided powers

07GK In this section we collect some results on divided power rings. We will use the convention $0! = 1$ (as empty products should give 1).

07GL **Definition 2.1.** Let A be a ring. Let I be an ideal of A. A collection of maps $\gamma_n : I \to I$, $n > 0$ is called a *divided power structure* on I if for all $n \geq 0$, $m > 0$, $x, y \in I$, and $a \in A$ we have

1. $\gamma_1(x) = x$, we also set $\gamma_0(x) = 1$,
2. $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$,
3. $\gamma_n(ax) = a^n\gamma_n(x)$,
4. $\gamma_n(x + y) = \sum_{i=0,\ldots,n} \gamma_i(x)\gamma_{n-i}(y)$,
5. $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$.

Note that the rational numbers $\frac{(n+m)!}{n!m!}$ and $\frac{(nm)!}{n!(m!)^n}$ occurring in the definition are in fact integers; the first is the number of ways to choose n out of $n + m$ and the second counts the number of ways to divide a group of nm objects into n groups

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This is a chapter of the Stacks Project, version 2c3bdd57, compiled on Jun 18, 2024.

of m. We make some remarks about the definition which show that $\gamma_n(x)$ is a replacement for $x^n/n!$ in I.

07GM **Lemma 2.2.** *Let A be a ring. Let I be an ideal of A.*

1. *If γ is a divided power structure*[1] *on I, then $n!\gamma_n(x) = x^n$ for $n \geq 1$, $x \in I$.*

Assume A is torsion free as a $\mathbf{Z}$-module.

2. *A divided power structure on I, if it exists, is unique.*
3. *If $\gamma_n : I \to I$ are maps then*
$$\gamma \textit{ is a divided power structure} \Leftrightarrow n!\gamma_n(x) = x^n \ \forall x \in I, n \geq 1.$$
4. *The ideal I has a divided power structure if and only if there exists a set of generators x_i of I as an ideal such that for all $n \geq 1$ we have $x_i^n \in (n!)I$.*

Proof. Proof of (1). If γ is a divided power structure, then condition (2) (applied to 1 and $n-1$ instead of n and m) implies that $n\gamma_n(x) = \gamma_1(x)\gamma_{n-1}(x)$. Hence by induction and condition (1) we get $n!\gamma_n(x) = x^n$.

Assume A is torsion free as a $\mathbf{Z}$-module. Proof of (2). This is clear from (1).

Proof of (3). Assume that $n!\gamma_n(x) = x^n$ for all $x \in I$ and $n \geq 1$. Since $A \subset A \otimes_{\mathbf{Z}} \mathbf{Q}$ it suffices to prove the axioms (1) – (5) of Definition 2.1 in case A is a $\mathbf{Q}$-algebra. In this case $\gamma_n(x) = x^n/n!$ and it is straightforward to verify (1) – (5); for example, (4) corresponds to the binomial formula

$$(x+y)^n = \sum_{i=0,\ldots,n} \frac{n!}{i!(n-i)!} x^i y^{n-i}$$

We encourage the reader to do the verifications to make sure that we have the coefficients correct.

Proof of (4). Assume we have generators x_i of I as an ideal such that $x_i^n \in (n!)I$ for all $n \geq 1$. We claim that for all $x \in I$ we have $x^n \in (n!)I$. If the claim holds then we can set $\gamma_n(x) = x^n/n!$ which is a divided power structure by (3). To prove the claim we note that it holds for $x = ax_i$. Hence we see that the claim holds for a set of generators of I as an abelian group. By induction on the length of an expression in terms of these, it suffices to prove the claim for $x + y$ if it holds for x and y. This follows immediately from the binomial theorem. ☐

07GN **Example 2.3.** Let p be a prime number. Let A be a ring such that every integer n not divisible by p is invertible, i.e., A is a $\mathbf{Z}_{(p)}$-algebra. Then $I = pA$ has a canonical divided power structure. Namely, given $x = pa \in I$ we set

$$\gamma_n(x) = \frac{p^n}{n!} a^n$$

The reader verifies immediately that $p^n/n! \in p\mathbf{Z}_{(p)}$ for $n \geq 1$ (for instance, this can be derived from the fact that the exponent of p in the prime factorization of $n!$ is $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \ldots$), so that the definition makes sense and gives us a sequence of maps $\gamma_n : I \to I$. It is a straightforward exercise to verify that conditions (1) – (5) of Definition 2.1 are satisfied. Alternatively, it is clear that the definition works for $A_0 = \mathbf{Z}_{(p)}$ and then the result follows from Lemma 4.2.

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[1] Here and in the following, γ stands short for a sequence of maps $\gamma_1, \gamma_2, \gamma_3, \ldots$ from I to I.

We notice that $\gamma_n(0) = 0$ for any ideal I of A and any divided power structure γ on I. (This follows from axiom (3) in Definition 2.1, applied to $a = 0$.)

07GP **Lemma 2.4.** *Let A be a ring. Let I be an ideal of A. Let $\gamma_n : I \to I$, $n \geq 1$ be a sequence of maps. Assume*

(a) *(1), (3), and (4) of Definition 2.1 hold for all $x, y \in I$, and*
(b) *properties (2) and (5) hold for x in some set of generators of I as an ideal.*

Then γ is a divided power structure on I.

Proof. The numbers (1), (2), (3), (4), (5) in this proof refer to the conditions listed in Definition 2.1. Applying (3) we see that if (2) and (5) hold for x then (2) and (5) hold for ax for all $a \in A$. Hence we see (b) implies (2) and (5) hold for a set of generators of I as an abelian group. Hence, by induction of the length of an expression in terms of these it suffices to prove that, given $x, y \in I$ such that (2) and (5) hold for x and y, then (2) and (5) hold for $x + y$.

Proof of (2) for $x + y$. By (4) we have

$$\gamma_n(x+y)\gamma_m(x+y) = \sum\nolimits_{i+j=n,\ k+l=m} \gamma_i(x)\gamma_k(x)\gamma_j(y)\gamma_l(y)$$

Using (2) for x and y this equals

$$\sum \frac{(i+k)!}{i!k!}\frac{(j+l)!}{j!l!}\gamma_{i+k}(x)\gamma_{j+l}(y)$$

Comparing this with the expansion

$$\gamma_{n+m}(x+y) = \sum \gamma_a(x)\gamma_b(y)$$

we see that we have to prove that given $a + b = n + m$ we have

$$\sum\nolimits_{i+k=a,\ j+l=b,\ i+j=n,\ k+l=m} \frac{(i+k)!}{i!k!}\frac{(j+l)!}{j!l!} = \frac{(n+m)!}{n!m!}.$$

Instead of arguing this directly, we note that the result is true for the ideal $I = (x, y)$ in the polynomial ring $\mathbf{Q}[x, y]$ because $\gamma_n(f) = f^n/n!$, $f \in I$ defines a divided power structure on I. Hence the equality of rational numbers above is true.

Proof of (5) for $x + y$ given that (1) – (4) hold and that (5) holds for x and y. We will again reduce the proof to an equality of rational numbers. Namely, using (4) we can write $\gamma_n(\gamma_m(x+y)) = \gamma_n(\sum \gamma_i(x)\gamma_j(y))$. Using (4) we can write $\gamma_n(\gamma_m(x+y))$ as a sum of terms which are products of factors of the form $\gamma_k(\gamma_i(x)\gamma_j(y))$. If $i > 0$ then

$$\begin{aligned}
\gamma_k(\gamma_i(x)\gamma_j(y)) &= \gamma_j(y)^k\gamma_k(\gamma_i(x)) \\
&= \frac{(ki)!}{k!(i!)^k}\gamma_j(y)^k\gamma_{ki}(x) \\
&= \frac{(ki)!}{k!(i!)^k}\frac{(kj)!}{(j!)^k}\gamma_{ki}(x)\gamma_{kj}(y)
\end{aligned}$$

using (3) in the first equality, (5) for x in the second, and (2) exactly k times in the third. Using (5) for y we see the same equality holds when $i = 0$. Continuing like this using all axioms but (5) we see that we can write

$$\gamma_n(\gamma_m(x+y)) = \sum\nolimits_{i+j=nm} c_{ij}\gamma_i(x)\gamma_j(y)$$

for certain universal constants $c_{ij} \in \mathbf{Z}$. Again the fact that the equality is valid in the polynomial ring $\mathbf{Q}[x, y]$ implies that the coefficients c_{ij} are all equal to $(nm)!/n!(m!)^n$ as desired. $\square$

07GQ **Lemma 2.5.** *Let A be a ring with two ideals $I, J \subset A$. Let γ be a divided power structure on I and let δ be a divided power structure on J. Then*

1. *γ and δ agree on IJ,*
2. *if γ and δ agree on $I \cap J$ then they are the restriction of a unique divided power structure ϵ on $I + J$.*

Proof. Let $x \in I$ and $y \in J$. Then

$$\gamma_n(xy) = y^n\gamma_n(x) = n!\delta_n(y)\gamma_n(x) = \delta_n(y)x^n = \delta_n(xy).$$

Hence γ and δ agree on a set of (additive) generators of IJ. By property (4) of Definition 2.1 it follows that they agree on all of IJ.

Assume γ and δ agree on $I \cap J$. Let $z \in I + J$. Write $z = x + y$ with $x \in I$ and $y \in J$. Then we set

$$\epsilon_n(z) = \sum \gamma_i(x)\delta_{n-i}(y)$$

for all $n \geq 1$. To see that this is well defined, suppose that $z = x' + y'$ is another representation with $x' \in I$ and $y' \in J$. Then $w = x - x' = y' - y \in I \cap J$. Hence

$$\begin{aligned}
\sum\nolimits_{i+j=n} \gamma_i(x)\delta_j(y) &= \sum\nolimits_{i+j=n} \gamma_i(x' + w)\delta_j(y) \\
&= \sum\nolimits_{i'+l+j=n} \gamma_{i'}(x')\gamma_l(w)\delta_j(y) \\
&= \sum\nolimits_{i'+l+j=n} \gamma_{i'}(x')\delta_l(w)\delta_j(y) \\
&= \sum\nolimits_{i'+j'=n} \gamma_{i'}(x')\delta_{j'}(y + w) \\
&= \sum\nolimits_{i'+j'=n} \gamma_{i'}(x')\delta_{j'}(y')
\end{aligned}$$

as desired. Hence, we have defined maps $\epsilon_n : I + J \to I + J$ for all $n \geq 1$; it is easy to see that $\epsilon_n |_I = \gamma_n$ and $\epsilon_n |_J = \delta_n$. Next, we prove conditions (1) – (5) of Definition 2.1 for the collection of maps ϵ_n. Properties (1) and (3) are clear. To see (4), suppose that $z = x + y$ and $z' = x' + y'$ with $x, x' \in I$ and $y, y' \in J$ and compute

$$\begin{aligned}
\epsilon_n(z + z') &= \sum\nolimits_{a+b=n} \gamma_a(x + x')\delta_b(y + y') \\
&= \sum\nolimits_{i+i'+j+j'=n} \gamma_i(x)\gamma_{i'}(x')\delta_j(y)\delta_{j'}(y') \\
&= \sum\nolimits_{k=0,\ldots,n} \sum\nolimits_{i+j=k} \gamma_i(x)\delta_j(y) \sum\nolimits_{i'+j'=n-k} \gamma_{i'}(x')\delta_{j'}(y') \\
&= \sum\nolimits_{k=0,\ldots,n} \epsilon_k(z)\epsilon_{n-k}(z')
\end{aligned}$$

as desired. Now we see that it suffices to prove (2) and (5) for elements of I or J, see Lemma 2.4. This is clear because γ and δ are divided power structures.

The existence of a divided power structure ϵ on $I + J$ whose restrictions to I and J are γ and δ is thus proven; its uniqueness is rather clear. $\square$

07GR **Lemma 2.6.** *Let p be a prime number. Let A be a ring, let $I \subset A$ be an ideal, and let γ be a divided power structure on I. Assume p is nilpotent in A/I. Then I is locally nilpotent if and only if p is nilpotent in A.*

Proof. If $p^N = 0$ in A, then for $x \in I$ we have $x^{pN} = (pN)!\gamma_{pN}(x) = 0$ because $(pN)!$ is divisible by p^N. Conversely, assume I is locally nilpotent. We've also assumed that p is nilpotent in A/I, hence $p^r \in I$ for some r, hence p^r nilpotent, hence p nilpotent. □

3. Divided power rings

07GT There is a category of divided power rings. Here is the definition.

07GU **Definition 3.1.** A *divided power ring* is a triple (A, I, γ) where A is a ring, $I \subset A$ is an ideal, and $\gamma = (\gamma_n)_{n \geq 1}$ is a divided power structure on I. A *homomorphism of divided power rings* $\varphi : (A, I, \gamma) \to (B, J, \delta)$ is a ring homomorphism $\varphi : A \to B$ such that $\varphi(I) \subset J$ and such that $\delta_n(\varphi(x)) = \varphi(\gamma_n(x))$ for all $x \in I$ and $n \geq 1$.

We sometimes say "let (B, J, δ) be a divided power algebra over (A, I, γ)" to indicate that (B, J, δ) is a divided power ring which comes equipped with a homomorphism of divided power rings $(A, I, \gamma) \to (B, J, \delta)$.

07GV **Lemma 3.2.** *The category of divided power rings has all limits and they agree with limits in the category of rings.*

Proof. The empty limit is the zero ring (that's weird but we need it). The product of a collection of divided power rings (A_t, I_t, γ_t), $t \in T$ is given by $(\prod A_t, \prod I_t, \gamma)$ where $\gamma_n((x_t)) = (\gamma_{t,n}(x_t))$. The equalizer of $\alpha, \beta : (A, I, \gamma) \to (B, J, \delta)$ is just $C = \{a \in A \mid \alpha(a) = \beta(a)\}$ with ideal $C \cap I$ and induced divided powers. It follows that all limits exist, see Categories, Lemma 14.11. □

The following lemma illustrates a very general category theoretic phenomenon in the case of divided power algebras.

