

DUALITY FOR SCHEMES

0DWE

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1. Introduction

0DWF This chapter studies relative duality for morphisms of schemes and the dualizing complex on a scheme. A reference is [Har66].

Dualizing complexes for Noetherian rings were defined and studied in Dualizing Complexes, Section 15 ff. In this chapter we continue this by studying dualizing complexes on schemes, see Section 2.

The bulk of this chapter is devoted to studying the right adjoint of pushforward in the setting of derived categories of sheaves of modules with quasi-coherent cohomology sheaves. See Sections 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, and 15. Here we follow the papers [Nee96], [LN07], [Lip09], and [Nee14].

We discuss the important and useful upper shriek functors $f^!$ for separated morphisms of finite type between Noetherian schemes in Sections 16, 17, and 18 culminating in the overview Section 19.

In Section 20 we explain alternative theory of duality and dualizing complexes when working over a fixed locally Noetherian base endowed with a dualizing complex (this section corresponds to a remark in Hartshorne's book).

In the remaining sections we give a few applications.

This chapter is continued by the chapter on duality on algebraic spaces, see Duality for Spaces, Section 1.

2. Dualizing complexes on schemes

0A85 We define a dualizing complex on a locally Noetherian scheme to be a complex which affine locally comes from a dualizing complex on the corresponding ring. This is not completely standard but agrees with all definitions in the literature on Noetherian schemes of finite dimension.

0A86 **Lemma 2.1.** *Let X be a locally Noetherian scheme. Let K be an object of $D(\mathcal{O}_X)$. The following are equivalent*

- (1) *For every affine open $U = \operatorname{Spec}(A) \subset X$ there exists a dualizing complex $\omega_A^\bullet$ for A such that $K|_U$ is isomorphic to the image of $\omega_A^\bullet$ by the functor $\tilde{\cdot}: D(A) \rightarrow D(\mathcal{O}_U)$.*
- (2) *There is an affine open covering $X = \bigcup U_i$, $U_i = \operatorname{Spec}(A_i)$ such that for each i there exists a dualizing complex $\omega_i^\bullet$ for A_i such that $K|_{U_i}$ is isomorphic to the image of $\omega_i^\bullet$ by the functor $\tilde{\cdot}: D(A_i) \rightarrow D(\mathcal{O}_{U_i})$.*

Proof. Assume (2) and let $U = \operatorname{Spec}(A)$ be an affine open of X . Since condition (2) implies that K is in $D_{Qcoh}(\mathcal{O}_X)$ we find an object $\omega_A^\bullet$ in $D(A)$ whose associated complex of quasi-coherent sheaves is isomorphic to $K|_U$, see Derived Categories of Schemes, Lemma 3.5. We will show that $\omega_A^\bullet$ is a dualizing complex for A which will finish the proof.

Since $X = \bigcup U_i$ is an open covering, we can find a standard open covering $U = D(f_1) \cup \dots \cup D(f_m)$ such that each $D(f_j)$ is a standard open in one of the affine opens U_i , see Schemes, Lemma 11.5. Say $D(f_j) = D(g_j)$ for $g_j \in A_{i_j}$. Then $A_{f_j} \cong (A_{i_j})_{g_j}$ and we have

$$(\omega_A^\bullet)_{f_j} \cong (\omega_{i_j}^\bullet)_{g_j}$$

in the derived category by Derived Categories of Schemes, Lemma 3.5. By Dualizing Complexes, Lemma 15.6 we find that the complex $(\omega_A^\bullet)_{f_j}$ is a dualizing complex over A_{f_j} for $j = 1, \dots, m$. This implies that $\omega_A^\bullet$ is dualizing by Dualizing Complexes, Lemma 15.7. $\square$

0A87 **Definition 2.2.** Let X be a locally Noetherian scheme. An object K of $D(\mathcal{O}_X)$ is called a *dualizing complex* if K satisfies the equivalent conditions of Lemma 2.1.

Please see remarks made at the beginning of this section.

0A88 **Lemma 2.3.** Let A be a Noetherian ring and let $X = \operatorname{Spec}(A)$. Let K, L be objects of $D(A)$. If $K \in D_{\operatorname{Coh}}(A)$ and L has finite injective dimension, then

$$R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L}) = R\widetilde{\operatorname{Hom}_A(K, L)}$$

in $D(\mathcal{O}_X)$.

Proof. We may assume that L is given by a finite complex $I^\bullet$ of injective A -modules. By induction on the length of $I^\bullet$ and compatibility of the constructions with distinguished triangles, we reduce to the case that $L = I[0]$ where I is an injective A -module. In this case, Derived Categories of Schemes, Lemma 10.8, tells us that the n th cohomology sheaf of $R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L})$ is the sheaf associated to the presheaf

$$D(f) \longmapsto \operatorname{Ext}_{A_f}^n(K \otimes_A A_f, I \otimes_A A_f)$$

Since A is Noetherian, the A_f -module $I \otimes_A A_f$ is injective (Dualizing Complexes, Lemma 3.8). Hence we see that

$$\begin{aligned} \operatorname{Ext}_{A_f}^n(K \otimes_A A_f, I \otimes_A A_f) &= \operatorname{Hom}_{A_f}(H^{-n}(K \otimes_A A_f), I \otimes_A A_f) \\ &= \operatorname{Hom}_{A_f}(H^{-n}(K) \otimes_A A_f, I \otimes_A A_f) \\ &= \operatorname{Hom}_A(H^{-n}(K), I) \otimes_A A_f \end{aligned}$$

The last equality because $H^{-n}(K)$ is a finite A -module, see Algebra, Lemma 10.2. This proves that the canonical map

$$R\widetilde{\operatorname{Hom}_A(K, L)} \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L})$$

is a quasi-isomorphism in this case and the proof is done. $\square$

0G4I **Lemma 2.4.** Let X be a Noetherian scheme. Let $K, L, M \in D_{\operatorname{QCoh}}(\mathcal{O}_X)$. Then the map

$$R\mathcal{H}om(L, M) \otimes_{\mathcal{O}_X}^L K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

of Cohomology, Lemma 42.9 is an isomorphism in the following two cases

- (1) $K \in D_{\operatorname{Coh}}^-(\mathcal{O}_X)$, $L \in D_{\operatorname{Coh}}^+(\mathcal{O}_X)$, and M affine locally has finite injective dimension (see proof), or
- (2) K and L are in $D_{\operatorname{Coh}}(\mathcal{O}_X)$, the object $R\mathcal{H}om(L, M)$ has finite tor dimension, and L and M affine locally have finite injective dimension (in particular L and M are bounded).

Proof. Proof of (1). We say M has affine locally finite injective dimension if X has an open covering by affines $U = \operatorname{Spec}(A)$ such that the object of $D(A)$ corresponding to $M|_U$ (Derived Categories of Schemes, Lemma 3.5) has finite injective dimension¹.

¹This condition is independent of the choice of the affine open cover of the Noetherian scheme X . Details omitted.

To prove the lemma we may replace X by U , i.e., we may assume $X = \operatorname{Spec}(A)$ for some Noetherian ring A . Observe that $R\mathcal{H}om(K, L)$ is in $D_{Coh}^+(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 11.5. Moreover, the formation of the left and right hand side of the arrow commutes with the functor $D(A) \rightarrow D_{QCoh}(\mathcal{O}_X)$ by Lemma 2.3 and Derived Categories of Schemes, Lemma 10.8 (to be sure this uses the assumptions on K, L, M and what we just proved about $R\mathcal{H}om(K, L)$). Then finally the arrow is an isomorphism by More on Algebra, Lemmas 98.1 part (2).

Proof of (2). We argue as above. A small change is that here we get $R\mathcal{H}om(K, L)$ in $D_{Coh}(\mathcal{O}_X)$ because affine locally (which is allowable by Lemma 2.3) we may appeal to Dualizing Complexes, Lemma 15.2. Then we finally conclude by More on Algebra, Lemma 98.2. $\square$

0A89 **Lemma 2.5.** *Let K be a dualizing complex on a locally Noetherian scheme X . Then K is an object of $D_{Coh}(\mathcal{O}_X)$ and $D = R\mathcal{H}om_{\mathcal{O}_X}(-, K)$ induces an anti-equivalence*

$$D : D_{Coh}(\mathcal{O}_X) \longrightarrow D_{Coh}(\mathcal{O}_X)$$

which comes equipped with a canonical isomorphism $\operatorname{id} \rightarrow D \circ D$. If X is quasi-compact, then D exchanges $D_{Coh}^+(\mathcal{O}_X)$ and $D_{Coh}^-(\mathcal{O}_X)$ and induces an anti-equivalence $D_{Coh}^b(\mathcal{O}_X) \rightarrow D_{Coh}^b(\mathcal{O}_X)$.