07GW **Lemma 3.3.** *Let $\mathcal{C}$ be the category of divided power rings. Let $F : \mathcal{C} \to Sets$ be a functor. Assume that*

1. *there exists a cardinal κ such that for every $f \in F(A, I, \gamma)$ there exists a morphism $(A', I', \gamma') \to (A, I, \gamma)$ of $\mathcal{C}$ such that f is the image of $f' \in F(A', I', \gamma')$ and $|A'| \leq \kappa$, and*
2. *F commutes with limits.*

Then F is representable, i.e., there exists an object (B, J, δ) of $\mathcal{C}$ such that

$$F(A, I, \gamma) = \mathrm{Hom}_{\mathcal{C}}((B, J, \delta), (A, I, \gamma))$$

functorially in (A, I, γ).

Proof. This is a special case of Categories, Lemma 25.1. □

07GX **Lemma 3.4.** *The category of divided power rings has all colimits.*

Proof. The empty colimit is $\mathbf{Z}$ with divided power ideal (0). Let's discuss general colimits. Let $\mathcal{C}$ be a category and let $c \mapsto (A_c, I_c, \gamma_c)$ be a diagram. Consider the functor

$$F(B, J, \delta) = \lim_{c \in \mathcal{C}} Hom((A_c, I_c, \gamma_c), (B, J, \delta))$$

Note that any $f = (f_c)_{c \in C} \in F(B, J, \delta)$ has the property that all the images $f_c(A_c)$ generate a subring B' of B of bounded cardinality κ and that all the images $f_c(I_c)$ generate a divided power sub ideal J' of B'. And we get a factorization of f as a f' in $F(B')$ followed by the inclusion $B' \to B$. Also, F commutes with limits. Hence we may apply Lemma 3.3 to see that F is representable and we win. ☐

07GY **Remark 3.5.** The forgetful functor $(A, I, \gamma) \mapsto A$ does not commute with colimits. For example, let

$$\begin{tikzcd}
(B, J, \delta) \arrow[r] & (B'', J'', \delta'') \\
(A, I, \gamma) \arrow[r] \arrow[u] & (B', J', \delta') \arrow[u]
\end{tikzcd}$$

be a pushout in the category of divided power rings. Then in general the map $B \otimes_A B' \to B''$ isn't an isomorphism. (It is always surjective.) An explicit example is given by $(A, I, \gamma) = (\mathbf{Z}, (0), \emptyset)$, $(B, J, \delta) = (\mathbf{Z}/4\mathbf{Z}, 2\mathbf{Z}/4\mathbf{Z}, \delta)$, and $(B', J', \delta') = (\mathbf{Z}/4\mathbf{Z}, 2\mathbf{Z}/4\mathbf{Z}, \delta')$ where $\delta_2(2) = 2$ and $\delta'_2(2) = 0$. More precisely, using Lemma 5.3 we let δ, resp. δ' be the unique divided power structure on J, resp. J' such that $\delta_2 : J \to J$, resp. $\delta'_2 : J' \to J'$ is the map $0 \mapsto 0, 2 \mapsto 2$, resp. $0 \mapsto 0, 2 \mapsto 0$. Then $(B'', J'', \delta'') = (\mathbf{F}_2, (0), \emptyset)$ which doesn't agree with the tensor product. However, note that it is always true that

$$B''/J'' = B/J \otimes_{A/I} B'/J'$$

as can be seen from the universal property of the pushout by considering maps into divided power algebras of the form $(C, (0), \emptyset)$.

4. Extending divided powers

07GZ Here is the definition.

07H0 **Definition 4.1.** Given a divided power ring (A, I, γ) and a ring map $A \to B$ we say γ *extends* to B if there exists a divided power structure $\bar{\gamma}$ on IB such that $(A, I, \gamma) \to (B, IB, \bar{\gamma})$ is a homomorphism of divided power rings.

07H1 **Lemma 4.2.** *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. If γ extends to B then it extends uniquely. Assume (at least) one of the following conditions holds*

1. *$IB = 0$,*
2. *I is principal, or*
3. *$A \to B$ is flat.*

Then γ extends to B.

Proof. Any element of IB can be written as a finite sum $\sum_{i=1}^{t} b_i x_i$ with $b_i \in B$ and $x_i \in I$. If γ extends to $\bar{\gamma}$ on IB then $\bar{\gamma}_n(x_i) = \gamma_n(x_i)$. Thus, conditions (3) and (4) in Definition 2.1 imply that

$$\bar{\gamma}_n(\sum\nolimits_{i=1}^{t} b_i x_i) = \sum\nolimits_{n_1 + \ldots + n_t = n} \prod\nolimits_{i=1}^{t} b_i^{n_i} \gamma_{n_i}(x_i)$$

Thus we see that $\bar{\gamma}$ is unique if it exists.

If $IB = 0$ then setting $\bar{\gamma}_n(0) = 0$ works. If $I = (x)$ then we define $\bar{\gamma}_n(bx) = b^n\gamma_n(x)$. This is well defined: if $b'x = bx$, i.e., $(b - b')x = 0$ then

$$\begin{aligned} b^n\gamma_n(x) - (b')^n\gamma_n(x) &= (b^n - (b')^n)\gamma_n(x) \\ &= (b^{n-1} + \ldots + (b')^{n-1})(b - b')\gamma_n(x) = 0 \end{aligned}$$

because $\gamma_n(x)$ is divisible by x (since $\gamma_n(I) \subset I$) and hence annihilated by $b - b'$. Next, we prove conditions (1) – (5) of Definition 2.1. Parts (1), (2), (3), (5) are obvious from the construction. For (4) suppose that $y, z \in IB$, say $y = bx$ and $z = cx$. Then $y + z = (b + c)x$ hence

$$\begin{aligned} \bar{\gamma}_n(y + z) &= (b + c)^n\gamma_n(x) \\ &= \sum \frac{n!}{i!(n-i)!} b^i c^{n-i}\gamma_n(x) \\ &= \sum b^i c^{n-i}\gamma_i(x)\gamma_{n-i}(x) \\ &= \sum \bar{\gamma}_i(y)\bar{\gamma}_{n-i}(z) \end{aligned}$$

as desired.

Assume $A \to B$ is flat. Suppose that $b_1, \ldots, b_r \in B$ and $x_1, \ldots, x_r \in I$. Then

$$\bar{\gamma}_n(\sum b_i x_i) = \sum b_1^{e_1} \ldots b_r^{e_r}\gamma_{e_1}(x_1)\ldots\gamma_{e_r}(x_r)$$

where the sum is over $e_1 + \ldots + e_r = n$ if $\bar{\gamma}_n$ exists. Next suppose that we have $c_1, \ldots, c_s \in B$ and $a_{ij} \in A$ such that $b_i = \sum a_{ij}c_j$. Setting $y_j = \sum a_{ij}x_i$ we claim that

$$\sum b_1^{e_1} \ldots b_r^{e_r}\gamma_{e_1}(x_1)\ldots\gamma_{e_r}(x_r) = \sum c_1^{d_1} \ldots c_s^{d_s}\gamma_{d_1}(y_1)\ldots\gamma_{d_s}(y_s)$$

in B where on the right hand side we are summing over $d_1 + \ldots + d_s = n$. Namely, using the axioms of a divided power structure we can expand both sides into a sum with coefficients in $\mathbf{Z}[a_{ij}]$ of terms of the form $c_1^{d_1} \ldots c_s^{d_s}\gamma_{e_1}(x_1)\ldots\gamma_{e_r}(x_r)$. To see that the coefficients agree we note that the result is true in $\mathbf{Q}[x_1, \ldots, x_r, c_1, \ldots, c_s, a_{ij}]$ with γ the unique divided power structure on $(x_1, \ldots, x_r)$. By Lazard's theorem (Algebra, Theorem 81.4) we can write B as a directed colimit of finite free A-modules. In particular, if $z \in IB$ is written as $z = \sum x_i b_i$ and $z = \sum x'_{i'}b'_{i'}$, then we can find $c_1, \ldots, c_s \in B$ and $a_{ij}, a'_{i'j} \in A$ such that $b_i = \sum a_{ij}c_j$ and $b'_{i'} = \sum a'_{i'j}c_j$ such that $y_j = \sum x_i a_{ij} = \sum x'_{i'}a'_{i'j}$ holds[2]. Hence the procedure above gives a well defined map $\bar{\gamma}_n$ on IB. By construction $\bar{\gamma}$ satisfies conditions (1), (3), and (4). Moreover, for $x \in I$ we have $\bar{\gamma}_n(x) = \gamma_n(x)$. Hence it follows from Lemma 2.4 that $\bar{\gamma}$ is a divided power structure on IB. ☐

07H2 **Lemma 4.3.** *Let (A, I, γ) be a divided power ring.*

1. *If $\varphi : (A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings, then* $\text{Ker}(\varphi) \cap I$ *is preserved by γ_n for all $n \geq 1$.*
2. *Let $\mathfrak{a} \subset A$ be an ideal and set $I' = I \cap \mathfrak{a}$. The following are equivalent*
   - (a) *I' is preserved by γ_n for all $n > 0$,*
   - (b) *γ extends to $A/\mathfrak{a}$, and*

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[2] This can also be proven without recourse to Algebra, Theorem 81.4. Indeed, if $z = \sum x_i b_i$ and $z = \sum x'_{i'}b'_{i'}$, then $\sum x_i b_i - \sum x'_{i'}b'_{i'} = 0$ is a relation in the A-module B. Thus, Algebra, Lemma 39.11 (applied to the x_i and $x'_{i'}$ taking the place of the f_i, and the b_i and $b'_{i'}$ taking the role of the x_i) yields the existence of the $c_1, \ldots, c_s \in B$ and $a_{ij}, a'_{i'j} \in A$ as required.

(c) *there exist a set of generators x_i of I' as an ideal such that $\gamma_n(x_i) \in I'$ for all $n > 0$.*

Proof. Proof of (1). This is clear. Assume (2)(a). Define $\bar{\gamma}_n(x \bmod I') = \gamma_n(x) \bmod I'$ for $x \in I$. This is well defined since $\gamma_n(x+y) = \gamma_n(x) \bmod I'$ for $y \in I'$ by Definition 2.1 (4) and the fact that $\gamma_j(y) \in I'$ by assumption. It is clear that $\bar{\gamma}$ is a divided power structure as γ is one. Hence (2)(b) holds. Also, (2)(b) implies (2)(a) by part (1). It is clear that (2)(a) implies (2)(c). Assume (2)(c). Note that $\gamma_n(x) = a^n\gamma_n(x_i) \in I'$ for $x = ax_i$. Hence we see that $\gamma_n(x) \in I'$ for a set of generators of I' as an abelian group. By induction on the length of an expression in terms of these, it suffices to prove $\forall n : \gamma_n(x+y) \in I'$ if $\forall n : \gamma_n(x), \gamma_n(y) \in I'$. This follows immediately from the fourth axiom of a divided power structure. □

07H3 **Lemma 4.4.** *Let (A, I, γ) be a divided power ring. Let $E \subset I$ be a subset. Then the smallest ideal $J \subset I$ preserved by γ and containing all $f \in E$ is the ideal J generated by $\gamma_n(f)$, $n \geq 1$, $f \in E$.*

Proof. Follows immediately from Lemma 4.3. □

07KD **Lemma 4.5.** *Let (A, I, γ) be a divided power ring. Let p be a prime. If p is nilpotent in A/I, then*

(1) *the p-adic completion $A^\wedge = \lim_e A/p^eA$ surjects onto A/I,*
(2) *the kernel of this map is the p-adic completion $I^\wedge$ of I, and*
(3) *each γ_n is continuous for the p-adic topology and extends to $\gamma_n^\wedge : I^\wedge \to I^\wedge$ defining a divided power structure on $I^\wedge$.*

If moreover A is a $\mathbf{Z}_{(p)}$-algebra, then

(4) *for e large enough the ideal $p^eA \subset I$ is preserved by the divided power structure γ and*
$$(A^\wedge, I^\wedge, \gamma^\wedge) = \lim_e(A/p^eA, I/p^eA, \bar{\gamma})$$
in the category of divided power rings.

Proof. Let $t \geq 1$ be an integer such that $p^tA/I = 0$, i.e., $p^tA \subset I$. The map $A^\wedge \to A/I$ is the composition $A^\wedge \to A/p^tA \to A/I$ which is surjective (for example by Algebra, Lemma 96.1). As $p^eI \subset p^eA \cap I \subset p^{e-t}I$ for $e \geq t$ we see that the kernel of the composition $A^\wedge \to A/I$ is the p-adic completion of I. The map γ_n is continuous because

$$\gamma_n(x + p^ey) = \sum\nolimits_{i+j=n} p^{je}\gamma_i(x)\gamma_j(y) = \gamma_n(x) \bmod p^eI$$

by the axioms of a divided power structure. It is clear that the axioms for divided power structures are inherited by the maps $\gamma_n^\wedge$ from the maps γ_n. Finally, to see the last statement say $e > t$. Then $p^eA \subset I$ and $\gamma_1(p^eA) \subset p^eA$ and for $n > 1$ we have

$$\gamma_n(p^ea) = p^n\gamma_n(p^{e-1}a) = \frac{p^n}{n!}p^{n(e-1)}a^n \in p^eA$$

as $p^n/n! \in \mathbf{Z}_{(p)}$ and as $n \geq 2$ and $e \geq 2$ so $n(e-1) \geq e$. This proves that γ extends to A/p^eA, see Lemma 4.3. The statement on limits is clear from the construction of limits in the proof of Lemma 3.2. □

5. Divided power polynomial algebras

07H4 A very useful example is the *divided power polynomial algebra*. Let A be a ring. Let $t \geq 1$. We will denote $A\langle x_1, \ldots, x_t\rangle$ the following A-algebra: As an A-module we set

$$A\langle x_1, \ldots, x_t\rangle = \bigoplus\nolimits_{n_1, \ldots, n_t \geq 0} Ax_1^{[n_1]} \ldots x_t^{[n_t]}$$

with multiplication given by

$$x_i^{[n]} x_i^{[m]} = \frac{(n+m)!}{n!m!} x_i^{[n+m]}.$$

We also set $x_i = x_i^{[1]}$. Note that $1 = x_1^{[0]} \ldots x_t^{[0]}$. There is a similar construction which gives the divided power polynomial algebra in infinitely many variables. There is an canonical A-algebra map $A\langle x_1, \ldots, x_t\rangle \to A$ sending $x_i^{[n]}$ to zero for $n > 0$. The kernel of this map is denoted $A\langle x_1, \ldots, x_t\rangle_+$.

07H5 **Lemma 5.1.** *Let (A, I, γ) be a divided power ring. There exists a unique divided power structure δ on*

$$J = IA\langle x_1, \ldots, x_t\rangle + A\langle x_1, \ldots, x_t\rangle_+$$

such that

1. *$\delta_n(x_i) = x_i^{[n]}$, and*
2. *$(A, I, \gamma) \to (A\langle x_1, \ldots, x_t\rangle, J, \delta)$ is a homomorphism of divided power rings.*

Moreover, $(A\langle x_1, \ldots, x_t\rangle, J, \delta)$ has the following universal property: A homomorphism of divided power rings $\varphi : (A\langle x_1, \ldots, x_t\rangle, J, \delta) \to (C, K, \epsilon)$ is the same thing as a homomorphism of divided power rings $A \to C$ and elements $k_1, \ldots, k_t \in K$.

Proof. We will prove the lemma in case of a divided power polynomial algebra in one variable. The result for the general case can be argued in exactly the same way, or by noting that $A\langle x_1, \ldots, x_t\rangle$ is isomorphic to the ring obtained by adjoining the divided power variables $x_1, \ldots, x_t$ one by one.

Let $A\langle x\rangle_+$ be the ideal generated by $x, x^{[2]}, x^{[3]}, \ldots$. Note that $J = IA\langle x\rangle + A\langle x\rangle_+$ and that

$$IA\langle x\rangle \cap A\langle x\rangle_+ = IA\langle x\rangle \cdot A\langle x\rangle_+$$

Hence by Lemma 2.5 it suffices to show that there exist divided power structures on the ideals $IA\langle x\rangle$ and $A\langle x\rangle_+$. The existence of the first follows from Lemma 4.2 as $A \to A\langle x\rangle$ is flat. For the second, note that if A is torsion free, then we can apply Lemma 2.2 (4) to see that δ exists. Namely, choosing as generators the elements $x^{[m]}$ we see that $(x^{[m]})^n = \frac{(nm)!}{(m!)^n} x^{[nm]}$ and $n!$ divides the integer $\frac{(nm)!}{(m!)^n}$. In general write $A = R/\mathfrak{a}$ for some torsion free ring R (e.g., a polynomial ring over $\mathbf{Z}$). The kernel of $R\langle x\rangle \to A\langle x\rangle$ is $\bigoplus \mathfrak{a}x^{[m]}$. Applying criterion (2)(c) of Lemma 4.3 we see that the divided power structure on $R\langle x\rangle_+$ extends to $A\langle x\rangle$ as desired.

Proof of the universal property. Given a homomorphism $\varphi : A \to C$ of divided power rings and $k_1, \ldots, k_t \in K$ we consider

$$A\langle x_1, \ldots, x_t\rangle \to C, \quad x_1^{[n_1]} \ldots x_t^{[n_t]} \longmapsto \epsilon_{n_1}(k_1) \ldots \epsilon_{n_t}(k_t)$$

using φ on coefficients. The only thing to check is that this is an A-algebra homomorphism (details omitted). The inverse construction is clear. $\square$

07H6 **Remark 5.2.** Let (A, I, γ) be a divided power ring. There is a variant of Lemma 5.1 for infinitely many variables. First note that if $s < t$ then there is a canonical map

$$A\langle x_1, \ldots, x_s\rangle \to A\langle x_1, \ldots, x_t\rangle$$

Hence if W is any set, then we set

$$A\langle x_w : w \in W\rangle = \text{colim}_{E \subset W} A\langle x_e : e \in E\rangle$$

(colimit over E finite subset of W) with transition maps as above. By the definition of a colimit we see that the universal mapping property of $A\langle x_w : w \in W\rangle$ is completely analogous to the mapping property stated in Lemma 5.1.

The following lemma can be found in [BO83].

07GS **Lemma 5.3.** *Let p be a prime number. Let A be a ring such that every integer n not divisible by p is invertible, i.e., A is a $\mathbf{Z}_{(p)}$-algebra. Let $I \subset A$ be an ideal. Two divided power structures γ, γ' on I are equal if and only if $\gamma_p = \gamma'_p$. Moreover, given a map $\delta : I \to I$ such that*

1. *$p!\delta(x) = x^p$ for all $x \in I$,*
2. *$\delta(ax) = a^p\delta(x)$ for all $a \in A$, $x \in I$, and*
3. *$\delta(x + y) = \delta(x) + \sum_{i+j=p, i,j\geq 1} \frac{1}{i!j!}x^iy^j + \delta(y)$ for all $x, y \in I$,*

then there exists a unique divided power structure γ on I such that $\gamma_p = \delta$.

Proof. If n is not divisible by p, then $\gamma_n(x) = cx\gamma_{n-1}(x)$ where c is a unit in $\mathbf{Z}_{(p)}$. Moreover,

$$\gamma_{pm}(x) = c\gamma_m(\gamma_p(x))$$

where c is a unit in $\mathbf{Z}_{(p)}$. Thus the first assertion is clear. For the second assertion, we can, working backwards, use these equalities to define all γ_n. More precisely, if $n = a_0 + a_1p + \ldots + a_ep^e$ with $a_i \in \{0, \ldots, p-1\}$ then we set

$$\gamma_n(x) = c_nx^{a_0}\delta(x)^{a_1}\ldots\delta^e(x)^{a_e}$$

for $c_n \in \mathbf{Z}_{(p)}$ defined by

$$c_n = (p!)^{a_1+a_2(1+p)+\ldots+a_e(1+\ldots+p^{e-1})}/n!.$$

Now we have to show the axioms (1) – (5) of a divided power structure, see Definition 2.1. We observe that (1) and (3) are immediate. Verification of (2) and (5) is by a direct calculation which we omit. Let $x, y \in I$. We claim there is a ring map

$$\varphi : \mathbf{Z}_{(p)}\langle u, v\rangle \longrightarrow A$$

which maps $u^{[n]}$ to $\gamma_n(x)$ and $v^{[n]}$ to $\gamma_n(y)$. By construction of $\mathbf{Z}_{(p)}\langle u, v\rangle$ this means we have to check that

$$\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$$

in A and similarly for y. This is true because (2) holds for γ. Let ϵ denote the divided power structure on the ideal $\mathbf{Z}_{(p)}\langle u, v\rangle_+$ of $\mathbf{Z}_{(p)}\langle u, v\rangle$. Next, we claim that $\varphi(\epsilon_n(f)) = \gamma_n(\varphi(f))$ for $f \in \mathbf{Z}_{(p)}\langle u, v\rangle_+$ and all n. This is clear for $n = 0, 1, \ldots, p-1$. For $n = p$ it suffices to prove it for a set of generators of the ideal $\mathbf{Z}_{(p)}\langle u, v\rangle_+$ because both ϵ_p and $\gamma_p = \delta$ satisfy properties (1) and (3) of the lemma. Hence it suffices to prove that $\gamma_p(\gamma_n(x)) = \frac{(pn)!}{p!(n!)^p}\gamma_{pn}(x)$ and similarly for y, which follows as (5) holds for γ. Now, if $n = a_0 + a_1 p + \ldots + a_e p^e$ is an arbitrary integer written in p-adic expansion as above, then

$$\epsilon_n(f) = c_n f^{a_0} \gamma_p(f)^{a_1} \ldots \gamma_p^e(f)^{a_e}$$

because ϵ is a divided power structure. Hence we see that $\varphi(\epsilon_n(f)) = \gamma_n(\varphi(f))$ holds for all n. Applying this for $f = u + v$ we see that axiom (4) for γ follows from the fact that ϵ is a divided power structure. $\square$

6. Tate resolutions

09PF In this section we briefly discuss the resolutions constructed in [Tat57] and [AH86] which combine divided power structures with differential graded algebras. In this section we will use *homological notation* for differential graded algebras. Our differential graded algebras will sit in nonnegative homological degrees. Thus our differential graded algebras (A, d) will be given as chain complexes

$$\ldots \to A_2 \to A_1 \to A_0 \to 0 \to \ldots$$

endowed with a multiplication.

Let R be a ring (commutative, as usual). In this section we will often consider graded R-algebras $A = \bigoplus_{d \geq 0} A_d$ whose components are zero in negative degrees. We will set $A_+ = \bigoplus_{d > 0} A_d$. We will write $A_{even} = \bigoplus_{d \geq 0} A_{2d}$ and $A_{odd} = \bigoplus_{d \geq 0} A_{2d+1}$. Recall that A is graded commutative if $xy = (-1)^{\deg(x)\deg(y)} yx$ for homogeneous elements x, y. Recall that A is strictly graded commutative if in addition $x^2 = 0$ for homogeneous elements x of odd degree. Finally, to understand the following definition, keep in mind that $\gamma_n(x) = x^n/n!$ if A is a $\mathbf{Q}$-algebra.

09PG **Definition 6.1.** Let R be a ring. Let $A = \bigoplus_{d \geq 0} A_d$ be a graded R-algebra which is strictly graded commutative. A collection of maps $\gamma_n : A_{even,+} \to A_{even,+}$ defined for all $n > 0$ is called a *divided power structure* on A if we have

1. $\gamma_n(x) \in A_{2nd}$ if $x \in A_{2d}$,
2. $\gamma_1(x) = x$ for any x, we also set $\gamma_0(x) = 1$,
3. $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!} \gamma_{n+m}(x)$,
4. $\gamma_n(xy) = x^n \gamma_n(y)$ for all $x \in A_{even}$ and $y \in A_{even,+}$,
5. $\gamma_n(xy) = 0$ if $x, y \in A_{odd}$ homogeneous and $n > 1$
6. if $x, y \in A_{even,+}$ then $\gamma_n(x + y) = \sum_{i=0,\ldots,n} \gamma_i(x)\gamma_{n-i}(y)$,
7. $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n} \gamma_{nm}(x)$ for $x \in A_{even,+}$.

Observe that conditions (2), (3), (4), (6), and (7) imply that γ is a "usual" divided power structure on the ideal $A_{even,+}$ of the (commutative) ring A_{even}, see Sections 2, 3, 4, and 5. In particular, we have $n!\gamma_n(x) = x^n$ for all $x \in A_{even,+}$. Condition (1) states that γ is compatible with grading and condition (5) tells us γ_n for $n > 1$ vanishes on products of homogeneous elements of odd degree. But note that it may happen that

$$\gamma_2(z_1 z_2 + z_3 z_4) = z_1 z_2 z_3 z_4$$

is nonzero if z_1, z_2, z_3, z_4 are homogeneous elements of odd degree.

09PH **Example 6.2** (Adjoining odd variable)**.** Let R be a ring. Let (A, γ) be a strictly graded commutative graded R-algebra endowed with a divided power structure as in the definition above. Let $d > 0$ be an odd integer. In this setting we can adjoin a variable T of degree d to A. Namely, set

$$A\langle T\rangle = A \oplus AT$$

with grading given by $A\langle T\rangle_m = A_m \oplus A_{m-d}T$. We claim there is a unique divided power structure on $A\langle T\rangle$ compatible with the given divided power structure on A. Namely, we set

$$\gamma_n(x + yT) = \gamma_n(x) + \gamma_{n-1}(x)yT$$

for $x \in A_{even,+}$ and $y \in A_{odd}$.

09PI **Example 6.3** (Adjoining even variable)**.** Let R be a ring. Let (A, γ) be a strictly graded commutative graded R-algebra endowed with a divided power structure as in the definition above. Let $d > 0$ be an even integer. In this setting we can adjoin a variable T of degree d to A. Namely, set

$$A\langle T\rangle = A \oplus AT \oplus AT^{(2)} \oplus AT^{(3)} \oplus \ldots$$

with multiplication given by

$$T^{(n)}T^{(m)} = \frac{(n+m)!}{n!m!}T^{(n+m)}$$

and with grading given by

$$A\langle T\rangle_m = A_m \oplus A_{m-d}T \oplus A_{m-2d}T^{(2)} \oplus \ldots$$

We claim there is a unique divided power structure on $A\langle T\rangle$ compatible with the given divided power structure on A such that $\gamma_n(T^{(i)}) = T^{(ni)}$. To define the divided power structure we first set

$$\gamma_n\left(\sum\nolimits_{i>0} x_i T^{(i)}\right) = \sum \prod\nolimits_{n=\sum e_i} x_i^{e_i} T^{(ie_i)}$$

if x_i is in A_{even}. If $x_0 \in A_{even,+}$ then we take

$$\gamma_n\left(\sum\nolimits_{i\geq 0} x_i T^{(i)}\right) = \sum\nolimits_{a+b=n} \gamma_a(x_0)\gamma_b\left(\sum\nolimits_{i>0} x_i T^{(i)}\right)$$

where γ_b is as defined above.

0F4I **Remark 6.4.** We can also adjoin a set (possibly infinite) of exterior or divided power generators in a given degree $d > 0$, rather than just one as in Examples 6.2 and 6.3. Namely, following Remark 5.2: for (A, γ) as above and a set J, let $A\langle T_j : j \in J\rangle$ be the directed colimit of the algebras $A\langle T_j : j \in S\rangle$ over all finite subsets S of J. It is immediate that this algebra has a unique divided power structure, compatible with the given structure on A and on each generator T_j.

At this point we tie in the definition of divided power structures with differentials. To understand the definition note that $\mathrm{d}(x^n/n!) = \mathrm{d}(x)x^{n-1}/(n-1)!$ if A is a $\mathbf{Q}$-algebra and $x \in A_{even,+}$.

09PJ **Definition 6.5.** Let R be a ring. Let $A = \bigoplus_{d\geq 0} A_d$ be a differential graded R-algebra which is strictly graded commutative. A divided power structure γ on A is *compatible with the differential graded structure* if $\mathrm{d}(\gamma_n(x)) = \mathrm{d}(x)\gamma_{n-1}(x)$ for all $x \in A_{even,+}$.

Warning: Let (A, d, γ) be as in Definition 6.5. It may not be true that $\gamma_n(x)$ is a boundary, if x is a boundary. Thus γ in general does not induce a divided power structure on the homology algebra $H(A)$. In some papers the authors put an additional compatibility condition in order to ensure that this is the case, but we elect not to do so.

09PK **Lemma 6.6.** *Let (A, d, γ) and (B, d, γ) be as in Definition 6.5. Let $f : A \to B$ be a map of differential graded algebras compatible with divided power structures. Assume*

1. *$H_k(A) = 0$ for $k > 0$, and*
2. *f is surjective.*

Then γ induces a divided power structure on the graded R-algebra $H(B)$.