Proof. Let $U \subset X$ be an affine open. Say $U = \operatorname{Spec}(A)$ and let $\omega_A^\bullet$ be a dualizing complex for A corresponding to $K|_U$ as in Lemma 2.1. By Lemma 2.3 the diagram

$$\begin{array}{ccc} D_{Coh}(A) & \longrightarrow & D_{Coh}(\mathcal{O}_U) \\ \downarrow R\mathcal{H}om_A(-, \omega_A^\bullet) & & \downarrow R\mathcal{H}om_{\mathcal{O}_X}(-, K|_U) \\ D_{Coh}(A) & \longrightarrow & D(\mathcal{O}_U) \end{array}$$

commutes. We conclude that D sends $D_{Coh}(\mathcal{O}_X)$ into $D_{Coh}(\mathcal{O}_X)$. Moreover, the canonical map

$$L \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(K, K) \otimes_{\mathcal{O}_X}^L L \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(L, K), K)$$

(using Cohomology, Lemma 42.9 for the second arrow) is an isomorphism for all L because this is true on affines by Dualizing Complexes, Lemma 15.3² and we have already seen on affines that we recover what happens in algebra. The statement on boundedness properties of the functor D in the quasi-compact case also follows from the corresponding statements of Dualizing Complexes, Lemma 15.3. $\square$

Let X be a locally ringed space. Recall that an object L of $D(\mathcal{O}_X)$ is *invertible* if it is an invertible object for the symmetric monoidal structure on $D(\mathcal{O}_X)$ given by derived tensor product. In Cohomology, Lemma 52.2 we have seen this means L is perfect and there is an open covering $X = \bigcup U_i$ such that $L|_{U_i} \cong \mathcal{O}_{U_i}[-n_i]$ for some integers n_i . In this case, the function

$$x \mapsto n_x, \quad \text{where } n_x \text{ is the unique integer such that } H^{n_x}(L_x) \neq 0$$

is locally constant on X . In particular, we have $L = \bigoplus H^n(L)[-n]$ which gives a well defined complex of $\mathcal{O}_X$ -modules (with zero differentials) representing L .

²An alternative is to first show that $R\mathcal{H}om_{\mathcal{O}_X}(K, K) = \mathcal{O}_X$ by working affine locally and then use Lemma 2.4 part (2) to see the map is an isomorphism.

0ATP **Lemma 2.6.** *Let X be a locally Noetherian scheme. If K and K' are dualizing complexes on X , then K' is isomorphic to $K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L$ for some invertible object L of $D(\mathcal{O}_X)$.*

Proof. Set

$$L = R\mathcal{H}om_{\mathcal{O}_X}(K, K')$$

This is an invertible object of $D(\mathcal{O}_X)$, because affine locally this is true, see Dualizing Complexes, Lemma 15.5 and its proof. The evaluation map $L \otimes_{\mathcal{O}_X}^{\mathbf{L}} K \rightarrow K'$ is an isomorphism for the same reason. $\square$

0AWF **Lemma 2.7.** *Let X be a locally Noetherian scheme. Let $\omega_X^\bullet$ be a dualizing complex on X . Then X is universally catenary and the function $X \rightarrow \mathbf{Z}$ defined by*

$$x \longmapsto \delta(x) \text{ such that } \omega_{X,x}^\bullet[-\delta(x)] \text{ is a normalized dualizing complex over } \mathcal{O}_{X,x}$$

is a dimension function.

Proof. Immediate from the affine case Dualizing Complexes, Lemma 17.3 and the definitions. $\square$

0ECM **Lemma 2.8.** *Let X be a locally Noetherian scheme. Let $\omega_X^\bullet$ be a dualizing complex on X with associated dimension function δ . Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ -module. Set $\mathcal{E}^i = \mathcal{E}xt_{\mathcal{O}_X}^{-i}(\mathcal{F}, \omega_X^\bullet)$. Then $\mathcal{E}^i$ is a coherent $\mathcal{O}_X$ -module and for $x \in X$ we have*

- (1) $\mathcal{E}_x^i$ is nonzero only for $\delta(x) \leq i \leq \delta(x) + \dim(\text{Supp}(\mathcal{F}_x))$,
- (2) $\dim(\text{Supp}(\mathcal{E}_x^{i+\delta(x)})) \leq i$,
- (3) $\text{depth}(\mathcal{F}_x)$ is the smallest integer $i \geq 0$ such that $\mathcal{E}_x^{i+\delta(x)} \neq 0$, and
- (4) we have $x \in \text{Supp}(\bigoplus_{j \leq i} \mathcal{E}^j) \Leftrightarrow \text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) + \delta(x) \leq i$.

Proof. Lemma 2.5 tells us that $\mathcal{E}^i$ is coherent. Choosing an affine neighbourhood of x and using Derived Categories of Schemes, Lemma 10.8 and More on Algebra, Lemma 99.2 part (3) we have

$$\mathcal{E}_x^i = \mathcal{E}xt_{\mathcal{O}_X}^{-i}(\mathcal{F}, \omega_X^\bullet)_x = \text{Ext}_{\mathcal{O}_{X,x}}^{-i}(\mathcal{F}_x, \omega_{X,x}^\bullet) = \text{Ext}_{\mathcal{O}_{X,x}}^{\delta(x)-i}(\mathcal{F}_x, \omega_{X,x}^\bullet[-\delta(x)])$$

By construction of δ in Lemma 2.7 this reduces parts (1), (2), and (3) to Dualizing Complexes, Lemma 16.5. Part (4) is a formal consequence of (3) and (1). $\square$

3. Right adjoint of pushforward

0A9D References for this section and the following are [Nee96], [LN07], [Lip09], and [Nee14].

Let $f : X \rightarrow Y$ be a morphism of schemes. In this section we consider the right adjoint to the functor $Rf_* : D_{Q\text{Coh}}(\mathcal{O}_X) \rightarrow D_{Q\text{Coh}}(\mathcal{O}_Y)$. In the literature, if this functor exists, then it is sometimes denoted $f^\times$. This notation is not universally accepted and we refrain from using it. We will not use the notation $f^!$ for such a functor, as this would clash (for general morphisms f) with the notation in [Har66].

0A9E **Lemma 3.1.** *Let $f : X \rightarrow Y$ be a morphism between quasi-separated and quasi-compact schemes. The functor $Rf_* : D_{Q\text{Coh}}(\mathcal{O}_X) \rightarrow D_{Q\text{Coh}}(\mathcal{O}_Y)$ has a right adjoint.*

This is almost the same as [Nee96, Example 4.2].

Proof. We will prove a right adjoint exists by verifying the hypotheses of Derived Categories, Proposition 38.2. First off, the category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, see Derived Categories of Schemes, Lemma 3.1. The category $D_{QCoh}(\mathcal{O}_X)$ is compactly generated by Derived Categories of Schemes, Theorem 15.3. Since X and Y are quasi-compact and quasi-separated, so is f , see Schemes, Lemmas 21.13 and 21.14. Hence the functor Rf_* commutes with direct sums, see Derived Categories of Schemes, Lemma 4.5. This finishes the proof. $\square$

0A9F **Example 3.2.** Let $A \rightarrow B$ be a ring map. Let $Y = \text{Spec}(A)$ and $X = \text{Spec}(B)$ and $f : X \rightarrow Y$ the morphism corresponding to $A \rightarrow B$. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ corresponds to restriction $D(B) \rightarrow D(A)$ via the equivalences $D(B) \rightarrow D_{QCoh}(\mathcal{O}_X)$ and $D(A) \rightarrow D_{QCoh}(\mathcal{O}_Y)$. Hence the right adjoint corresponds to the functor $K \mapsto R\text{Hom}(B, K)$ of Dualizing Complexes, Section 13.

0A9G **Example 3.3.** If $f : X \rightarrow Y$ is a separated finite type morphism of Noetherian schemes, then the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ does not map $D_{Coh}(\mathcal{O}_Y)$ into $D_{Coh}(\mathcal{O}_X)$. Namely, let k be a field and consider the morphism $f : \mathbf{A}_k^1 \rightarrow \text{Spec}(k)$. By Example 3.2 this corresponds to the question of whether $R\text{Hom}(B, -)$ maps $D_{Coh}(A)$ into $D_{Coh}(B)$ where $A = k$ and $B = k[x]$. This is not true because

$$R\text{Hom}(k[x], k) = \left(\prod_{n \geq 0} k \right) [0]$$

which is not a finite $k[x]$ -module. Hence $a(\mathcal{O}_Y)$ does not have coherent cohomology sheaves.

0A9H **Example 3.4.** If $f : X \rightarrow Y$ is a proper or even finite morphism of Noetherian schemes, then the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ does not map $D_{QCoh}^-(\mathcal{O}_Y)$ into $D_{QCoh}^-(\mathcal{O}_X)$. Namely, let k be a field, let $k[\epsilon]$ be the dual numbers over k , let $X = \text{Spec}(k)$, and let $Y = \text{Spec}(k[\epsilon])$. Then $\text{Ext}_{k[\epsilon]}^i(k, k)$ is nonzero for all $i \geq 0$. Hence $a(\mathcal{O}_Y)$ is not bounded above by Example 3.2.

0A9I **Lemma 3.5.** Let $f : X \rightarrow Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_X)$ be the right adjoint to Rf_* of Lemma 3.1. Then a maps $D_{QCoh}^+(\mathcal{O}_Y)$ into $D_{QCoh}^+(\mathcal{O}_X)$. In fact, there exists an integer N such that $H^i(K) = 0$ for $i \leq c$ implies $H^i(a(K)) = 0$ for $i \leq c - N$.