Proof. Suppose that x and x' are homogeneous of the same degree $2d$ and define the same cohomology class in $H(B)$. Say $x' - x = \mathrm{d}(w)$. Choose a lift $y \in A_{2d}$ of x and a lift $z \in A_{2d+1}$ of w. Then $y' = y + \mathrm{d}(z)$ is a lift of x'. Hence

$$\gamma_n(y') = \sum \gamma_i(y)\gamma_{n-i}(\mathrm{d}(z)) = \gamma_n(y) + \sum\nolimits_{i<n} \gamma_i(y)\gamma_{n-i}(\mathrm{d}(z))$$

Since A is acyclic in positive degrees and since $\mathrm{d}(\gamma_j(\mathrm{d}(z))) = 0$ for all j we can write this as

$$\gamma_n(y') = \gamma_n(y) + \sum\nolimits_{i<n} \gamma_i(y)\mathrm{d}(z_i)$$

for some z_i in A. Moreover, for $0 < i < n$ we have

$$\mathrm{d}(\gamma_i(y)z_i) = \mathrm{d}(\gamma_i(y))z_i + \gamma_i(y)\mathrm{d}(z_i) = \mathrm{d}(y)\gamma_{i-1}(y)z_i + \gamma_i(y)\mathrm{d}(z_i)$$

and the first term maps to zero in B as $\mathrm{d}(y)$ maps to zero in B. Hence $\gamma_n(x')$ and $\gamma_n(x)$ map to the same element of $H(B)$. Thus we obtain a well defined map $\gamma_n : H_{2d}(B) \to H_{2nd}(B)$ for all $d > 0$ and $n > 0$. We omit the verification that this defines a divided power structure on $H(B)$. □

09PL **Lemma 6.7.** *Let (A, d, γ) be as in Definition 6.5. Let $R \to R'$ be a ring map. Then d and γ induce similar structures on $A' = A \otimes_R R'$ such that (A', d, γ) is as in Definition 6.5.*

Proof. Observe that $A'_{even} = A_{even} \otimes_R R'$ and $A'_{even,+} = A_{even,+} \otimes_R R'$. Hence we are trying to show that the divided powers γ extend to A'_{even} (terminology as in Definition 4.1). Once we have shown γ extends it follows easily that this extension has all the desired properties.

Choose a polynomial R-algebra P (on any set of generators) and a surjection of R-algebras $P \to R'$. The ring map $A_{even} \to A_{even} \otimes_R P$ is flat, hence the divided powers γ extend to $A_{even} \otimes_R P$ uniquely by Lemma 4.2. Let $J = \mathrm{Ker}(P \to R')$. To show that γ extends to $A \otimes_R R'$ it suffices to show that $I' = \mathrm{Ker}(A_{even,+} \otimes_R P \to A_{even,+} \otimes_R R')$ is generated by elements z such that $\gamma_n(z) \in I'$ for all $n > 0$. This is clear as I' is generated by elements of the form $x \otimes f$ with $x \in A_{even,+}$ and $f \in \mathrm{Ker}(P \to R')$. □

09PM **Lemma 6.8.** *Let (A, d, γ) be as in Definition 6.5. Let $d \geq 1$ be an integer. Let $A\langle T\rangle$ be the graded divided power polynomial algebra on T with $\deg(T) = d$ constructed in Example 6.2 or 6.3. Let $f \in A_{d-1}$ be an element with $d(f) = 0$. There exists a unique differential d on $A\langle T\rangle$ such that $d(T) = f$ and such that d is compatible with the divided power structure on $A\langle T\rangle$.*

Proof. This is proved by a direct computation which is omitted. ☐

In Lemma 12.3 we will compute the cohomology of $A\langle T\rangle$ in some special cases. Here is Tate's construction, as extended by Avramov and Halperin.

09PN **Lemma 6.9.** *Let $R \to S$ be a homomorphism of commutative rings. There exists a factorization*

$$R \to A \to S$$

with the following properties:

1. *(A, d, γ) is as in Definition 6.5,*
2. *$A \to S$ is a quasi-isomorphism (if we endow S with the zero differential),*
3. *$A_0 = R[x_j : j \in J] \to S$ is any surjection of a polynomial ring onto S, and*
4. *A is a graded divided power polynomial algebra over R.*

The last condition means that A is constructed out of A_0 by successively adjoining a set of variables T in each degree > 0 as in Example 6.2 or 6.3. Moreover, if R is Noetherian and $R \to S$ is of finite type, then A can be taken to have only finitely many generators in each degree.

Proof. We write out the construction for the case that R is Noetherian and $R \to S$ is of finite type. Without those assumptions, the proof is the same, except that we have to use some set (possibly infinite) of generators in each degree.

Start of the construction: Let $A(0) = R[x_1, \ldots, x_n]$ be a (usual) polynomial ring and let $A(0) \to S$ be a surjection. As grading we take $A(0)_0 = A(0)$ and $A(0)_d = 0$ for $d \neq 0$. Thus $\mathrm{d} = 0$ and γ_n, $n > 0$, is zero as well.

Choose generators $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$ for the kernel of the given map $A(0) = R[x_1, \ldots, x_n] \to S$. We apply Example 6.2 m times to get

$$A(1) = A(0)\langle T_1, \ldots, T_m\rangle$$

with $\deg(T_i) = 1$ as a graded divided power polynomial algebra. We set $\mathrm{d}(T_i) = f_i$. Since $A(1)$ is a divided power polynomial algebra over $A(0)$ and since $\mathrm{d}(f_i) = 0$ this extends uniquely to a differential on $A(1)$ by Lemma 6.8.

Induction hypothesis: Assume we are given factorizations

$$R \to A(0) \to A(1) \to \ldots \to A(m) \to S$$

where $A(0)$ and $A(1)$ are as above and each $R \to A(m') \to S$ for $2 \leq m' \leq m$ satisfies properties (1) and (4) of the statement of the lemma and (2) replaced by the condition that $H_i(A(m')) \to H_i(S)$ is an isomorphism for $m' > i \geq 0$. The base case is $m = 1$.

Induction step: Assume we have $R \to A(m) \to S$ as in the induction hypothesis. Consider the group $H_m(A(m))$. This is a module over $H_0(A(m)) = S$. In fact, it is a subquotient of $A(m)_m$ which is a finite type module over $A(m)_0 = R[x_1, \ldots, x_n]$. Thus we can pick finitely many elements

$$e_1, \ldots, e_t \in \mathrm{Ker}(\mathrm{d} : A(m)_m \to A(m)_{m-1})$$

which map to generators of this module. Applying Example 6.2 or 6.3 t times we get

$$A(m+1) = A(m)\langle T_1, \ldots, T_t\rangle$$

with $\deg(T_i) = m + 1$ as a graded divided power algebra. We set $\mathrm{d}(T_i) = e_i$. Since $A(m+1)$ is a divided power polynomial algebra over $A(m)$ and since $\mathrm{d}(e_i) = 0$ this extends uniquely to a differential on $A(m+1)$ compatible with the divided power structure. Since we've added only material in degree $m+1$ and higher we see that $H_i(A(m+1)) = H_i(A(m))$ for $i < m$. Moreover, it is clear that $H_m(A(m+1)) = 0$ by construction.

To finish the proof we observe that we have shown there exists a sequence of maps

$$R \to A(0) \to A(1) \to \ldots \to A(m) \to A(m+1) \to \ldots \to S$$

and to finish the proof we set $A = \mathrm{colim}\, A(m)$. □

0BZ9 **Lemma 6.10.** *Let $R \to S$ be a pseudo-coherent ring map (More on Algebra, Definition 82.1). Then Lemma 6.9 holds, with the resolution A of S having finitely many generators in each degree.*

Proof. This is proved in exactly the same way as Lemma 6.9. The only additional twist is that, given $A(m) \to S$ we have to show that $H_m = H_m(A(m))$ is a finite $R[x_1, \ldots, x_m]$-module (so that in the next step we need only add finitely many variables). Consider the complex

$$\ldots \to A(m)_{m-1} \to A(m)_m \to A(m)_{m-1} \to \ldots \to A(m)_0 \to S \to 0$$

Since S is a pseudo-coherent $R[x_1, \ldots, x_n]$-module and since $A(m)_i$ is a finite free $R[x_1, \ldots, x_n]$-module we conclude that this is a pseudo-coherent complex, see More on Algebra, Lemma 64.9. Since the complex is exact in (homological) degrees $> m$ we conclude that H_m is a finite R-module by More on Algebra, Lemma 64.3. □

09PP **Lemma 6.11.** *Let R be a commutative ring. Suppose that (A, d, γ) and (B, d, γ) are as in Definition 6.5. Let $\overline{\varphi} : H_0(A) \to H_0(B)$ be an R-algebra map. Assume*

1. *A is a graded divided power polynomial algebra over R.*
2. *$H_k(B) = 0$ for $k > 0$.*

Then there exists a map $\varphi : A \to B$ of differential graded R-algebras compatible with divided powers that lifts $\overline{\varphi}$.

Proof. The assumption means that A is obtained from R by successively adjoining some set of polynomial generators in degree zero, exterior generators in positive odd degrees, and divided power generators in positive even degrees. So we have a filtration $R \subset A(0) \subset A(1) \subset \ldots$ of A such that $A(m+1)$ is obtained from $A(m)$ by adjoining generators of the appropriate type (which we simply call "divided power generators") in degree $m+1$. In particular, $A(0) \to H_0(A)$ is a surjection from a (usual) polynomial algebra over R onto $H_0(A)$. Thus we can lift $\overline{\varphi}$ to an R-algebra map $\varphi(0) : A(0) \to B_0$.

Write $A(1) = A(0)\langle T_j : j \in J\rangle$ for some set J of divided power variables T_j of degree 1. Let $f_j \in B_0$ be $f_j = \varphi(0)(\mathrm{d}(T_j))$. Observe that f_j maps to zero in $H_0(B)$ as $\mathrm{d}T_j$ maps to zero in $H_0(A)$. Thus we can find $b_j \in B_1$ with $\mathrm{d}(b_j) = f_j$. By the universal property of divided power polynomial algebras from Lemma 5.1, we find a lift $\varphi(1) : A(1) \to B$ of $\varphi(0)$ mapping T_j to f_j.

Having constructed $\varphi(m)$ for some $m \geq 1$ we can construct $\varphi(m+1) : A(m+1) \to B$ in exactly the same manner. We omit the details. □

09PQ **Lemma 6.12.** *Let R be a commutative ring. Let S and T be commutative R-algebras. Then there is a canonical structure of a strictly graded commutative R-algebra with divided powers on*

$$\mathrm{Tor}_*^R(S, T).$$

Proof. Choose a factorization $R \to A \to S$ as above. Since $A \to S$ is a quasi-isomorphism and since A_d is a free R-module, we see that the differential graded algebra $B = A \otimes_R T$ computes the Tor groups displayed in the lemma. Choose a surjection $R[y_j : j \in J] \to T$. Then we see that B is a quotient of the differential graded algebra $A[y_j : j \in J]$ whose homology sits in degree 0 (it is equal to $S[y_j : j \in J]$). By Lemma 6.7 the differential graded algebras B and $A[y_j : j \in J]$ have divided power structures compatible with the differentials. Hence we obtain our divided power structure on $H(B)$ by Lemma 6.6.

The divided power algebra structure constructed in this way is independent of the choice of A. Namely, if A' is a second choice, then Lemma 6.11 implies there is a map $A \to A'$ preserving all structure and the augmentations towards S. Then the induced map $B = A \otimes_R T \to A' \otimes_R T' = B'$ also preserves all structure and is a quasi-isomorphism. The induced isomorphism of Tor algebras is therefore compatible with products and divided powers. ☐

7. Application to complete intersections

09PR Let R be a ring. Let (A, d, γ) be as in Definition 6.5. A *derivation* of degree 2 is an R-linear map $\theta : A \to A$ with the following properties

1. $\theta(A_d) \subset A_{d-2}$,
2. $\theta(xy) = \theta(x)y + x\theta(y)$,
3. θ commutes with d,
4. $\theta(\gamma_n(x)) = \theta(x)\gamma_{n-1}(x)$ for all $x \in A_{2d}$ all d.

In the following lemma we construct a derivation.

09PS **Lemma 7.1.** *Let R be a ring. Let (A, d, γ) be as in Definition 6.5. Let $R' \to R$ be a surjection of rings whose kernel has square zero and is generated by one element f. If A is a graded divided power polynomial algebra over R with finitely many variables in each degree, then we obtain a derivation $\theta : A/IA \to A/IA$ where I is the annihilator of f in R.*

Proof. Since A is a divided power polynomial algebra, we can find a divided power polynomial algebra A' over R' such that $A = A' \otimes_{R'} R$. Moreover, we can lift d to an R-linear operator d on A' such that

1. $\mathrm{d}(xy) = \mathrm{d}(x)y + (-1)^{\deg(x)}x\mathrm{d}(y)$ for $x, y \in A'$ homogeneous, and
2. $\mathrm{d}(\gamma_n(x)) = \mathrm{d}(x)\gamma_{n-1}(x)$ for $x \in A'_{even,+}$.

We omit the details (hint: proceed one variable at the time). However, it may not be the case that d^2 is zero on A'. It is clear that d^2 maps A' into $fA' \cong A/IA$. Hence d^2 annihilates fA' and factors as a map $A \to A/IA$. Since d^2 is R-linear we obtain our map $\theta : A/IA \to A/IA$. The verification of the properties of a derivation is immediate. ☐

09PT **Lemma 7.2.** *Assumption and notation as in Lemma 7.1. Suppose $S = H_0(A)$ is isomorphic to $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ for some n, m, and $f_j \in R[x_1, \ldots, x_n]$.*

Moreover, suppose given a relation

$$\sum r_j f_j = 0$$

with $r_j \in R[x_1, \ldots, x_n]$*. Choose* $r'_j, f'_j \in R'[x_1, \ldots, x_n]$ *lifting* r_j, f_j*. Write* $\sum r'_j f'_j = gf$ *for some* $g \in R/I[x_1, \ldots, x_n]$*. If* $H_1(A) = 0$ *and all the coefficients of each* r_j *are in* I*, then there exists an element* $\xi \in H_2(A/IA)$ *such that* $\theta(\xi) = g$ *in* S/IS*.*

Proof. Let $A(0) \subset A(1) \subset A(2) \subset \ldots$ be the filtration of A such that $A(m)$ is gotten from $A(m-1)$ by adjoining divided power variables of degree m. Then $A(0)$ is a polynomial algebra over R equipped with an R-algebra surjection $A(0) \to S$. Thus we can choose a map

$$\varphi : R[x_1, \ldots, x_n] \to A(0)$$

lifting the augmentations to S. Next, $A(1) = A(0)\langle T_1, \ldots, T_t\rangle$ for some divided power variables T_i of degree 1. Since $H_0(A) = S$ we can pick $\xi_j \in \sum A(0)T_i$ with $\mathrm{d}(\xi_j) = \varphi(f_j)$. Then

$$\mathrm{d}\left(\sum \varphi(r_j)\xi_j\right) = \sum \varphi(r_j)\varphi(f_j) = \sum \varphi(r_j f_j) = 0$$

Since $H_1(A) = 0$ we can pick $\xi \in A_2$ with $\mathrm{d}(\xi) = \sum \varphi(r_j)\xi_j$. If the coefficients of r_j are in I, then the same is true for $\varphi(r_j)$. In this case $\mathrm{d}(\xi)$ dies in A_1/IA_1 and hence ξ defines a class in $H_2(A/IA)$.