Proof. By Derived Categories of Schemes, Lemma 4.1 the functor Rf_* has finite cohomological dimension. In other words, there exist an integer N such that $H^i(Rf_*L) = 0$ for $i \geq N + c$ if $H^i(L) = 0$ for $i \geq c$. Say $K \in D_{QCoh}^+(\mathcal{O}_Y)$ has $H^i(K) = 0$ for $i \leq c$. Then

$$\text{Hom}_{D(\mathcal{O}_X)}(\tau_{\leq c-N}a(K), a(K)) = \text{Hom}_{D(\mathcal{O}_Y)}(Rf_*\tau_{\leq c-N}a(K), K) = 0$$

by what we said above. Clearly, this implies that $H^i(a(K)) = 0$ for $i \leq c - N$. $\square$

Let $f : X \rightarrow Y$ be a morphism of quasi-separated and quasi-compact schemes. Let a denote the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$. For every $K \in D_{QCoh}(\mathcal{O}_Y)$ and $L \in D_{QCoh}(\mathcal{O}_X)$ we obtain a canonical map

0B6H (3.5.1) $Rf_*R\text{Hom}_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\text{Hom}_{\mathcal{O}_Y}(Rf_*L, K)$

Namely, this map is constructed as the composition

$$Rf_*R\text{Hom}_{\mathcal{O}_X}(L, a(K)) \rightarrow R\text{Hom}_{\mathcal{O}_Y}(Rf_*L, Rf_*a(K)) \rightarrow R\text{Hom}_{\mathcal{O}_Y}(Rf_*L, K)$$

where the first arrow is Cohomology, Remark 42.11 and the second arrow is the counit $Rf_*a(K) \rightarrow K$ of the adjunction.

0A9Q **Lemma 3.6.** *Let $f : X \rightarrow Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$. Let $L \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$. Then the map (3.5.1)*

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

becomes an isomorphism after applying the functor $DQ_Y : D(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ discussed in Derived Categories of Schemes, Section 21.

Proof. The statement makes sense as DQ_Y exists by Derived Categories of Schemes, Lemma 21.1. Since DQ_Y is the right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_Y) \rightarrow D(\mathcal{O}_Y)$ to prove the lemma we have to show that for any $M \in D_{QCoh}(\mathcal{O}_Y)$ the map (3.5.1) induces an bijection

$$\mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) \longrightarrow \mathrm{Hom}_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K))$$

To see this we use the following string of equalities

$$\begin{aligned} \mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) &= \mathrm{Hom}_X(Lf^*M, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) \\ &= \mathrm{Hom}_X(Lf^*M \otimes_{\mathcal{O}_X}^{\mathbf{L}} L, a(K)) \\ &= \mathrm{Hom}_Y(Rf_*(Lf^*M \otimes_{\mathcal{O}_X}^{\mathbf{L}} L), K) \\ &= \mathrm{Hom}_Y(M \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*L, K) \\ &= \mathrm{Hom}_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)) \end{aligned}$$

The first equality holds by Cohomology, Lemma 28.1. The second equality by Cohomology, Lemma 42.2. The third equality by construction of a . The fourth equality by Derived Categories of Schemes, Lemma 22.1 (this is the important step). The fifth by Cohomology, Lemma 42.2. $\square$

0GEU **Example 3.7.** The statement of Lemma 3.6 is not true without applying the “coherator” DQ_Y . Indeed, suppose $Y = \mathrm{Spec}(R)$ and $X = \mathbf{A}_R^1$. Take $L = \mathcal{O}_X$ and $K = \mathcal{O}_Y$. The left hand side of the arrow is in $D_{QCoh}(\mathcal{O}_Y)$ but the right hand side of the arrow is isomorphic to $\prod_{n \geq 0} \mathcal{O}_Y$ which is not quasi-coherent.

0GEV **Remark 3.8.** In the situation of Lemma 3.6 we have

$$DQ_Y(Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = Rf_*DQ_X(R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

by Derived Categories of Schemes, Lemma 21.2. Thus if $R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \in D_{QCoh}(\mathcal{O}_X)$, then we can “erase” the DQ_Y on the left hand side of the arrow. On the other hand, if we know that $R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K) \in D_{QCoh}(\mathcal{O}_Y)$, then we can “erase” the DQ_Y from the right hand side of the arrow. If both are true then we see that (3.5.1) is an isomorphism. Combining this with Derived Categories of Schemes, Lemma 10.8 we see that $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \rightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$ is an isomorphism if

- (1) L and Rf_*L are perfect, or
- (2) K is bounded below and L and Rf_*L are pseudo-coherent.

For (2) we use that $a(K)$ is bounded below if K is bounded below, see Lemma 3.5.

0GEW **Example 3.9.** Let $f : X \rightarrow Y$ be a proper morphism of Noetherian schemes, $L \in D_{\text{Coh}}^-(X)$ and $K \in D_{\text{QCoh}}^+(\mathcal{O}_Y)$. Then the map $Rf_* R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \rightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* L, K)$ is an isomorphism. Namely, the complexes L and $Rf_* L$ are pseudo-coherent by Derived Categories of Schemes, Lemmas 10.3 and 11.3 and the discussion in Remark 3.8 applies.

0B6I **Lemma 3.10.** Let $f : X \rightarrow Y$ be a morphism of quasi-separated and quasi-compact schemes. For all $L \in D_{\text{QCoh}}(\mathcal{O}_X)$ and $K \in D_{\text{QCoh}}(\mathcal{O}_Y)$ (3.5.1) induces an isomorphism $R\mathcal{H}om_X(L, a(K)) \rightarrow R\mathcal{H}om_Y(Rf_* L, K)$ of global derived homs.

Proof. By the construction in Cohomology, Section 44 we have

$$R\mathcal{H}om_X(L, a(K)) = R\Gamma(X, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = R\Gamma(Y, Rf_* R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

and

$$R\mathcal{H}om_Y(Rf_* L, K) = R\Gamma(Y, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* L, K))$$

Thus the lemma is a consequence of Lemma 3.6. Namely, a map $E \rightarrow E'$ in $D(\mathcal{O}_Y)$ which induces an isomorphism $DQ_Y(E) \rightarrow DQ_Y(E')$ induces a quasi-isomorphism $R\Gamma(Y, E) \rightarrow R\Gamma(Y, E')$. Indeed we have $H^i(Y, E) = \text{Ext}_Y^i(\mathcal{O}_Y, E) = \text{Hom}(\mathcal{O}_Y[-i], E) = \text{Hom}(\mathcal{O}_Y[-i], DQ_Y(E))$ because $\mathcal{O}_Y[-i]$ is in $D_{\text{QCoh}}(\mathcal{O}_Y)$ and DQ_Y is the right adjoint to the inclusion functor $D_{\text{QCoh}}(\mathcal{O}_Y) \rightarrow D(\mathcal{O}_Y)$. $\square$

4. Right adjoint of pushforward and restriction to opens

0E4G In this section we study the question to what extent the right adjoint of pushforward commutes with restriction to open subschemes. This is a base change question, so let's first discuss this more generally.

We often want to know whether the right adjoints to pushforward commutes with base change. Thus we consider a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array} \quad (4.0.1)$$

of quasi-compact and quasi-separated schemes. Denote

$$a : D_{\text{QCoh}}(\mathcal{O}_Y) \rightarrow D_{\text{QCoh}}(\mathcal{O}_X) \quad \text{and} \quad a' : D_{\text{QCoh}}(\mathcal{O}_{Y'}) \rightarrow D_{\text{QCoh}}(\mathcal{O}_{X'})$$

the right adjoints to Rf_* and Rf'_* (Lemma 3.1). Consider the base change map of Cohomology, Remark 28.3. It induces a transformation of functors

$$Lg^* \circ Rf_* \longrightarrow Rf'_* \circ L(g')^*$$

on derived categories of sheaves with quasi-coherent cohomology. Hence a transformation between the right adjoints in the opposite direction

$$a \circ Rg_* \longleftarrow Rg'_* \circ a'$$

0A9K **Lemma 4.1.** In diagram (4.0.1) assume that g is flat or more generally that f and g are Tor independent. Then $a \circ Rg_* \longleftarrow Rg'_* \circ a'$ is an isomorphism.

Proof. In this case the base change map $Lg^* \circ Rf_* K \longrightarrow Rf'_* \circ L(g')^* K$ is an isomorphism for every K in $D_{\text{QCoh}}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 22.5. Thus the corresponding transformation between adjoint functors is an isomorphism as well. $\square$

Let $f : X \rightarrow Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $V \subset Y$ be a quasi-compact open subscheme and set $U = f^{-1}(V)$. This gives a cartesian square

$$\begin{array}{ccc} U & \xrightarrow{\quad} & X \\ f|_U \downarrow & j' & \downarrow f \\ V & \xrightarrow{\quad j \quad} & Y \end{array}$$

as in (4.0.1). By Lemma 4.1 the map $\xi : a \circ Rj_* \leftarrow Rj'_* \circ a'$ is an isomorphism where a and a' are the right adjoints to Rf_* and $R(f|_U)_*$. We obtain a transformation of functors $D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_U)$

$$(4.1.1) \quad (j')^* \circ a \rightarrow (j')^* \circ a \circ Rj_* \circ j^* \xrightarrow{\xi^{-1}} (j')^* \circ Rj'_* \circ a' \circ j^* \rightarrow a' \circ j^*$$

where the first arrow comes from $\text{id} \rightarrow Rj_* \circ j^*$ and the final arrow from the isomorphism $(j')^* \circ Rj'_* \rightarrow \text{id}$. In particular, we see that (4.1.1) is an isomorphism when evaluated on K if and only if $a(K)|_U \rightarrow a(Rj_*(K|_V))|_U$ is an isomorphism.

0A9M Example 4.2. There is a finite morphism $f : X \rightarrow Y$ of Noetherian schemes such that (4.1.1) is not an isomorphism when evaluated on some $K \in D_{Coh}(\mathcal{O}_Y)$. Namely, let $X = \text{Spec}(B) \rightarrow Y = \text{Spec}(A)$ with $A = k[x, \epsilon]$ where k is a field and $\epsilon^2 = 0$ and $B = k[x] = A/(\epsilon)$. For $n \in \mathbb{N}$ set $M_n = A/(\epsilon, x^n)$. Observe that

$$\text{Ext}_A^i(B, M_n) = M_n, \quad i \geq 0$$

because B has the free periodic resolution $\dots \rightarrow A \rightarrow A \rightarrow A$ with maps given by multiplication by ϵ . Consider the object $K = \bigoplus M_n[n] = \prod M_n[n]$ of $D_{Coh}(A)$ (equality in $D(A)$ by Derived Categories, Lemmas 33.5 and 34.2). Then we see that $a(K)$ corresponds to $R\text{Hom}(B, K)$ by Example 3.2 and

$$H^0(R\text{Hom}(B, K)) = \text{Ext}_A^0(B, K) = \prod_{n \geq 1} \text{Ext}_A^n(B, M_n) = \prod_{n \geq 1} M_n$$

by the above. But this module has elements which are not annihilated by any power of x , whereas the complex K does have every element of its cohomology annihilated by a power of x . In other words, for the map (4.1.1) with $V = D(x)$ and $U = D(x)$ and the complex K cannot be an isomorphism because $(j')^*(a(K))$ is nonzero and $a'(j^*K)$ is zero.

0A9N Lemma 4.3. Let $f : X \rightarrow Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$. Let $V \subset Y$ be quasi-compact open with inverse image $U \subset X$.

- (1) For every $Q \in D_{QCoh}^+(\mathcal{O}_Y)$ supported on $Y \setminus V$ the image $a(Q)$ is supported on $X \setminus U$ if and only if (4.1.1) is an isomorphism on all K in $D_{QCoh}^+(\mathcal{O}_Y)$.
- (2) For every $Q \in D_{QCoh}(\mathcal{O}_Y)$ supported on $Y \setminus V$ the image $a(Q)$ is supported on $X \setminus U$ if and only if (4.1.1) is an isomorphism on all K in $D_{QCoh}(\mathcal{O}_Y)$.
- (3) If a commutes with direct sums, then the equivalent conditions of (1) imply the equivalent conditions of (2).

Proof. Proof of (1). Let $K \in D_{QCoh}^+(\mathcal{O}_Y)$. Choose a distinguished triangle

$$K \rightarrow Rj_*K|_V \rightarrow Q \rightarrow K[1]$$

Observe that Q is in $D_{QCoh}^+(\mathcal{O}_Y)$ (Derived Categories of Schemes, Lemma 4.1) and is supported on $Y \setminus V$ (Derived Categories of Schemes, Definition 6.1). Applying a

we obtain a distinguished triangle

$$a(K) \rightarrow a(Rj_*K|_V) \rightarrow a(Q) \rightarrow a(K)[1]$$

on X . If $a(Q)$ is supported on $X \setminus U$, then restricting to U the map $a(K)|_U \rightarrow a(Rj_*K|_V)|_U$ is an isomorphism, i.e., (4.1.1) is an isomorphism on K . The converse is immediate.

The proof of (2) is exactly the same as the proof of (1).

Proof of (3). Assume the equivalent conditions of (1) hold. Set $T = Y \setminus V$. We will use the notation $D_{QCoh,T}(\mathcal{O}_Y)$ and $D_{QCoh,f^{-1}(T)}(\mathcal{O}_X)$ to denote complexes whose cohomology sheaves are supported on T and $f^{-1}(T)$. Since a commutes with direct sums, the strictly full, saturated, triangulated subcategory $\mathcal{D}$ with objects

$$\{Q \in D_{QCoh,T}(\mathcal{O}_Y) \mid a(Q) \in D_{QCoh,f^{-1}(T)}(\mathcal{O}_X)\}$$

is preserved by direct sums and hence derived colimits. On the other hand, the category $D_{QCoh,T}(\mathcal{O}_Y)$ is generated by a perfect object E (see Derived Categories of Schemes, Lemma 15.4). By assumption we see that $E \in \mathcal{D}$. By Derived Categories, Lemma 37.3 every object Q of $D_{QCoh,T}(\mathcal{O}_Y)$ is a derived colimit of a system $Q_1 \rightarrow Q_2 \rightarrow Q_3 \rightarrow \dots$ such that the cones of the transition maps are direct sums of shifts of E . Arguing by induction we see that $Q_n \in \mathcal{D}$ for all n and finally that Q is in $\mathcal{D}$. Thus the equivalent conditions of (2) hold. $\square$

0A9P **Lemma 4.4.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \rightarrow Y$ be a proper morphism. If³*

- (1) *f is flat and of finite presentation, or*
- (2) *Y is Noetherian*

then the equivalent conditions of Lemma 4.3 part (1) hold for all quasi-compact opens V of Y .

Proof. Let $Q \in D_{QCoh}^+(\mathcal{O}_Y)$ be supported on $Y \setminus V$. To get a contradiction, assume that $a(Q)$ is not supported on $X \setminus U$. Then we can find a perfect complex P_U on U and a nonzero map $P_U \rightarrow a(Q)|_U$ (follows from Derived Categories of Schemes, Theorem 15.3). Then using Derived Categories of Schemes, Lemma 13.11 we may assume there is a perfect complex P on X and a map $P \rightarrow a(Q)$ whose restriction to U is nonzero. By definition of a this map is adjoint to a map $Rf_*P \rightarrow Q$.

The complex Rf_*P is pseudo-coherent. In case (1) this follows from Derived Categories of Schemes, Lemma 30.5. In case (2) this follows from Derived Categories of Schemes, Lemmas 11.3 and 10.3. Thus we may apply Derived Categories of Schemes, Lemma 17.5 and get a map $I \rightarrow \mathcal{O}_Y$ of perfect complexes whose restriction to V is an isomorphism such that the composition $I \otimes_{\mathcal{O}_Y}^L Rf_*P \rightarrow Rf_*P \rightarrow Q$ is zero. By Derived Categories of Schemes, Lemma 22.1 we have $I \otimes_{\mathcal{O}_Y}^L Rf_*P = Rf_*(Lf^*I \otimes_{\mathcal{O}_X}^L P)$. We conclude that the composition

$$Lf^*I \otimes_{\mathcal{O}_X}^L P \rightarrow P \rightarrow a(Q)$$

is zero. However, the restriction to U is the map $P|_U \rightarrow a(Q)|_U$ which we assumed to be nonzero. This contradiction finishes the proof. $\square$

³This proof works for those morphisms of quasi-compact and quasi-separated schemes such that Rf_*P is pseudo-coherent for all P perfect on X . It follows easily from a theorem of Kiehl [Kie72] that this holds if f is proper and pseudo-coherent. This is the correct generality for this lemma and some of the other results in this chapter.

5. Right adjoint of pushforward and base change, I

0AA5 The map (4.1.1) is a special case of a base change map. Namely, suppose that we have a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

of quasi-compact and quasi-separated schemes, i.e., a diagram as in (4.0.1). Assume f and g are **Tor independent**. Then we can consider the morphism of functors $D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_{X'})$ given by the composition

0AA6 (5.0.1) $L(g')^* \circ a \rightarrow L(g')^* \circ a \circ Rg_* \circ Lg^* \leftarrow L(g')^* \circ Rg'_* \circ a' \circ Lg^* \rightarrow a' \circ Lg^*$

The first arrow comes from the adjunction map $\text{id} \rightarrow Rg_* Lg^*$ and the last arrow from the adjunction map $L(g')^* Rg'_* \rightarrow \text{id}$. We need the assumption on Tor independence to invert the arrow in the middle, see Lemma 4.1. Alternatively, we can think of (5.0.1) by adjointness of $L(g')^*$ and $R(g')_*$ as a natural transformation

$$a \rightarrow a \circ Rg_* \circ Lg^* \leftarrow Rg'_* \circ a' \circ Lg^*$$

were again the second arrow is invertible. If $M \in D_{QCoh}(\mathcal{O}_X)$ and $K \in D_{QCoh}(\mathcal{O}_Y)$ then on Yoneda functors this map is given by

$$\begin{aligned} \text{Hom}_X(M, a(K)) &= \text{Hom}_Y(Rf_* M, K) \\ &\rightarrow \text{Hom}_Y(Rf_* M, Rg_* Lg^* K) \\ &= \text{Hom}_{Y'}(Lg^* Rf_* M, Lg^* K) \\ &\leftarrow \text{Hom}_{Y'}(Rf'_* L(g')^* M, Lg^* K) \\ &= \text{Hom}_{X'}(L(g')^* M, a'(Lg^* K)) \\ &= \text{Hom}_X(M, Rg'_* a'(Lg^* K)) \end{aligned}$$

(were the arrow pointing left is invertible by the base change theorem given in Derived Categories of Schemes, Lemma 22.5) which makes things a little bit more explicit.

In this section we first prove that the base change map satisfies some natural compatibilities with regards to stacking squares as in Cohomology, Remarks 28.4 and 28.5 for the usual base change map. We suggest the reader skip the rest of this section on a first reading.