The construction of θ in the proof of Lemma 7.1 proceeds by successively lifting $A(i)$ to $A'(i)$ and lifting the differential d. We lift φ to $\varphi' : R'[x_1, \ldots, x_n] \to A'(0)$. Next, we have $A'(1) = A'(0)\langle T_1, \ldots, T_t\rangle$. Moreover, we can lift ξ_j to $\xi'_j \in \sum A'(0)T_i$. Then $\mathrm{d}(\xi'_j) = \varphi'(f'_j) + fa_j$ for some $a_j \in A'(0)$. Consider a lift $\xi' \in A'_2$ of ξ. Then we know that

$$\mathrm{d}(\xi') = \sum \varphi'(r'_j)\xi'_j + \sum f b_i T_i$$

for some $b_i \in A(0)$. Applying d again we find

$$\theta(\xi) = \sum \varphi'(r'_j)\varphi'(f'_j) + \sum f\varphi'(r'_j)a_j + \sum f b_i \mathrm{d}(T_i)$$

The first term gives us what we want. The second term is zero because the coefficients of r_j are in I and hence are annihilated by f. The third term maps to zero in H_0 because $\mathrm{d}(T_i)$ maps to zero. □

The method of proof of the following lemma is apparently due to Gulliksen.

09PU **Lemma 7.3.** *Let* $R' \to R$ *be a surjection of Noetherian rings whose kernel has square zero and is generated by one element* f*. Let* $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$*. Let* $\sum r_j f_j = 0$ *be a relation in* $R[x_1, \ldots, x_n]$*. Assume that*

1. *each* r_j *has coefficients in the annihilator* I *of* f *in* R*,*
2. *for some lifts* $r'_j, f'_j \in R'[x_1, \ldots, x_n]$ *we have* $\sum r'_j f'_j = gf$ *where* g *is not nilpotent in* S/IS*.*

Then S *does not have finite tor dimension over* R *(i.e.,* S *is not a perfect* R*-algebra).*

Proof. Choose a Tate resolution $R \to A \to S$ as in Lemma 6.9. Let $\xi \in H_2(A/IA)$ and $\theta : A/IA \to A/IA$ be the element and derivation found in Lemmas 7.1 and 7.2. Observe that

$$\theta^n(\gamma_n(\xi)) = g^n$$

in $H_0(A/IA) = S/IS$. Hence if g is not nilpotent in S/IS, then ξ^n is nonzero in $H_{2n}(A/IA)$ for all $n > 0$. Since $H_{2n}(A/IA) = \mathrm{Tor}^R_{2n}(S, R/I)$ we conclude. □

The following result can be found in [Rod88].

09PV **Lemma 7.4.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset J \subset A$ be proper ideals. If A/J has finite tor dimension over A/I, then $I/\mathfrak{m}I \to J/\mathfrak{m}J$ is injective.*

Proof. Let $f \in I$ be an element mapping to a nonzero element of $I/\mathfrak{m}I$ which is mapped to zero in $J/\mathfrak{m}J$. We can choose an ideal I' with $\mathfrak{m}I \subset I' \subset I$ such that I/I' is generated by the image of f. Set $R = A/I$ and $R' = A/I'$. Let $J = (a_1, \ldots, a_m)$ for some $a_j \in A$. Then $f = \sum b_j a_j$ for some $b_j \in \mathfrak{m}$. Let $r_j, f_j \in R$ resp. $r'_j, f'_j \in R'$ be the image of b_j, a_j. Then we see we are in the situation of Lemma 7.3 (with the ideal I of that lemma equal to $\mathfrak{m}_R$) and the lemma is proved. ☐

09PW **Lemma 7.5.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset J \subset A$ be proper ideals. Assume*

1. *A/J has finite tor dimension over A/I, and*
2. *J is generated by a regular sequence.*

Then I is generated by a regular sequence and J/I is generated by a regular sequence.

Proof. By Lemma 7.4 we see that $I/\mathfrak{m}I \to J/\mathfrak{m}J$ is injective. Thus we can find $s \leq r$ and a minimal system of generators $f_1, \ldots, f_r$ of J such that $f_1, \ldots, f_s$ are in I and form a minimal system of generators of I. The lemma follows as any minimal system of generators of J is a regular sequence by More on Algebra, Lemmas 30.15 and 30.7. ☐

09PX **Lemma 7.6.** *Let $R \to S$ be a local ring map of Noetherian local rings. Let $I \subset R$ and $J \subset S$ be ideals with $IS \subset J$. If $R \to S$ is flat and $S/\mathfrak{m}_R S$ is regular, then the following are equivalent*

1. *J is generated by a regular sequence and S/J has finite tor dimension as a module over R/I,*
2. *J is generated by a regular sequence and $Tor_p^{R/I}(S/J, R/\mathfrak{m}_R)$ is nonzero for only finitely many p,*
3. *I is generated by a regular sequence and J/IS is generated by a regular sequence in S/IS.*

Proof. If (3) holds, then J is generated by a regular sequence, see for example More on Algebra, Lemmas 30.13 and 30.7. Moreover, if (3) holds, then $S/J = (S/I)/(J/I)$ has finite projective dimension over S/IS because the Koszul complex will be a finite free resolution of S/J over S/IS. Since $R/I \to S/IS$ is flat, it then follows that S/J has finite tor dimension over R/I by More on Algebra, Lemma 66.11. Thus (3) implies (1).

The implication (1) $\Rightarrow$ (2) is trivial. Assume (2). By More on Algebra, Lemma 77.6 we find that S/J has finite tor dimension over S/IS. Thus we can apply Lemma 7.5 to conclude that IS and J/IS are generated by regular sequences. Let $f_1, \ldots, f_r \in I$ be a minimal system of generators of I. Since $R \to S$ is flat, we see that $f_1, \ldots, f_r$ form a minimal system of generators for IS in S. Thus $f_1, \ldots, f_r \in R$ is a sequence of elements whose images in S form a regular sequence by More on Algebra, Lemmas 30.15 and 30.7. Thus $f_1, \ldots, f_r$ is a regular sequence in R by Algebra, Lemma 68.5. ☐

8. Local complete intersection rings

09PY Let $(A, \mathfrak{m})$ be a Noetherian complete local ring. By the Cohen structure theorem (see Algebra, Theorem 160.8) we can write A as the quotient of a regular Noetherian complete local ring R. Let us say that A is a *complete intersection* if there exists some surjection $R \to A$ with R a regular local ring such that the kernel is generated by a regular sequence. The following lemma shows this notion is independent of the choice of the surjection.

09PZ **Lemma 8.1.** *Let $(A, \mathfrak{m})$ be a Noetherian complete local ring. The following are equivalent*

1. *for every surjection of local rings $R \to A$ with R a regular local ring, the kernel of $R \to A$ is generated by a regular sequence, and*
2. *for some surjection of local rings $R \to A$ with R a regular local ring, the kernel of $R \to A$ is generated by a regular sequence.*

Proof. Let k be the residue field of A. If the characteristic of k is $p > 0$, then we denote Λ a Cohen ring (Algebra, Definition 160.5) with residue field k (Algebra, Lemma 160.6). If the characteristic of k is 0 we set $\Lambda = k$. Recall that $\Lambda[[x_1, \ldots, x_n]]$ for any n is formally smooth over $\mathbf{Z}$, resp. $\mathbf{Q}$ in the $\mathfrak{m}$-adic topology, see More on Algebra, Lemma 39.1. Fix a surjection $\Lambda[[x_1, \ldots, x_n]] \to A$ as in the Cohen structure theorem (Algebra, Theorem 160.8).

Let $R \to A$ be a surjection from a regular local ring R. Let $f_1, \ldots, f_r$ be a minimal sequence of generators of $\text{Ker}(R \to A)$. We will use without further mention that an ideal in a Noetherian local ring is generated by a regular sequence if and only if any minimal set of generators is a regular sequence. Observe that $f_1, \ldots, f_r$ is a regular sequence in R if and only if $f_1, \ldots, f_r$ is a regular sequence in the completion $R^\wedge$ by Algebra, Lemmas 68.5 and 97.2. Moreover, we have

$$R^\wedge/(f_1, \ldots, f_r)R^\wedge = (R/(f_1, \ldots, f_n))^\wedge = A^\wedge = A$$

because A is $\mathfrak{m}_A$-adically complete (first equality by Algebra, Lemma 97.1). Finally, the ring $R^\wedge$ is regular since R is regular (More on Algebra, Lemma 43.4). Hence we may assume R is complete.

If R is complete we can choose a map $\Lambda[[x_1, \ldots, x_n]] \to R$ lifting the given map $\Lambda[[x_1, \ldots, x_n]] \to A$, see More on Algebra, Lemma 37.5. By adding some more variables $y_1, \ldots, y_m$ mapping to generators of the kernel of $R \to A$ we may assume that $\Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m]] \to R$ is surjective (some details omitted). Then we can consider the commutative diagram

$$\begin{tikzcd}
\Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m]] \arrow[r] \arrow[d] & R \arrow[d] \\
\Lambda[[x_1, \ldots, x_n]] \arrow[r] & A
\end{tikzcd}$$

By Algebra, Lemma 135.6 we see that the condition for $R \to A$ is equivalent to the condition for the fixed chosen map $\Lambda[[x_1, \ldots, x_n]] \to A$. This finishes the proof of the lemma. $\square$

The following two lemmas are sanity checks on the definition given above.

09Q0 **Lemma 8.2.** *Let R be a regular ring. Let $\mathfrak p \subset R$ be a prime. Let $f_1, \ldots, f_r \in \mathfrak p$ be a regular sequence. Then the completion of*

$$A = (R/(f_1, \ldots, f_r))_{\mathfrak p} = R_{\mathfrak p}/(f_1, \ldots, f_r)R_{\mathfrak p}$$

is a complete intersection in the sense defined above.

Proof. The completion of A is equal to $A^\wedge = R_{\mathfrak p}^\wedge/(f_1, \ldots, f_r)R_{\mathfrak p}^\wedge$ because completion for finite modules over the Noetherian ring $R_{\mathfrak p}$ is exact (Algebra, Lemma 97.1). The image of the sequence $f_1, \ldots, f_r$ in $R_{\mathfrak p}$ is a regular sequence by Algebra, Lemmas 97.2 and 68.5. Moreover, $R_{\mathfrak p}^\wedge$ is a regular local ring by More on Algebra, Lemma 43.4. Hence the result holds by our definition of complete intersection for complete local rings. ☐

The following lemma is the analogue of Algebra, Lemma 135.4.

09Q1 **Lemma 8.3.** *Let R be a regular ring. Let $\mathfrak p \subset R$ be a prime. Let $I \subset \mathfrak p$ be an ideal. Set $A = (R/I)_{\mathfrak p} = R_{\mathfrak p}/I_{\mathfrak p}$. The following are equivalent*

1. *the completion of A is a complete intersection in the sense above,*
2. *$I_{\mathfrak p} \subset R_{\mathfrak p}$ is generated by a regular sequence,*
3. *the module $(I/I^2)_{\mathfrak p}$ can be generated by $\dim(R_{\mathfrak p}) - \dim(A)$ elements,*
4. *add more here.*

Proof. We may and do replace R by its localization at $\mathfrak p$. Then $\mathfrak p = \mathfrak m$ is the maximal ideal of R and $A = R/I$. Let $f_1, \ldots, f_r \in I$ be a minimal sequence of generators. The completion of A is equal to $A^\wedge = R^\wedge/(f_1, \ldots, f_r)R^\wedge$ because completion for finite modules over the Noetherian ring $R_{\mathfrak p}$ is exact (Algebra, Lemma 97.1).

If (1) holds, then the image of the sequence $f_1, \ldots, f_r$ in $R^\wedge$ is a regular sequence by assumption. Hence it is a regular sequence in R by Algebra, Lemmas 97.2 and 68.5. Thus (1) implies (2).

Assume (3) holds. Set $c = \dim(R) - \dim(A)$ and let $f_1, \ldots, f_c \in I$ map to generators of I/I^2. by Nakayama's lemma (Algebra, Lemma 20.1) we see that $I = (f_1, \ldots, f_c)$. Since R is regular and hence Cohen-Macaulay (Algebra, Proposition 103.4) we see that $f_1, \ldots, f_c$ is a regular sequence by Algebra, Proposition 103.4. Thus (3) implies (2). Finally, (2) implies (1) by Lemma 8.2. ☐

The following result is due to Avramov, see [Avr75].

09Q2 **Proposition 8.4.** *Let $A \to B$ be a flat local homomorphism of Noetherian local rings. Then the following are equivalent*

1. *$B^\wedge$ is a complete intersection,*
2. *$A^\wedge$ and $(B/\mathfrak m_A B)^\wedge$ are complete intersections.*

Proof. Consider the diagram

$$\begin{tikzcd}
B \arrow[r] & B^\wedge \\
A \arrow[r] \arrow[u] & A^\wedge \arrow[u]
\end{tikzcd}$$

Since the horizontal maps are faithfully flat (Algebra, Lemma 97.3) we conclude that the right vertical arrow is flat (for example by Algebra, Lemma 99.15). Moreover, we have $(B/\mathfrak{m}_A B)^\wedge = B^\wedge/\mathfrak{m}_{A^\wedge} B^\wedge$ by Algebra, Lemma 97.1. Thus we may assume A and B are complete local Noetherian rings.