0ATQ **Lemma 5.1.** *Consider a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{k} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{l} & Y \\ g' \downarrow & & \downarrow g \\ Z' & \xrightarrow{m} & Z \end{array}$$

of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where f and l as well as g and m are Tor independent. Then the maps (5.0.1)

for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

Proof. It follows from the assumptions that $g \circ f$ and m are Tor independent (details omitted), hence the statement makes sense. In this proof we write k^* in place of Lk^* and f_* instead of Rf_* . Let a, b , and c be the right adjoints of Lemma 3.1 for f, g , and $g \circ f$ and similarly for the primed versions. The arrow corresponding to the top square is the composition

$$\gamma_{top} : k^* \circ a \rightarrow k^* \circ a \circ l_* \circ l^* \xleftarrow{\xi_{top}} k^* \circ k_* \circ a' \circ l^* \rightarrow a' \circ l^*$$

where $\xi_{top} : k_* \circ a' \rightarrow a \circ l_*$ is an isomorphism (hence can be inverted) and is the arrow “dual” to the base change map $l^* \circ f_* \rightarrow f'_* \circ k^*$. The outer arrows come from the canonical maps $1 \rightarrow l_* \circ l^*$ and $k^* \circ k_* \rightarrow 1$. Similarly for the second square we have

$$\gamma_{bot} : l^* \circ b \rightarrow l^* \circ b \circ m_* \circ m^* \xleftarrow{\xi_{bot}} l^* \circ l_* \circ b' \circ m^* \rightarrow b' \circ m^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ c \rightarrow k^* \circ c \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ c' \circ m^* \rightarrow c' \circ m^*$$

We have $(g \circ f)_* = g_* \circ f_*$ and hence $c = a \circ b$ and similarly $c' = a' \circ b'$. The statement of the lemma is that γ_{rect} is equal to the composition

$$k^* \circ c = k^* \circ a \circ b \xrightarrow{\gamma_{top}} a' \circ l^* \circ b \xrightarrow{\gamma_{bot}} a' \circ b' \circ m^* = c' \circ m^*$$

To see this we contemplate the following diagram:

$$\begin{array}{ccccc}
 & & & & k^* \circ a \circ b \\
 & & & & \downarrow \\
 & & & & k^* \circ a \circ l_* \circ l^* \circ b \\
 & & & \swarrow & \uparrow \xi_{top} \\
 & & & k^* \circ k_* \circ a' \circ l^* \circ b & \\
 & & & \downarrow & \\
 k^* \circ a \circ b \circ m_* \circ m^* & \longrightarrow & k^* \circ a \circ l_* \circ l^* \circ b \circ m_* \circ m^* & \xleftarrow{\xi_{top}} & k^* \circ k_* \circ a' \circ l^* \circ b \circ m_* \circ m^* \\
 \uparrow \xi_{rect} & & \uparrow \xi_{bot} & & \downarrow \xi_{bot} \\
 k^* \circ k_* \circ a' \circ b' \circ m^* & \longleftarrow & k^* \circ k_* \circ a' \circ l^* \circ l_* \circ b' \circ m^* & \longrightarrow & a' \circ l^* \circ b \circ m_* \circ m^* \\
 & & \downarrow & & \downarrow \\
 & & a' \circ l^* \circ l_* \circ b' \circ m^* & \xrightarrow{\xi_{bot}} & a' \circ b' \circ m^*
 \end{array}$$

Going down the right hand side we have the composition and going down the left hand side we have γ_{rect} . All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 28.2 or more simply the discussion preceding

Categories, Definition 28.1. Hence we see that it suffices to show the diagram

$$\begin{array}{ccc}
 a \circ l_* \circ l^* \circ b \circ m_* & \longleftarrow & a \circ b \circ m_* \\
 \uparrow \xi_{top} & & \uparrow \xi_{rect} \\
 k_* \circ a' \circ l^* \circ b \circ m_* & & \\
 \uparrow \xi_{bot} & & \\
 k_* \circ a' \circ l^* \circ l_* \circ b' & \longrightarrow & k_* \circ a' \circ b'
 \end{array}$$

becomes commutative if we invert the arrows ξ_{top} , ξ_{bot} , and ξ_{rect} (note that this is different from asking the diagram to be commutative). However, the diagram

$$\begin{array}{ccccc}
 & & a \circ l_* \circ l^* \circ b \circ m_* & & \\
 & \nearrow \xi_{bot} & & \nwarrow \xi_{top} & \\
 a \circ l_* \circ l^* \circ l_* \circ b' & & & & k_* \circ a' \circ l^* \circ b \circ m_* \\
 & \nwarrow \xi_{top} & & \nearrow \xi_{bot} & \\
 & & k_* \circ a' \circ l^* \circ l_* \circ b' & &
 \end{array}$$

commutes by Categories, Lemma 28.2. Since the diagrams

$$\begin{array}{ccc}
 a \circ l_* \circ l^* \circ b \circ m_* & \longleftarrow & a \circ b \circ m \\
 \uparrow & & \uparrow \\
 a \circ l_* \circ l^* \circ l_* \circ b' & \longleftarrow & a \circ l_* \circ b'
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 a \circ l_* \circ l^* \circ l_* \circ b' & \longrightarrow & a \circ l_* \circ b' \\
 \uparrow & & \uparrow \\
 k_* \circ a' \circ l^* \circ l_* \circ b' & \longrightarrow & k_* \circ a' \circ b'
 \end{array}$$

commute (see references cited) and since the composition of $l_* \rightarrow l_* \circ l^* \circ l_* \rightarrow l_*$ is the identity, we find that it suffices to prove that

$$k \circ a' \circ b' \xrightarrow{\xi_{bot}} a \circ l_* \circ b \xrightarrow{\xi_{top}} a \circ b \circ m_*$$

is equal to ξ_{rect} (via the identifications $a \circ b = c$ and $a' \circ b' = c'$). This is the statement dual to Cohomology, Remark 28.4 and the proof is complete. $\square$

0ATR **Lemma 5.2.** *Consider a commutative diagram*

$$\begin{array}{ccccc}
 X'' & \xrightarrow{g'} & X' & \xrightarrow{g} & X \\
 f'' \downarrow & & f' \downarrow & & \downarrow f \\
 Y'' & \xrightarrow{h'} & Y' & \xrightarrow{h} & Y
 \end{array}$$

of quasi-compact and quasi-separated schemes where both diagrams are cartesian and where f and h as well as f' and h' are Tor independent. Then the maps (5.0.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

Proof. It follows from the assumptions that f and $h \circ h'$ are Tor independent (details omitted), hence the statement makes sense. In this proof we write g^* in place of Lg^* and f_* instead of Rf_* . Let a , a' , and a'' be the right adjoints of

Lemma 3.1 for f , f' , and f'' . The arrow corresponding to the right square is the composition

$$\gamma_{right} : g^* \circ a \rightarrow g^* \circ a \circ h_* \circ h^* \xleftarrow{\xi_{right}} g^* \circ g_* \circ a' \circ h^* \rightarrow a' \circ h^*$$

where $\xi_{right} : g_* \circ a' \rightarrow a \circ h_*$ is an isomorphism (hence can be inverted) and is the arrow “dual” to the base change map $h^* \circ f_* \rightarrow f'_* \circ g^*$. The outer arrows come from the canonical maps $1 \rightarrow h_* \circ h^*$ and $g^* \circ g_* \rightarrow 1$. Similarly for the left square we have

$$\gamma_{left} : (g')^* \circ a' \rightarrow (g')^* \circ a' \circ (h')_* \circ (h')^* \xleftarrow{\xi_{left}} (g')^* \circ (g')_* \circ a'' \circ (h')^* \rightarrow a'' \circ (h')^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ a \rightarrow k^* \circ a \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ a'' \circ m^* \rightarrow a'' \circ m^*$$

where $k = g \circ g'$ and $m = h \circ h'$. We have $k^* = (g')^* \circ g^*$ and $m^* = (h')^* \circ h^*$. The statement of the lemma is that γ_{rect} is equal to the composition

$$k^* \circ a = (g')^* \circ g^* \circ a \xrightarrow{\gamma_{right}} (g')^* \circ a' \circ h^* \xrightarrow{\gamma_{left}} a'' \circ (h')^* \circ h^* = a'' \circ m^*$$

To see this we contemplate the following diagram

$$\begin{array}{ccccc}
 & & & & (g')^* \circ g^* \circ a \\
 & & & & \downarrow \\
 & & & & (g')^* \circ g^* \circ a \circ h_* \circ h^* \\
 & & & \swarrow & \uparrow \xi_{right} \\
 & & & (g')^* \circ g^* \circ g_* \circ a' \circ h^* & \downarrow \\
 (g')^* \circ g^* \circ a \circ h_* \circ (h')_* \circ (h')^* \circ h^* & \swarrow & & & (g')^* \circ a' \circ h^* \\
 \uparrow \xi_{right} & & & & \downarrow \\
 (g')^* \circ g^* \circ g_* \circ a' \circ (h')_* \circ (h')^* \circ h^* & \swarrow & & & (g')^* \circ a' \circ (h')_* \circ (h')^* \circ h^* \\
 \uparrow \xi_{left} & & & & \downarrow \\
 (g')^* \circ g^* \circ g_* \circ (g')_* \circ a'' \circ (h')^* \circ h^* & \swarrow & & & (g')^* \circ (g')_* \circ a'' \circ (h')^* \circ h^* \\
 & \searrow & & & \uparrow \xi_{left} \\
 & & & & a'' \circ (h')^* \circ h^*
 \end{array}$$

Going down the right hand side we have the composition and going down the left hand side we have γ_{rect} . All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 28.2 or more simply the discussion preceding Categories, Definition 28.1. Hence we see that it suffices to show that

$$g_* \circ (g')_* \circ a'' \xrightarrow{\xi_{left}} g_* \circ a' \circ (h')_* \xrightarrow{\xi_{right}} a \circ h_* \circ (h')_*$$

is equal to ξ_{rect} . This is the statement dual to Cohomology, Remark 28.5 and the proof is complete. $\square$

0ATS **Remark 5.3.** Consider a commutative diagram

$$\begin{array}{ccccc}
 X'' & \xrightarrow{\quad k' \quad} & X' & \xrightarrow{\quad k \quad} & X \\
 f'' \downarrow & & f' \downarrow & & f \downarrow \\
 Y'' & \xrightarrow{\quad l' \quad} & Y' & \xrightarrow{\quad l \quad} & Y \\
 g'' \downarrow & & g' \downarrow & & g \downarrow \\
 Z'' & \xrightarrow{\quad m' \quad} & Z' & \xrightarrow{\quad m \quad} & Z
 \end{array}$$

of quasi-compact and quasi-separated schemes where all squares are cartesian and where (f, l) , (g, m) , (f', l') , (g', m') are Tor independent pairs of maps. Let a , a' , a'' , b , b' , b'' be the right adjoints of Lemma 3.1 for f , f' , f'' , g , g' , g'' . Let us label the squares of the diagram A , B , C , D as follows

$$\begin{array}{cc}
 A & B \\
 C & D
 \end{array}$$

Then the maps (5.0.1) for the squares are (where we use $k^* = Lk^*$, etc)

$$\begin{array}{ll}
 \gamma_A : (k')^* \circ a' \rightarrow a'' \circ (l')^* & \gamma_B : k^* \circ a \rightarrow a' \circ l^* \\
 \gamma_C : (l')^* \circ b' \rightarrow b'' \circ (m')^* & \gamma_D : l^* \circ b \rightarrow b' \circ m^*
 \end{array}$$

For the 2×1 and 1×2 rectangles we have four further base change maps

$$\begin{array}{l}
 \gamma_{A+B} : (k \circ k')^* \circ a \rightarrow a'' \circ (l \circ l')^* \\
 \gamma_{C+D} : (l \circ l')^* \circ b \rightarrow b'' \circ (m \circ m')^* \\
 \gamma_{A+C} : (k')^* \circ (a' \circ b') \rightarrow (a'' \circ b'') \circ (m')^* \\
 \gamma_{B+D} : k^* \circ (a \circ b) \rightarrow (a' \circ b') \circ m^*
 \end{array}$$

By Lemma 5.2 we have

$$\gamma_{A+B} = \gamma_A \circ \gamma_B, \quad \gamma_{C+D} = \gamma_C \circ \gamma_D$$

and by Lemma 5.1 we have

$$\gamma_{A+C} = \gamma_C \circ \gamma_A, \quad \gamma_{B+D} = \gamma_D \circ \gamma_B$$

Here it would be more correct to write $\gamma_{A+B} = (\gamma_A \star \text{id}_{l^*}) \circ (\text{id}_{(k')^*} \star \gamma_B)$ with notation as in Categories, Section 28 and similarly for the others. However, we continue the abuse of notation used in the proofs of Lemmas 5.1 and 5.2 of dropping $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Having said all of this we find (a priori) two transformations

$$(k')^* \circ k^* \circ a \circ b \longrightarrow a'' \circ b'' \circ (m')^* \circ m^*$$

namely

$$\gamma_C \circ \gamma_A \circ \gamma_D \circ \gamma_B = \gamma_{A+C} \circ \gamma_{B+D}$$

and

$$\gamma_C \circ \gamma_D \circ \gamma_A \circ \gamma_B = \gamma_{C+D} \circ \gamma_{A+B}$$

The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that

$$\begin{array}{ccc} (k')^* \circ a' \circ l^* \circ b & \xrightarrow{\gamma_D} & (k')^* \circ a' \circ b' \circ m^* \\ \gamma_A \downarrow & & \downarrow \gamma_A \\ a'' \circ (l')^* \circ l^* \circ b & \xrightarrow{\gamma_D} & a'' \circ (l')^* \circ b' \circ m^* \end{array}$$

commutes. This is true by Categories, Lemma 28.2 or more simply the discussion preceding Categories, Definition 28.1.

6. Right adjoint of pushforward and base change, II

0BZF In this section we prove that the base change map of Section 5 is an isomorphism in some cases. We first observe that it suffices to check over affine opens, provided formation of the right adjoint of pushforward commutes with restriction to opens.

0E9S **Remark 6.1.** Consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

of quasi-compact and quasi-separated schemes with (g, f) Tor independent. Let $V \subset Y$ and $V' \subset Y'$ be affine opens with $g(V') \subset V$. Form the cartesian diagrams

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ V' & \longrightarrow & Y' \end{array}$$

Assume (4.1.1) with respect to K and the first diagram and (4.1.1) with respect to Lg^*K and the second diagram are isomorphisms. Then the restriction of the base change map (5.0.1)

$$L(g')^*a(K) \longrightarrow a'(Lg^*K)$$

to U' is isomorphic to the base change map (5.0.1) for $K|_V$ and the cartesian diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

This follows from the fact that (4.1.1) is a special case of the base change map (5.0.1) and that the base change maps compose correctly if we stack squares horizontally, see Lemma 5.2. Thus in order to check the base change map restricted to U' is an isomorphism it suffices to work with the last diagram.

0AA8 **Lemma 6.2.** In diagram (4.0.1) assume

- (1) $g : Y' \rightarrow Y$ is a morphism of affine schemes,
- (2) $f : X \rightarrow Y$ is proper, and
- (3) f and g are Tor independent.

Then the base change map (5.0.1) induces an isomorphism

$$L(g')^*a(K) \longrightarrow a'(Lg^*K)$$

in the following cases

- (1) for all $K \in D_{Q\text{Coh}}(\mathcal{O}_Y)$ if f is flat of finite presentation,
- (2) for all $K \in D_{Q\text{Coh}}(\mathcal{O}_Y)$ if f is perfect and Y Noetherian,
- (3) for $K \in D_{Q\text{Coh}}^+(\mathcal{O}_Y)$ if g has finite Tor dimension and Y Noetherian.

Proof. Write $Y = \text{Spec}(A)$ and $Y' = \text{Spec}(A')$. As a base change of an affine morphism, the morphism g' is affine. Let M be a perfect generator for $D_{Q\text{Coh}}(\mathcal{O}_X)$, see Derived Categories of Schemes, Theorem 15.3. Then $L(g')^*M$ is a generator for $D_{Q\text{Coh}}(\mathcal{O}_{X'})$, see Derived Categories of Schemes, Remark 16.4. Hence it suffices to show that (5.0.1) induces an isomorphism

$$(6.2.1) \quad R\text{Hom}_{X'}(L(g')^*M, L(g')^*a(K)) \longrightarrow R\text{Hom}_{X'}(L(g')^*M, a'(Lg^*K))$$

of global hom complexes, see Cohomology, Section 44, as this will imply the cone of $L(g')^*a(K) \rightarrow a'(Lg^*K)$ is zero. The structure of the proof is as follows: we will first show that these Hom complexes are isomorphic and in the last part of the proof we will show that the isomorphism is induced by (6.2.1).

The left hand side. Because M is perfect, the canonical map

$$R\text{Hom}_X(M, a(K)) \otimes_A^{\mathbf{L}} A' \longrightarrow R\text{Hom}_{X'}(L(g')^*M, L(g')^*a(K))$$

is an isomorphism by Derived Categories of Schemes, Lemma 22.6. We can combine this with the isomorphism $R\text{Hom}_Y(Rf_*M, K) = R\text{Hom}_X(M, a(K))$ of Lemma 3.10 to get that the left hand side equals $R\text{Hom}_Y(Rf_*M, K) \otimes_A^{\mathbf{L}} A'$.

The right hand side. Here we first use the isomorphism

$$R\text{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) = R\text{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K)$$

of Lemma 3.10. Then we use the base change map $Lg^*Rf_*M \rightarrow Rf'_*L(g')^*M$ is an isomorphism by Derived Categories of Schemes, Lemma 22.5. Hence we may rewrite this as $R\text{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$. Since Y, Y' are affine and K, Rf_*M are in $D_{Q\text{Coh}}(\mathcal{O}_Y)$ (Derived Categories of Schemes, Lemma 4.1) we have a canonical map

$$\beta : R\text{Hom}_Y(Rf_*M, K) \otimes_A^{\mathbf{L}} A' \longrightarrow R\text{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$$

in $D(A')$. This is the arrow More on Algebra, Equation (99.1.1) where we have used Derived Categories of Schemes, Lemmas 3.5 and 10.8 to translate back and forth into algebra.

- (1) If f is flat and of finite presentation, the complex Rf_*M is perfect on Y by Derived Categories of Schemes, Lemma 30.4 and β is an isomorphism by More on Algebra, Lemma 99.2 part (1).
- (2) If f is perfect and Y Noetherian, the complex Rf_*M is perfect on Y by More on Morphisms, Lemma 61.13 and β is an isomorphism as before.
- (3) If g has finite tor dimension and Y is Noetherian, the complex Rf_*M is pseudo-coherent on Y (Derived Categories of Schemes, Lemmas 11.3 and 10.3) and β is an isomorphism by More on Algebra, Lemma 99.2 part (4).

We conclude that we obtain the same answer as in the previous paragraph.

In the rest of the proof we show that the identifications of the left and right hand side of (6.2.1) given in the second and third paragraph are in fact given by (6.2.1).