Assume A and B are complete local Noetherian rings. Choose a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in More on Algebra, Lemma 39.3. Let $I = \text{Ker}(R \to A)$ and $J = \text{Ker}(S \to B)$. Note that since $R/I = A \to B = S/J$ is flat the map $J/IS \otimes_R R/\mathfrak{m}_R \to J/J \cap \mathfrak{m}_R S$ is an isomorphism. Hence a minimal system of generators of J/IS maps to a minimal system of generators of $\text{Ker}(S/\mathfrak{m}_R S \to B/\mathfrak{m}_A B)$. Finally, $S/\mathfrak{m}_R S$ is a regular local ring.

Assume (1) holds, i.e., J is generated by a regular sequence. Since $A = R/I \to B = S/J$ is flat we see Lemma 7.6 applies and we deduce that I and J/IS are generated by regular sequences. We have $\dim(B) = \dim(A) + \dim(B/\mathfrak{m}_A B)$ and $\dim(S/IS) = \dim(A) + \dim(S/\mathfrak{m}_R S)$ (Algebra, Lemma 112.7). Thus J/IS is generated by

$$\dim(S/IS) - \dim(S/J) = \dim(S/\mathfrak{m}_R S) - \dim(B/\mathfrak{m}_A B)$$

elements (Algebra, Lemma 60.13). It follows that $\text{Ker}(S/\mathfrak{m}_R S \to B/\mathfrak{m}_A B)$ is generated by the same number of elements (see above). Hence $\text{Ker}(S/\mathfrak{m}_R S \to B/\mathfrak{m}_A B)$ is generated by a regular sequence, see for example Lemma 8.3. In this way we see that (2) holds.

If (2) holds, then I and $J/J \cap \mathfrak{m}_R S$ are generated by regular sequences. Lifting these generators (see above), using flatness of $R/I \to S/IS$, and using Grothendieck's lemma (Algebra, Lemma 99.3) we find that J/IS is generated by a regular sequence in S/IS. Thus Lemma 7.6 tells us that J is generated by a regular sequence, whence (1) holds. ☐

09Q3 **Definition 8.5.** Let A be a Noetherian ring.

(1) If A is local, then we say A is a *complete intersection* if its completion is a complete intersection in the sense above.
(2) In general we say A is a *local complete intersection* if all of its local rings are complete intersections.

We will check below that this does not conflict with the terminology introduced in Algebra, Definitions 135.1 and 135.5. But first, we show this "makes sense" by showing that if A is a Noetherian local complete intersection, then A is a local complete intersection, i.e., all of its local rings are complete intersections.

09Q4 **Lemma 8.6.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $\mathfrak{p} \subset A$ be a prime ideal. If A is a complete intersection, then $A_\mathfrak{p}$ is a complete intersection too.*

Proof. Choose a prime $\mathfrak{q}$ of $A^\wedge$ lying over $\mathfrak{p}$ (this is possible as $A \to A^\wedge$ is faithfully flat by Algebra, Lemma 97.3). Then $A_\mathfrak{p} \to (A^\wedge)_\mathfrak{q}$ is a flat local ring homomorphism. Thus by Proposition 8.4 we see that $A_\mathfrak{p}$ is a complete intersection if and only if $(A^\wedge)_\mathfrak{q}$ is a complete intersection. Thus it suffices to prove the lemma in case A is complete (this is the key step of the proof).

Assume A is complete. By definition we may write $A = R/(f_1, \ldots, f_r)$ for some regular sequence $f_1, \ldots, f_r$ in a regular local ring R. Let $\mathfrak{q} \subset R$ be the prime corresponding to $\mathfrak{p}$. Observe that $f_1, \ldots, f_r \in \mathfrak{q}$ and that $A_\mathfrak{p} = R_\mathfrak{q}/(f_1, \ldots, f_r)R_\mathfrak{q}$. Hence $A_\mathfrak{p}$ is a complete intersection by Lemma 8.2. ☐

09Q5 **Lemma 8.7.** *Let A be a Noetherian ring. Then A is a local complete intersection if and only if $A_\mathfrak{m}$ is a complete intersection for every maximal ideal $\mathfrak{m}$ of A.*

Proof. This follows immediately from Lemma 8.6 and the definitions. ☐

09Q6 **Lemma 8.8.** *Let S be a finite type algebra over a field k.*

1. *for a prime $\mathfrak{q} \subset S$ the local ring $S_\mathfrak{q}$ is a complete intersection in the sense of Algebra, Definition 135.5 if and only if $S_\mathfrak{q}$ is a complete intersection in the sense of Definition 8.5, and*
2. *S is a local complete intersection in the sense of Algebra, Definition 135.1 if and only if S is a local complete intersection in the sense of Definition 8.5.*

Proof. Proof of (1). Let $k[x_1, \ldots, x_n] \to S$ be a surjection. Let $\mathfrak{p} \subset k[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\mathfrak{q}$. Let $I \subset k[x_1, \ldots, x_n]$ be the kernel of our surjection. Note that $k[x_1, \ldots, x_n]_\mathfrak{p} \to S_\mathfrak{q}$ is surjective with kernel $I_\mathfrak{p}$. Observe that $k[x_1, \ldots, x_n]$ is a regular ring by Algebra, Proposition 114.2. Hence the equivalence of the two notions in (1) follows by combining Lemma 8.3 with Algebra, Lemma 135.7.

Having proved (1) the equivalence in (2) follows from the definition and Algebra, Lemma 135.9. ☐

09Q7 **Lemma 8.9.** *Let $A \to B$ be a flat local homomorphism of Noetherian local rings. Then the following are equivalent*

1. *B is a complete intersection,*
2. *A and $B/\mathfrak{m}_A B$ are complete intersections.*

Proof. Now that the definition makes sense this is a trivial reformulation of the (nontrivial) Proposition 8.4. ☐

9. Local complete intersection maps

09Q9 Let $A \to B$ be a local homomorphism of Noetherian complete local rings. A consequence of the Cohen structure theorem is that we can find a commutative diagram

$$\begin{tikzcd}
S \arrow[r] & B \\
 & A \arrow[lu] \arrow[u]
\end{tikzcd}$$

of Noetherian complete local rings with $S \to B$ surjective, $A \to S$ flat, and $S/\mathfrak{m}_A S$ a regular local ring. This follows from More on Algebra, Lemma 39.3. Let us (temporarily) say $A \to S \to B$ is a *good factorization* of $A \to B$ if S is a Noetherian local ring, $A \to S \to B$ are local ring maps, $S \to B$ surjective, $A \to S$ flat, and $S/\mathfrak{m}_A S$ regular. Let us say that $A \to B$ is a *complete intersection homomorphism* if there exists some good factorization $A \to S \to B$ such that the kernel of $S \to B$ is generated by a regular sequence. The following lemma shows this notion is independent of the choice of the diagram.

09QA **Lemma 9.1.** *Let $A \to B$ be a local homomorphism of Noetherian complete local rings. The following are equivalent*

1. *for some good factorization $A \to S \to B$ the kernel of $S \to B$ is generated by a regular sequence, and*
2. *for every good factorization $A \to S \to B$ the kernel of $S \to B$ is generated by a regular sequence.*

Proof. Let $A \to S \to B$ be a good factorization. As B is complete we obtain a factorization $A \to S^\wedge \to B$ where $S^\wedge$ is the completion of S. Note that this is also a good factorization: The ring map $S \to S^\wedge$ is flat (Algebra, Lemma 97.2), hence $A \to S^\wedge$ is flat. The ring $S^\wedge/\mathfrak{m}_A S^\wedge = (S/\mathfrak{m}_A S)^\wedge$ is regular since $S/\mathfrak{m}_A S$ is regular (More on Algebra, Lemma 43.4). Let $f_1, \ldots, f_r$ be a minimal sequence of generators of $\text{Ker}(S \to B)$. We will use without further mention that an ideal in a Noetherian local ring is generated by a regular sequence if and only if any minimal set of generators is a regular sequence. Observe that $f_1, \ldots, f_r$ is a regular sequence in S if and only if $f_1, \ldots, f_r$ is a regular sequence in the completion $S^\wedge$ by Algebra, Lemma 68.5. Moreover, we have

$$S^\wedge/(f_1, \ldots, f_r)R^\wedge = (S/(f_1, \ldots, f_n))^\wedge = B^\wedge = B$$

because B is $\mathfrak{m}_B$-adically complete (first equality by Algebra, Lemma 97.1). Thus the kernel of $S \to B$ is generated by a regular sequence if and only if the kernel of $S^\wedge \to B$ is generated by a regular sequence. Hence it suffices to consider good factorizations where S is complete.

Assume we have two factorizations $A \to S \to B$ and $A \to S' \to B$ with S and S' complete. By More on Algebra, Lemma 39.4 the ring $S \times_B S'$ is a Noetherian complete local ring. Hence, using More on Algebra, Lemma 39.3 we can choose a good factorization $A \to S'' \to S \times_B S'$ with S'' complete. Thus it suffices to show: If $A \to S' \to S \to B$ are comparable good factorizations, then $\text{Ker}(S \to B)$ is generated by a regular sequence if and only if $\text{Ker}(S' \to B)$ is generated by a regular sequence.

Let $A \to S' \to S \to B$ be comparable good factorizations. First, since $S'/\mathfrak{m}_R S' \to S/\mathfrak{m}_R S$ is a surjection of regular local rings, the kernel is generated by a regular sequence $\overline{x}_1, \ldots, \overline{x}_c \in \mathfrak{m}_{S'}/\mathfrak{m}_R S'$ which can be extended to a regular system of parameters for the regular local ring $S'/\mathfrak{m}_R S'$, see (Algebra, Lemma 106.4). Set $I = \text{Ker}(S' \to S)$. By flatness of S over R we have

$$I/\mathfrak{m}_R I = \text{Ker}(S'/\mathfrak{m}_R S' \to S/\mathfrak{m}_R S) = (\overline{x}_1, \ldots, \overline{x}_c).$$

Choose lifts $x_1, \ldots, x_c \in I$. These lifts form a regular sequence generating I as S' is flat over R, see Algebra, Lemma 99.3.

We conclude that if also $\text{Ker}(S \to B)$ is generated by a regular sequence, then so is $\text{Ker}(S' \to B)$, see More on Algebra, Lemmas 30.13 and 30.7.

Conversely, assume that $J = \text{Ker}(S' \to B)$ is generated by a regular sequence. Because the generators $x_1, \ldots, x_c$ of I map to linearly independent elements of $\mathfrak{m}_{S'}/\mathfrak{m}_{S'}^2$ we see that $I/\mathfrak{m}_{S'}I \to J/\mathfrak{m}_{S'}J$ is injective. Hence there exists a minimal system of generators $x_1, \ldots, x_c, y_1, \ldots, y_d$ for J. Then $x_1, \ldots, x_c, y_1, \ldots, y_d$ is a regular sequence and it follows that the images of $y_1, \ldots, y_d$ in S form a regular sequence generating $\text{Ker}(S \to B)$. This finishes the proof of the lemma. ☐

In the following proposition observe that the condition on vanishing of Tor's applies in particular if B has finite tor dimension over A and thus in particular if B is flat over A.

09QB **Proposition 9.2.** *Let $A \to B$ be a local homomorphism of Noetherian local rings. Then the following are equivalent*

1. *B is a complete intersection and $Tor_p^A(B, A/\mathfrak{m}_A)$ is nonzero for only finitely many p,*
2. *A is a complete intersection and $A^\wedge \to B^\wedge$ is a complete intersection homomorphism in the sense defined above.*

Proof. Let $F_\bullet \to A/\mathfrak{m}_A$ be a resolution by finite free A-modules. Observe that $\text{Tor}_p^A(B, A/\mathfrak{m}_A)$ is the pth homology of the complex $F_\bullet \otimes_A B$. Let $F_\bullet^\wedge = F_\bullet \otimes_A A^\wedge$ be the completion. Then $F_\bullet^\wedge$ is a resolution of $A^\wedge/\mathfrak{m}_{A^\wedge}$ by finite free $A^\wedge$-modules (as $A \to A^\wedge$ is flat and completion on finite modules is exact, see Algebra, Lemmas 97.1 and 97.2). It follows that

$$F_\bullet^\wedge \otimes_{A^\wedge} B^\wedge = F_\bullet \otimes_A B \otimes_B B^\wedge$$

By flatness of $B \to B^\wedge$ we conclude that

$$\text{Tor}_p^{A^\wedge}(B^\wedge, A^\wedge/\mathfrak{m}_{A^\wedge}) = \text{Tor}_p^A(B, A/\mathfrak{m}_A) \otimes_B B^\wedge$$

In this way we see that the condition in (1) on the local ring map $A \to B$ is equivalent to the same condition for the local ring map $A^\wedge \to B^\wedge$. Thus we may assume A and B are complete local Noetherian rings (since the other conditions are formulated in terms of the completions in any case).

Assume A and B are complete local Noetherian rings. Choose a diagram

$$\begin{tikzcd}
S \arrow[r] & B \\
R \arrow[r] \arrow[u] & A \arrow[u]
\end{tikzcd}$$

as in More on Algebra, Lemma 39.3. Let $I = \text{Ker}(R \to A)$ and $J = \text{Ker}(S \to B)$. The proposition now follows from Lemma 7.6. ☐

09QC **Remark 9.3.** It appears difficult to define an good notion of "local complete intersection homomorphisms" for maps between general Noetherian rings. The reason is that, for a local Noetherian ring A, the fibres of $A \to A^\wedge$ are not local complete intersection rings. Thus, if $A \to B$ is a local homomorphism of local Noetherian rings, and the map of completions $A^\wedge \to B^\wedge$ is a complete intersection homomorphism in the sense defined above, then $(A_\mathfrak{p})^\wedge \to (B_\mathfrak{q})^\wedge$ is in general **not** a complete intersection homomorphism in the sense defined above. A solution can be had by working exclusively with excellent Noetherian rings. More generally, one could work with those Noetherian rings whose formal fibres are complete intersections, see [Rod87]. We will develop this theory in Dualizing Complexes, Section 23.

To finish of this section we compare the notion defined above with the notion introduced in More on Algebra, Section 8.