To make our formulas manageable we will use $(-, -)_X = R\mathrm{Hom}_X(-, -)$, use $-\otimes A'$ in stead of $-\otimes_A^{\mathbf{L}} A'$, and we will abbreviate $g^* = Lg^*$ and $f_* = Rf_*$. Consider the following commutative diagram

$$\begin{array}{ccccc}
((g')^*M, (g')^*a(K))_{X'} & \xleftarrow{\alpha} & (M, a(K))_X \otimes A' & \xlongequal{\quad} & (f_*M, K)_Y \otimes A' \\
\downarrow & & \downarrow & & \downarrow \\
((g')^*M, (g')^*a(g_*g^*K))_{X'} & \xleftarrow{\alpha} & (M, a(g_*g^*K))_X \otimes A' & \xlongequal{\quad} & (f_*M, g_*g^*K)_Y \otimes A' \\
\uparrow & & \uparrow & & \uparrow \\
((g')^*M, (g')^*g'_*a'(g^*K))_{X'} & \xleftarrow{\alpha} & (M, g'_*a'(g^*K))_X \otimes A' & \xrightarrow{\mu'} & (f_*M, K)_Y \otimes A' \\
\downarrow & \swarrow \mu & \downarrow & & \downarrow \beta \\
((g')^*M, a'(g^*K))_{X'} & \xlongequal{\quad} & (f'_*(g')^*M, g^*K)_{Y'} & \xrightarrow{\quad} & (g^*f_*M, g^*K)_{Y'}
\end{array}$$

The arrows labeled α are the maps from Derived Categories of Schemes, Lemma 22.6 for the diagram with corners X', X, Y', Y . The upper part of the diagram is commutative as the horizontal arrows are functorial in the entries. The middle vertical arrows come from the invertible transformation $g'_* \circ a' \rightarrow a \circ g_*$ of Lemma 4.1 and therefore the middle square is commutative. Going down the left hand side is (6.2.1). The upper horizontal arrows provide the identifications used in the second paragraph of the proof. The lower horizontal arrows including β provide the identifications used in the third paragraph of the proof. Given $E \in D(A)$, $E' \in D(A')$, and $c : E \rightarrow E'$ in $D(A)$ we will denote $\mu_c : E \otimes A' \rightarrow E'$ the map induced by c and the adjointness of restriction and base change; if c is clear we write $\mu = \mu_c$, i.e., we drop c from the notation. The map μ in the diagram is of this form with c given by the identification $(M, g'_*a'(g^*K))_X = ((g')^*M, a'(g^*K))_{X'}$; the triangle involving μ is commutative by Derived Categories of Schemes, Remark 22.7.

Observe that

$$\begin{array}{ccccc}
(M, a(g_*g^*K))_X & \xlongequal{\quad} & (f_*M, g_*g^*K)_Y & \xlongequal{\quad} & (g^*f_*M, g^*K)_{Y'} \\
\uparrow & & & & \uparrow \\
(M, g'_*a'(g^*K))_X & \xlongequal{\quad} & ((g')^*M, a'(g^*K))_{X'} & \xlongequal{\quad} & (f'_*(g')^*M, g^*K)_{Y'}
\end{array}$$

is commutative by the very definition of the transformation $g'_* \circ a' \rightarrow a \circ g_*$. Letting μ' be as above corresponding to the identification $(f_*M, g_*g^*K)_X = (g^*f_*M, g^*K)_{Y'}$, then the hexagon commutes as well. Thus it suffices to show that β is equal to the composition of $(f_*M, K)_Y \otimes A' \rightarrow (f_*M, g_*g^*K)_X \otimes A'$ and μ' . To do this, it suffices to prove the two induced maps $(f_*M, K)_Y \rightarrow (g^*f_*M, g^*K)_{Y'}$ are the same. In other words, it suffices to show the diagram

$$\begin{array}{ccc}
R\mathrm{Hom}_A(E, K) & \xrightarrow{\text{induced by } \beta} & R\mathrm{Hom}_{A'}(E \otimes_A^{\mathbf{L}} A', K \otimes_A^{\mathbf{L}} A') \\
& \searrow & \swarrow \\
& R\mathrm{Hom}_A(E, K \otimes_A^{\mathbf{L}} A') &
\end{array}$$

commutes for all $E, K \in D(A)$. Since this is how β is constructed in More on Algebra, Section 99 the proof is complete. $\square$

7. Right adjoint of pushforward and trace maps

0AWG Let $f : X \rightarrow Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_X)$ be the right adjoint as in Lemma 3.1. By Categories, Section 24 we obtain a transformation of functors

$$\mathrm{Tr}_f : Rf_* \circ a \longrightarrow \mathrm{id}$$

The corresponding map $\mathrm{Tr}_{f,K} : Rf_*a(K) \rightarrow K$ for $K \in D_{QCoh}(\mathcal{O}_Y)$ is sometimes called the *trace map*. This is the map which has the property that the bijection

$$\mathrm{Hom}_X(L, a(K)) \longrightarrow \mathrm{Hom}_Y(Rf_*L, K)$$

for $L \in D_{QCoh}(\mathcal{O}_X)$ which characterizes the right adjoint is given by

$$\varphi \longmapsto \mathrm{Tr}_{f,K} \circ Rf_*\varphi$$

The map (3.5.1)

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

comes about by composition with $\mathrm{Tr}_{f,K}$. Every trace map we are going to consider in this section will be a special case of this trace map. Before we discuss some special cases we show that formation of the trace map commutes with base change.

0B6J **Lemma 7.1** (Trace map and base change). *Suppose we have a diagram (4.0.1) where f and g are tor independent. Then the maps $1 \star \mathrm{Tr}_f : Lg^* \circ Rf_* \circ a \rightarrow Lg^*$ and $\mathrm{Tr}_{f'} \star 1 : Rf'_* \circ a' \circ Lg^* \rightarrow Lg^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \rightarrow Rf'_* \circ L(g')^*$ (Cohomology, Remark 28.3) and $\alpha : L(g')^* \circ a \rightarrow a' \circ Lg^*$ (5.0.1). More precisely, the diagram*

$$\begin{array}{ccc} Lg^* \circ Rf_* \circ a & \xrightarrow{1 \star \mathrm{Tr}_f} & Lg^* \\ \beta \star 1 \downarrow & & \uparrow \mathrm{Tr}_{f'} \star 1 \\ Rf'_* \circ L(g')^* \circ a & \xrightarrow{1 \star \alpha} & Rf'_* \circ a' \circ Lg^* \end{array}$$

of transformations of functors commutes.

Proof. In this proof we write f_* for Rf_* and g^* for Lg^* and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \rightarrow f'_* \circ (g')^*$ is an isomorphism and that α is defined using the isomorphism $\beta^\vee : g'_* \circ a' \rightarrow a \circ g_*$ which is the adjoint of β , see Lemma 4.1 and its proof. First we note that the top horizontal arrow of the diagram in the lemma is equal to the composition

$$g^* \circ f_* \circ a \rightarrow g^* \circ f_* \circ a \circ g_* \circ g^* \rightarrow g^* \circ g_* \circ g^* \rightarrow g^*$$

where the first arrow is the unit for (g^*, g_*) , the second arrow is Tr_f , and the third arrow is the counit for (g^*, g_*) . This is a simple consequence of the fact that the composition $g^* \rightarrow g^* \circ g_* \circ g^* \rightarrow g^*$ of unit and counit is the identity. Consider the

diagram

$$\begin{array}{ccccc}
 & g^* \circ f_* \circ a & \xrightarrow{\text{Tr}_f} & g^* & \\
 \beta \swarrow & \downarrow & & \nwarrow & \text{Tr}_{f'} \\
 f'_* \circ (g')^* \circ a & g^* \circ f_* \circ a \circ g_* \circ g^* & \xleftarrow{\beta^\vee} & g^* \circ f_* \circ g'_* \circ a' \circ g^* & f'_* \circ a' \circ g^* \\
 & \downarrow \beta & & \downarrow \beta & \\
 & f'_* \circ (g')^* \circ a \circ g_* \circ g^* & \xleftarrow{\beta^\vee} & f'_* \circ (g')^* \circ g'_* \circ a' \circ g^* &
 \end{array}$$

In this diagram the two squares commute Categories, Lemma 28.2 or more simply the discussion preceding Categories, Definition 28.1. The triangle commutes by the discussion above. By Categories, Lemma 24.8 the square

$$\begin{array}{ccc}
 g^* \circ f_* \circ g'_* \circ a' & \xrightarrow{\beta} & f'_* \circ (g')^* \circ g'_* \circ a' \\
 \beta^\vee \downarrow & & \downarrow \\
 g^* \circ f_* \circ a \circ g_* & \xrightarrow{\quad} & \text{id}
 \end{array}$$

commutes which implies the pentagon in the big diagram commutes. Since β and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\text{Tr}_f \circ \alpha \circ \beta$ by definition this proves the lemma. $\square$

Let $f : X \rightarrow Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_X)$ be the right adjoint of Rf_* as in Lemma 3.1. By Categories, Section 24 we obtain a transformation of functors

$$\eta_f : \text{id} \rightarrow a \circ Rf_*$$

which is called the unit of the adjunction.

0B6K **Lemma 7.2.** *Suppose we have a diagram (4.0.1) where f and g are tor independent. Then the maps $1 \star \eta_f : L(g')^* \rightarrow L(g')^* \circ a \circ Rf_*$ and $\eta_{f'} \star 1 : L(g')^* \rightarrow a' \circ Rf'_* \circ L(g')^*$ agree via the base change maps $\beta : Lg^* \circ Rf_* \rightarrow Rf'_* \circ L(g')^*$ (Cohomology, Remark 28.3) and $\alpha : L(g')^* \circ a \rightarrow a' \circ Lg^*$ (5.0.1). More precisely, the diagram*

$$\begin{array}{ccc}
 L(g')^* & \xrightarrow{1 \star \eta_f} & L(g')^* \circ a \circ Rf_* \\
 \eta_{f'} \star 1 \downarrow & & \downarrow \alpha \\
 a' \circ Rf'_* \circ L(g')^* & \xleftarrow{\beta} & a' \circ Lg^* \circ Rf_*
 \end{array}$$

of transformations of functors commutes.