09QD **Lemma 9.4.** *Consider a commutative diagram*

$$\begin{tikzcd}S \arrow[r] & B \\ & A \arrow[lu] \arrow[u]\end{tikzcd}$$

of Noetherian local rings with $S \to B$ surjective, $A \to S$ flat, and $S/\mathfrak{m}_A S$ a regular local ring. The following are equivalent

1. $\text{Ker}(S \to B)$ *is generated by a regular sequence, and*
2. $A^\wedge \to B^\wedge$ *is a complete intersection homomorphism as defined above.*

Proof. Omitted. Hint: the proof is identical to the argument given in the first paragraph of the proof of Lemma 9.1. ☐

09QE **Lemma 9.5.** *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. The following are equivalent*

1. *$A \to B$ is a local complete intersection in the sense of More on Algebra, Definition 33.2,*
2. *for every prime $\mathfrak{q} \subset B$ and with $\mathfrak{p} = A \cap \mathfrak{q}$ the ring map $(A_\mathfrak{p})^\wedge \to (B_\mathfrak{q})^\wedge$ is a complete intersection homomorphism in the sense defined above.*

Proof. Choose a surjection $R = A[x_1, \ldots, x_n] \to B$. Observe that $A \to R$ is flat with regular fibres. Let I be the kernel of $R \to B$. Assume (2). Then we see that I is locally generated by a regular sequence by Lemma 9.4 and Algebra, Lemma 68.6. In other words, (1) holds. Conversely, assume (1). Then after localizing on R and B we can assume that I is generated by a Koszul regular sequence. By More on Algebra, Lemma 30.7 we find that I is locally generated by a regular sequence. Hence (2) hold by Lemma 9.4. Some details omitted. ☐

09QF **Lemma 9.6.** *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map such that the image of* $\text{Spec}(B) \to \text{Spec}(A)$ *contains all closed points of* $\text{Spec}(A)$. *Then the following are equivalent*

1. *B is a complete intersection and $A \to B$ has finite tor dimension,*
2. *A is a complete intersection and $A \to B$ is a local complete intersection in the sense of More on Algebra, Definition 33.2.*

Proof. This is a reformulation of Proposition 9.2 via Lemma 9.5. We omit the details. ☐

10. Smooth ring maps and diagonals

0FCV In this section we use the material above to characterize smooth ring maps as those whose diagonal is perfect.

0FCW **Lemma 10.1.** *Let $A \to B$ be a local ring homomorphism of Noetherian local rings such that B is flat and essentially of finite type over A. If*

$$B \otimes_A B \longrightarrow B$$

is a perfect ring map, i.e., if B has finite tor dimension over $B \otimes_A B$, then B is the localization of a smooth A-algebra.

Proof. As B is essentially of finite type over A, so is $B \otimes_A B$ and in particular $B \otimes_A B$ is Noetherian. Hence the quotient B of $B \otimes_A B$ is pseudo-coherent over $B \otimes_A B$ (More on Algebra, Lemma 64.17) which explains why perfectness of the ring map (More on Algebra, Definition 82.1) agrees with the condition of finite tor dimension.

We may write $B = R/K$ where R is the localization of $A[x_1, \ldots, x_n]$ at a prime ideal and $K \subset R$ is an ideal. Denote $\mathfrak{m} \subset R \otimes_A R$ the maximal ideal which is the inverse image of the maximal ideal of B via the surjection $R \otimes_A R \to B \otimes_A B \to B$. Then we have surjections

$$(R \otimes_A R)_\mathfrak{m} \to (B \otimes_A B)_\mathfrak{m} \to B$$

and hence ideals $I \subset J \subset (R \otimes_A R)_\mathfrak{m}$ as in Lemma 7.4. We conclude that $I/\mathfrak{m}I \to J/\mathfrak{m}J$ is injective.

Let $K = (f_1, \ldots, f_r)$ with r minimal. We may and do assume that $f_i \in R$ is the image of an element of $A[x_1, \ldots, x_n]$ which we also denote f_i. Observe that I is generated by $f_1 \otimes 1, \ldots, f_r \otimes 1$ and $1 \otimes f_1, \ldots, 1 \otimes f_r$. We claim that this is a minimal set of generators of I. Namely, if κ is the common residue field of R, B, $(R \otimes_A R)_\mathfrak{m}$, and $(B \otimes_A B)_\mathfrak{m}$ then we have a map $R \otimes_A R \to R \otimes_A \kappa \oplus \kappa \otimes_A R$ which factors through $(R \otimes_A R)_\mathfrak{m}$. Since B is flat over A and since we have the short exact sequence $0 \to K \to R \to B \to 0$ we see that $K \otimes_A \kappa \subset R \otimes_A \kappa$, see Algebra, Lemma 39.12. Thus restricting the map $(R \otimes_A R)_\mathfrak{m} \to R \otimes_A \kappa \oplus \kappa \otimes_A R$ to I we obtain a map

$$I \to K \otimes_A \kappa \oplus \kappa \otimes_A K \to K \otimes_B \kappa \oplus \kappa \otimes_B K.$$

The elements $f_1 \otimes 1, \ldots, f_r \otimes 1, 1 \otimes f_1, \ldots, 1 \otimes f_r$ map to a basis of the target of this map, since by Nakayama's lemma (Algebra, Lemma 20.1) $f_1, \ldots, f_r$ map to a basis of $K \otimes_B \kappa$. This proves our claim.

The ideal J is generated by $f_1 \otimes 1, \ldots, f_r \otimes 1$ and the elements $x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n$ (for the proof it suffices to see that these elements are contained in the ideal J). Now we can write

$$f_i \otimes 1 - 1 \otimes f_i = \sum g_{ij}(x_j \otimes 1 - 1 \otimes x_j)$$

for some g_{ij} in $(R \otimes_A R)_\mathfrak{m}$. This is a general fact about elements of $A[x_1, \ldots, x_n]$ whose proof we omit. Denote $a_{ij} \in \kappa$ the image of g_{ij}. Another computation shows that a_{ij} is the image of $\partial f_i / \partial x_j$ in κ. The injectivity of $I/\mathfrak{m}I \to J/\mathfrak{m}J$ and the remarks made above forces the matrix (a_{ij}) to have maximal rank r. Set

$$C = A[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

and consider the naive cotangent complex $NL_{C/A} \cong (C^{\oplus r} \to C^{\oplus n})$ where the map is given by the matrix of partial derivatives. Thus $NL_{C/A} \otimes_A B$ is isomorphic to a free B-module of rank $n - r$ placed in degree 0. Hence C_g is smooth over A for some $g \in C$ mapping to a unit in B, see Algebra, Lemma 137.12. This finishes the proof. ☐

0FCX **Lemma 10.2.** *Let $A \to B$ be a flat finite type ring map of Noetherian rings. If*

$$B \otimes_A B \longrightarrow B$$

is a perfect ring map, i.e., if B has finite tor dimension over $B \otimes_A B$, then B is a smooth A-algebra.

Proof. This follows from Lemma 10.1 and general facts about smooth ring maps, see Algebra, Lemmas 137.12 and 137.13. Alternatively, the reader can slightly modify the proof of Lemma 10.1 to prove this lemma. ☐

11. Freeness of the conormal module

0FJP Tate resolutions and derivations on them can be used to prove (stronger) versions of the results in this section, see [Iye01]. Two more elementary references are [Vas67] and [Fer67].

0FJQ **Lemma 11.1.** *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of finite projective dimension over R. If $F \subset I/I^2$ is a direct summand isomorphic to R/I, then there exists a nonzerodivisor $x \in I$ such that the image of x in I/I^2 generates F.* [Vas67]

Proof. By assumption we may choose a finite free resolution

$$0 \to R^{\oplus n_e} \to R^{\oplus n_{e-1}} \to \ldots \to R^{\oplus n_1} \to R \to R/I \to 0$$

Then $\varphi_1 : R^{\oplus n_1} \to R$ has rank 1 and we see that I contains a nonzerodivisor y by Algebra, Proposition 102.9. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the associated primes of R, see Algebra, Lemma 63.5. Let $I^2 \subset J \subset I$ be an ideal such that $J/I^2 = F$. Then $J \not\subset \mathfrak{p}_i$ for all i as $y^2 \in J$ and $y^2 \notin \mathfrak{p}_i$, see Algebra, Lemma 63.9. By Nakayama's lemma (Algebra, Lemma 20.1) we have $J \not\subset \mathfrak{m}J + I^2$. By Algebra, Lemma 15.2 we can pick $x \in J$, $x \notin \mathfrak{m}J + I^2$ and $x \notin \mathfrak{p}_i$ for $i = 1, \ldots, n$. Then x is a nonzerodivisor and the image of x in I/I^2 generates (by Nakayama's lemma) the summand $J/I^2 \cong R/I$. ☐

0FJR **Lemma 11.2.** *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of finite projective dimension over R. If $F \subset I/I^2$ is a direct summand free of rank r, then there exists a regular sequence $x_1, \ldots, x_r \in I$ such that $x_1 \bmod I^2, \ldots, x_r \bmod I^2$ generate F.* Local version of [Vas67, Theorem 1.1]

Proof. If $r = 0$ there is nothing to prove. Assume $r > 0$. We may pick $x \in I$ such that x is a nonzerodivisor and $x \bmod I^2$ generates a summand of F isomorphic to R/I, see Lemma 11.1. Consider the ring $R' = R/(x)$ and the ideal $I' = I/(x)$. Of course $R'/I' = R/I$. The short exact sequence

$$0 \to R/I \xrightarrow{x} I/xI \to I' \to 0$$

splits because the map $I/xI \to I/I^2$ sends xR/xI to a direct summand. Now $I/xI = I \otimes_R^{\mathbf{L}} R'$ has finite projective dimension over R', see More on Algebra, Lemmas 74.3 and 74.9. Hence the summand I' has finite projective dimension over R'. On the other hand, we have the short exact sequence $0 \to xR/xI \to I/I^2 \to I'/(I')^2 \to 0$ and we conclude $I'/(I')^2$ has the free direct summand $F' = F/(R/I \cdot x)$ of rank $r - 1$. By induction on r we may we pick a regular sequence $x'_2, \ldots, x'_r \in I'$ such that there congruence classes freely generate F'. If $x_1 = x$ and $x_2, \ldots, x_r$ are any elements lifting $x'_1, \ldots, x'_r$ in R, then we see that the lemma holds. ☐

0FJS **Proposition 11.3.** *Let R be a Noetherian ring. Let $I \subset R$ be an ideal which has finite projective dimension and such that I/I^2 is finite locally free over R/I. Then I is a regular ideal (More on Algebra, Definition 32.1).* Variant of [Vas67, Corollary 1]. See also [Iye01] and [Fer67].

Proof. By Algebra, Lemma 68.6 it suffices to show that $I_\mathfrak{p} \subset R_\mathfrak{p}$ is generated by a regular sequence for every $\mathfrak{p} \supset I$. Thus we may assume R is local. If I/I^2 has rank r, then by Lemma 11.2 we find a regular sequence $x_1, \ldots, x_r \in I$ generating I/I^2. By Nakayama (Algebra, Lemma 20.1) we conclude that I is generated by $x_1, \ldots, x_r$. □

For any local complete intersection homomorphism $A \to B$ of rings, the naive cotangent complex $NL_{B/A}$ is perfect of tor-amplitude in $[-1, 0]$, see More on Algebra, Lemma 85.4. Using the above, we can show that this sometimes characterizes local complete intersection homomorphisms.

0FJT **Lemma 11.4.** *Let $A \to B$ be a perfect (More on Algebra, Definition 82.1) ring homomorphism of Noetherian rings. Then the following are equivalent*

1. *$NL_{B/A}$ has tor-amplitude in $[-1, 0]$,*
2. *$NL_{B/A}$ is a perfect object of $D(B)$ with tor-amplitude in $[-1, 0]$, and*
3. *$A \to B$ is a local complete intersection (More on Algebra, Definition 33.2).*

Proof. Write $B = A[x_1, \ldots, x_n]/I$. Then $NL_{B/A}$ is represented by the complex

$$I/I^2 \longrightarrow \bigoplus B\text{d}x_i$$

of B-modules with I/I^2 placed in degree -1. Since the term in degree 0 is finite free, this complex has tor-amplitude in $[-1, 0]$ if and only if I/I^2 is a flat B-module, see More on Algebra, Lemma 66.2. Since I/I^2 is a finite B-module and B is Noetherian, this is true if and only if I/I^2 is a finite locally free B-module (Algebra, Lemma 78.2). Thus the equivalence of (1) and (2) is clear. Moreover, the equivalence of (1) and (3) also follows if we apply Proposition 11.3 (and the observation that a regular ideal is a Koszul regular ideal as well as a quasi-regular ideal, see More on Algebra, Section 32). □

0FJV **Lemma 11.5.** *Let $A \to B$ be a flat ring map of finite presentation. Then the following are equivalent*

1. *$NL_{B/A}$ has tor-amplitude in $[-1, 0]$,*
2. *$NL_{B/A}$ is a perfect object of $D(B)$ with tor-amplitude in $[-1, 0]$,*
3. *$A \to B$ is syntomic (Algebra, Definition 136.1), and*
4. *$A \to B$ is a local complete intersection (More on Algebra, Definition 33.2).*

Proof. The equivalence of (3) and (4) is More on Algebra, Lemma 33.5.

If $A \to B$ is syntomic, then we can find a cocartesian diagram

$$\begin{tikzcd}
B_0 \arrow[r] & B \\
A_0 \arrow[r] \arrow[u] & A \arrow[u]
\end{tikzcd}$$

such that $A_0 \to B_0$ is syntomic and A_0 is Noetherian, see Algebra, Lemmas 127.18 and 168.9. By Lemma 11.4 we see that NL_{B_0/A_0} is perfect of tor-amplitude in $[-1, 0]$. By More on Algebra, Lemma 85.3 we conclude the same thing is true for $NL_{B/A} = NL_{B_0/A_0} \otimes_{B_0}^{\mathbf{L}} B$ (see also More on Algebra, Lemmas 66.13 and 74.9). This proves that (3) implies (2).

Assume (1). By More on Algebra, Lemma 85.3 for every ring map $A \to k$ where k is a field, we see that $NL_{B \otimes_A k/k}$ has tor-amplitude in $[-1, 0]$ (see More on Algebra, Lemma 66.13). Hence by Lemma 11.4 we see that $k \to B \otimes_A k$ is a local complete intersection homomorphism. Thus $A \to B$ is syntomic by definition. This proves (1) implies (3) and finishes the proof. ☐

12. Koszul complexes and Tate resolutions

0GZ3 In this section we "lift" the result of More on Algebra, Lemma 94.1 to the category of differential graded algebras endowed with divided powers compatible with the differential graded structure (beware that in this section we represent Koszul complexes as chain complexes whereas in locus citatus we use cochain complexes).

Let R be a ring. Let $I \subset R$ be an ideal generated by $f_1, \ldots, f_r \in R$. For $n \geq 1$ we denote

$$K_n = K_{n,\bullet} = R\langle \xi_1, \ldots, \xi_r \rangle$$

the differential graded Koszul algebra with ξ_i in degree 1 and $\mathrm{d}(\xi_i) = f_i^n$. There exists a unique divided power structure on this (as in Definition 6.5), see Example 6.2. For $m > n$ the transition map $K_m \to K_n$ is the differential graded algebra map compatible with divided powers given by sending ξ_i to $f_i^{m-n}\xi_i$.

0GZ4 **Lemma 12.1.** *In the situation above, if R is Noetherian, then for every n there exists an $N \geq n$ and maps*

$$K_N \to A \to R/(f_1^N, \ldots, f_r^N) \quad \textit{and} \quad A \to K_n$$

with the following properties

1. (A, d, γ) *is as in Definition 6.5,*
2. $A \to R/(f_1^N, \ldots, f_r^N)$ *is a quasi-isomorphism,*
3. *the composition* $K_N \to A \to R/(f_1^N, \ldots, f_r^N)$ *is the canonical map,*
4. *the composition* $K_N \to A \to K_n$ *is the transition map,*
5. $A_0 = R \to R/(f_1^N, \ldots, f_r^N)$ *is the canonical surjection,*
6. *A is a graded divided power polynomial algebra over R with finitely many generators in each degree, and*
7. *$A \to K_n$ is a homomorphism of differential graded R-algebras compatible with divided powers which induces the canonical map $R/(f_1^N, \ldots, f_r^N) \to R/(f_1^n, \ldots, f_r^n)$ on homology in degree* 0.

Condition (4) means that A is constructed out of A_0 by successively adjoining a finite set of variables T in each degree > 0 as in Example 6.2 or 6.3.

Proof. Fix n. If $r = 0$, then we can just pick $A = R$. Assume $r > 0$. By More on Algebra, Lemma 94.1 (translated into the language of chain complexes) we can choose

$$n_r > n_{r-1} > \ldots > n_1 > n_0 = n$$

such that the transition maps $K_{n_{i+1}} \to K_{n_i}$ on Koszul algebras (see above) induce the zero map on homology in degrees > 0. We will prove the lemma with $N = n_r$.

We will construct A exactly as in the statement and proof of Lemma 6.9. Thus we will have

$$A = \operatorname{colim} A(m), \quad \text{and} \quad A(0) \to A(1) \to A(2) \to \ldots \to R/(f_1^N, \ldots, f_r^N)$$

This will immediately give us properties (1), (2), (5), and (6). To finish the proof we will construct the R-algebra maps $K_N \to A \to K_n$. To do this we will construct

1. an isomorphism $A(1) \to K_N = K_{n_r}$,

(2) a map $A(2) \to K_{n_{r-1}}$,
(3) $\ldots$
(4) a map $A(r) \to K_{n_1}$,
(5) a map $A(r+1) \to K_{n_0} = K_n$, and
(6) a map $A \to K_n$.

In each of these steps the map constructed will be between differential graded algebras compatibly endowed with divided powers and each of the maps will be compatible with the previous one via the transition maps between the Koszul algebras and each of the maps will induce the obvious canonical map on homology in degree 0.

Recall that $A(0) = R$. For $m = 1$, the proof of Lemma 6.9 chooses $A(1) = R\langle T_1, \ldots, T_r\rangle$ with T_i of degree 1 and with $\text{d}(T_i) = f_i^N$. Namely, the f_i^N are generators of the kernel of $A(0) \to R/(f_1^N, \ldots, f_r^N)$. Thus for $A(1) \to K_N = K_{n_r}$ we use the map

$$\varphi_1 : A(1) \longrightarrow K_{n_r}, \quad T_i \longmapsto \xi_i$$

which is an isomorphism.

For $m = 2$, the construction in the proof of Lemma 6.9 chooses generators $e_1, \ldots, e_t \in \text{Ker}(\text{d} : A(1)_1 \to A(1)_0)$. The construction proceeds by taking $A(2) = A(1)\langle T_1, \ldots, T_t\rangle$ as a divided power polynomial algebra with T_i of degree 2 and with $\text{d}(T_i) = e_i$. Since $\varphi_1(e_i)$ is a cocycle in K_{n_r} we see that its image in $K_{n_{r-1}}$ is a coboundary by our choice of n_r and n_{r-1} above. Hence we can construct the following commutative diagram

$$\begin{tikzcd}
A(1) \arrow[r, "\varphi_1"'] \arrow[d] & K_{n_r} \arrow[d] \\
A(2) \arrow[r, "\varphi_2"] & K_{n_{r-1}}
\end{tikzcd}$$

by sending T_i to an element in degree 2 whose boundary is the image of $\varphi_1(e_i)$. The map φ_2 exists and is compatible with the differential and the divided powers by the universal of the divided power polynomial algebra.

The algebra $A(m)$ and the map $\varphi_m : A(m) \to K_{n_{r+1-m}}$ are constructed in exactly the same manner for $m = 2, \ldots, r$.

Given the map $A(r) \to K_{n_1}$ we see that the composition $H_r(A(r)) \to H_r(K_{n_1}) \to H_r(K_{n_0}) \subset (K_{n_0})_r$ is zero, hence we can extend this to $A(r+1) \to K_{n_0} = K_n$ by sending the new polynomial generators of $A(r+1)$ to zero.

Having constructed $A(r+1) \to K_{n_0} = K_n$ we can simply extend to $A(r+2), A(r+3), \ldots$ in the only possible way by sending the new polynomial generators to zero. This finishes the proof. $\square$

0GZ5 **Remark 12.2.** In the situation above, if R is Noetherian, we can inductively choose a sequence of integers $1 = n_0 < n_1 < n_2 < \ldots$ such that for $i = 1, 2, 3, \ldots$ we have maps $K_{n_i} \to A_i \to R/(f_1^{n_i}, \ldots, f_r^{n_i})$ and $A_i \to K_{n_{i-1}}$ as in Lemma 12.1.

Denote $A_{i+1} \to A_i$ the composition $A_{i+1} \to K_{n_i} \to A_i$. Then the diagram

$$\begin{tikzcd}
K_{n_1} \arrow[d] & K_{n_2} \arrow[l] \arrow[d] & K_{n_3} \arrow[l] \arrow[d] & \ldots \arrow[l] \\
A_1 \arrow[d] & A_2 \arrow[l] \arrow[d] & A_3 \arrow[l] \arrow[d] & \ldots \arrow[l] \\
K_1 & K_{n_1} \arrow[l] & K_{n_2} \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

commutes. In this way we see that the inverse systems (K_n) and (A_n) are pro-isomorphic in the category of differential graded R-algebras with compatible divided powers.

0GZ6 **Lemma 12.3.** *Let (A, d, γ), $d \geq 1$, $f \in A_{d-1}$, and $A\langle T\rangle$ be as in Lemma 6.8.*

1. *If $d = 1$, then there is a long exact sequences*
$$\ldots \to H_0(A) \xrightarrow{f} H_0(A) \to H_0(A\langle T\rangle) \to 0$$
2. *For $d = 2$ there is a bounded spectral sequence $(E_1)_{i,j} = H_{j-i}(A) \cdot T^{[i]}$ converging to $H_{i+j}(A\langle T\rangle)$. The differential $(d_1)_{i,j} : H_{j-i}(A) \cdot T^{[i]} \to H_{j-i+1}(A) \cdot T^{[i-1]}$ sends $\xi \cdot T^{[i]}$ to the class of $f\xi \cdot T^{[i-1]}$.*
3. *Add more here for other degrees as needed.*

Proof. For $d = 1$, we have a short exact sequence of complexes

$$0 \to A \to A\langle T\rangle \to A \cdot T \to 0$$

and the result (1) follows easily from this. For $d = 2$ we view $A\langle T\rangle$ as a filtered chain complex with subcomplexes

$$F^p A\langle T\rangle = \bigoplus\nolimits_{i \leq p} A \cdot T^{[i]}$$

Applying the spectral sequence of Homology, Section 24 (translated into chain complexes) we obtain (2). ☐

The following lemma will be needed later.

0GZ7 **Lemma 12.4.** *In the situation above, for all $n \geq t \geq 1$ there exists an $N > n$ and a map*

$$K_t \longrightarrow K_n \otimes_R K_t$$

in the derived category of left differential graded K_N-modules whose composition with the multiplication map is the transition map (in either direction).

Proof. We first prove this for $r = 1$. Set $f = f_1$. Write $K_t = R\langle x\rangle$, $K_n = R\langle y\rangle$, and $K_N = R\langle z\rangle$ with x, y, z of degree 1 and $\mathrm{d}(x) = f^t$, $\mathrm{d}(y) = f^n$, and $\mathrm{d}(z) = f^N$. For all $N > t$ we claim there is a quasi-isomorphism

$$B_{N,t} = R\langle x, z, u\rangle \longrightarrow K_t, \quad x \mapsto x, \quad z \mapsto f^{N-t}x, \quad u \mapsto 0$$

Here the left hand side denotes the divided power polynomial algebra in variables x and z of degree 1 and u of degree 2 with $\mathrm{d}(x) = f^t$, $\mathrm{d}(z) = f^N$, and $\mathrm{d}(u) = z - f^{N-t}x$. To prove the claim, we observe that the following three submodules of $H_*(R\langle x, z\rangle)$ are the same

1. the kernel of $H_*(R\langle x, z\rangle) \to H_*(K_t)$,

(2) the image of $z - f^{N-t}x : H_*(R\langle x, z\rangle) \to H_*(R\langle x, z\rangle)$, and
(3) the kernel of $z - f^{N-t}x : H_*(R\langle x, z\rangle) \to H_*(R\langle x, z\rangle)$.

This observation is proved by a direct computation[3] which we omit. Then we can apply Lemma 12.3 part (2) to see that the claim is true.

Via the homomorphism $K_N \to B_{N,t}$ of differential graded R-algebras sending z to z, we may view $B_{N,t} \to K_t$ as a quasi-isomorphism of left differential graded K_N-modules. To define the arrow in the statement of the lemma we use the homomorphism

$$B_{N,t} = R\langle x, z, u\rangle \to K_n \otimes_R K_t, \quad x \mapsto 1 \otimes x, \quad z \mapsto f^{N-n} y \otimes 1, \quad u \mapsto -f^{N-n-t} y \otimes x$$

This makes sense as long as we assume $N \geq n + t$. It is a pleasant computation to show that the (pre or post) composition with the multiplication map is the transition map.

For $r > 1$ we proceed by writing each of the Koszul algebras as a tensor product of Koszul algebras in 1 variable and we apply the previous construction. In other words, we write

$$K_t = R\langle x_1, \ldots, x_r\rangle = R\langle x_1\rangle \otimes_R \ldots \otimes_R R\langle x_r\rangle$$

where x_i is in degree 1 and $\mathrm{d}(x_i) = f_i^t$. In the case $r > 1$ we then use

$$B_{N,t} = R\langle x_1, z_1, u_1\rangle \otimes_R \ldots \otimes_R R\langle x_r, z_r, u_r\rangle$$

where x_i, z_i have degree 1 and u_i has degree 2 and we have $\mathrm{d}(x_i) = f_i^t$, $\mathrm{d}(z_i) = f_i^N$, and $\mathrm{d}(u_i) = z_i - f_i^{N-t} x_i$. The tensor product map $B_{N,t} \to K_t$ will be a quasi-isomorphism as it is a tensor product of quasi-isomorphisms between bounded above complexes of free R-modules. Finally, we define the map

$$B_{N,t} \to K_n \otimes_R K_t = R\langle y_1, \ldots, y_r\rangle \otimes_R R\langle x_1, \ldots, x_r\rangle$$

as the tensor product of the maps constructed in the case of $r = 1$ or simply by the rules $x_i \mapsto 1 \otimes x_i$, $z_i \mapsto f_i^{N-n} y_i \otimes 1$, and $u_i \mapsto -f_i^{N-n-t} y_i \otimes x_i$ which makes sense as long as $N \geq n + t$. We omit the details. ☐

13. Other chapters

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[3] Hint: setting $z' = z - f^{N-t}x$ we see that $R\langle x, z\rangle = R\langle x, z'\rangle$ with $\mathrm{d}(z') = 0$ and moreover the map $R\langle x, z'\rangle \to K_t$ is the map killing z'.

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(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes
(31) Divisors
(32) Limits of Schemes
(33) Varieties
(34) Topologies on Schemes
(35) Descent
(36) Derived Categories of Schemes
(37) More on Morphisms
(38) More on Flatness
(39) Groupoid Schemes
(40) More on Groupoid Schemes
(41) Étale Morphisms of Schemes

Topics in Scheme Theory

(42) Chow Homology
(43) Intersection Theory
(44) Picard Schemes of Curves
(45) Weil Cohomology Theories
(46) Adequate Modules
(47) Dualizing Complexes
(48) Duality for Schemes
(49) Discriminants and Differents
(50) de Rham Cohomology
(51) Local Cohomology
(52) Algebraic and Formal Geometry
(53) Algebraic Curves
(54) Resolution of Surfaces
(55) Semistable Reduction
(56) Functors and Morphisms
(57) Derived Categories of Varieties
(58) Fundamental Groups of Schemes
(59) Étale Cohomology
(60) Crystalline Cohomology
(61) Pro-étale Cohomology
(62) Relative Cycles
(63) More Étale Cohomology
(64) The Trace Formula

Algebraic Spaces

(65) Algebraic Spaces
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(68) Decent Algebraic Spaces
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Topics in Geometry

(82) Chow Groups of Spaces
(83) Quotients of Groupoids
(84) More on Cohomology of Spaces
(85) Simplicial Spaces
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Deformation Theory

(90) Formal Deformation Theory
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Algebraic Stacks

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Topics in Moduli Theory

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Miscellany

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