Proof. This proof is dual to the proof of Lemma 7.1. In this proof we write f_* for Rf_* and g^* for Lg^* and we drop $\star$ products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that $\beta : g^* \circ f_* \rightarrow f'_* \circ (g')^*$ is an isomorphism and that α is defined using the isomorphism $\beta^\vee : g'_* \circ a' \rightarrow a \circ g_*$ which is the adjoint of β , see Lemma 4.1 and its proof. First we note that the left vertical arrow of the diagram in the lemma is equal to the composition

$$(g')^* \rightarrow (g')^* \circ g'_* \circ (g')^* \rightarrow (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* \rightarrow a' \circ f'_* \circ (g')^*$$

where the first arrow is the unit for $((g')^*, g'_*)$, the second arrow is $\eta_{f'}$, and the third arrow is the counit for $((g')^*, g'_*)$. This is a simple consequence of the fact that the composition $(g')^* \rightarrow (g')^* \circ (g')_* \circ (g')^* \rightarrow (g')^*$ of unit and counit is the identity. Consider the diagram

$$\begin{array}{ccccc}
 & & (g')^* \circ a \circ f_* & \xrightarrow{\quad} & (g')^* \circ a \circ g_* \circ g^* \circ f_* \\
 & \nearrow \eta_f & & \nwarrow \beta & \uparrow \beta^\vee \\
 (g')^* & & (g')^* \circ a \circ g_* \circ f'_* \circ (g')^* & & (g')^* \circ g'_* \circ a' \circ g^* \circ f_* \\
 & \searrow \eta_{f'} & \uparrow \beta^\vee & \nwarrow \beta & \downarrow \beta^\vee \\
 & & (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* & & a' \circ g^* \circ f_* \\
 & \searrow \eta_{f'} & & \nearrow \beta & \\
 & & a' \circ f'_* \circ (g')^* & &
 \end{array}$$

In this diagram the two squares commute Categories, Lemma 28.2 or more simply the discussion preceding Categories, Definition 28.1. The triangle commutes by the discussion above. By the dual of Categories, Lemma 24.8 the square

$$\begin{array}{ccc}
 \text{id} & \xrightarrow{\quad} & g'_* \circ a' \circ g^* \circ f_* \\
 \downarrow & & \downarrow \beta \\
 g'_* \circ a' \circ g^* \circ f_* & \xrightarrow{\beta^\vee} & a \circ g_* \circ f'_* \circ (g')^*
 \end{array}$$

commutes which implies the pentagon in the big diagram commutes. Since β and $\beta^\vee$ are isomorphisms, and since going on the outside of the big diagram equals $\beta \circ \alpha \circ \eta_f$ by definition this proves the lemma. $\square$

0B6L **Example 7.3.** Let $A \rightarrow B$ be a ring map. Let $Y = \text{Spec}(A)$ and $X = \text{Spec}(B)$ and $f : X \rightarrow Y$ the morphism corresponding to $A \rightarrow B$. As seen in Example 3.2 the right adjoint of $Rf_* : D_{Q\text{Coh}}(\mathcal{O}_X) \rightarrow D_{Q\text{Coh}}(\mathcal{O}_Y)$ sends an object K of $D(A) = D_{Q\text{Coh}}(\mathcal{O}_Y)$ to $R\text{Hom}(B, K)$ in $D(B) = D_{Q\text{Coh}}(\mathcal{O}_X)$. The trace map is the map

$$\text{Tr}_{f,K} : R\text{Hom}(B, K) \longrightarrow R\text{Hom}(A, K) = K$$

induced by the A -module map $A \rightarrow B$.

8. Right adjoint of pushforward and pullback

0B6N Let $f : X \rightarrow Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the right adjoint of pushforward as in Lemma 3.1. For $K, L \in D_{Q\text{Coh}}(\mathcal{O}_Y)$ there is a canonical map

$$Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(L) \longrightarrow a(K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} L)$$

Namely, this map is adjoint to a map

$$Rf_*(Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(L)) = K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*(a(L)) \longrightarrow K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} L$$

(equality by Derived Categories of Schemes, Lemma 22.1) for which we use the trace map $Rf_*a(L) \rightarrow L$. When $L = \mathcal{O}_Y$ we obtain a map

0A9S (8.0.1)
$$Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(\mathcal{O}_Y) \longrightarrow a(K)$$

functorial in K and compatible with distinguished triangles.

0A9T **Lemma 8.1.** *Let $f : X \rightarrow Y$ be a morphism of quasi-compact and quasi-separated schemes. The map $Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(L) \rightarrow a(K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} L)$ defined above for $K, L \in D_{QCoh}(\mathcal{O}_Y)$ is an isomorphism if K is perfect. In particular, (8.0.1) is an isomorphism if K is perfect.*

Proof. Let $K^\vee$ be the “dual” to K , see Cohomology, Lemma 50.5. For $M \in D_{QCoh}(\mathcal{O}_X)$ we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M, K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} L) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M \otimes_{\mathcal{O}_Y}^{\mathbf{L}} K^\vee, L) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(M \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*K^\vee, a(L)) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(M, Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(L)) \end{aligned}$$

Second equality by the definition of a and the projection formula (Cohomology, Lemma 54.3) or the more general Derived Categories of Schemes, Lemma 22.1. Hence the result by the Yoneda lemma. $\square$

0B6P **Lemma 8.2.** *Suppose we have a diagram (4.0.1) where f and g are tor independent. Let $K \in D_{QCoh}(\mathcal{O}_Y)$. The diagram*

$$\begin{array}{ccc} L(g')^*(Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(\mathcal{O}_Y)) & \longrightarrow & L(g')^*a(K) \\ \downarrow & & \downarrow \\ L(f')^*Lg^*K \otimes_{\mathcal{O}_{X'}}^{\mathbf{L}} a'(\mathcal{O}_{Y'}) & \longrightarrow & a'(Lg^*K) \end{array}$$

*commutes where the horizontal arrows are the maps (8.0.1) for K and Lg^*K and the vertical maps are constructed using Cohomology, Remark 28.3 and (5.0.1).*

Proof. In this proof we will write f_* for Rf_* and f^* for Lf^* , etc, and we will write $\otimes$ for $\otimes_{\mathcal{O}_X}^{\mathbf{L}}$, etc. Let us write (8.0.1) as the composition

$$\begin{aligned} f^*K \otimes a(\mathcal{O}_Y) &\rightarrow a(f_*(f^*K \otimes a(\mathcal{O}_Y))) \\ &\leftarrow a(K \otimes f_*a(\mathcal{O}_K)) \\ &\rightarrow a(K \otimes \mathcal{O}_Y) \\ &\rightarrow a(K) \end{aligned}$$

Here the first arrow is the unit η_f , the second arrow is a applied to Cohomology, Equation (54.2.1) which is an isomorphism by Derived Categories of Schemes, Lemma 22.1, the third arrow is a applied to $\mathrm{id}_K \otimes \mathrm{Tr}_f$, and the fourth arrow is a applied to the isomorphism $K \otimes \mathcal{O}_Y = K$. The proof of the lemma consists in showing that each of these maps gives rise to a commutative square as in the statement of the lemma. For η_f and Tr_f this is Lemmas 7.2 and 7.1. For the arrow using Cohomology, Equation (54.2.1) this is Cohomology, Remark 54.5. For the multiplication map it is clear. This finishes the proof. $\square$

0B6Q **Lemma 8.3.** *Let $f : X \rightarrow Y$ be a proper morphism of Noetherian schemes. Let $V \subset Y$ be an open such that $f^{-1}(V) \rightarrow V$ is an isomorphism. Then for $K \in D_{QCoh}^+(\mathcal{O}_Y)$ the map (8.0.1) restricts to an isomorphism over $f^{-1}(V)$.*

Proof. By Lemma 4.4 the map (4.1.1) is an isomorphism for objects of $D_{QCoh}^+(\mathcal{O}_Y)$. Hence Lemma 8.2 tells us the restriction of (8.0.1) for K to $f^{-1}(V)$ is the map (8.0.1)

for $K|_V$ and $f^{-1}(V) \rightarrow V$. Thus it suffices to show that the map is an isomorphism when f is the identity morphism. This is clear. $\square$

0B6R **Lemma 8.4.** *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be composable morphisms of quasi-compact and quasi-separated schemes and set $h = g \circ f$. Let a, b, c be the adjoints of Lemma 3.1 for f, g, h . For any $K \in D_{QCoh}(\mathcal{O}_Z)$ the diagram*

$$\begin{array}{ccccc} Lf^*(Lg^*K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} b(\mathcal{O}_Z)) \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(\mathcal{O}_Y) & \longrightarrow & a(Lg^*K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} b(\mathcal{O}_Z)) & \longrightarrow & a(b(K)) \\ \parallel & & & & \parallel \\ Lh^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*b(\mathcal{O}_Z) \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(\mathcal{O}_Y) & \longrightarrow & Lh^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} c(\mathcal{O}_Z) & \longrightarrow & c(K) \end{array}$$

is commutative where the arrows are (8.0.1) and we have used $Lh^* = Lf^* \circ Lg^*$ and $c = a \circ b$.

Proof. In this proof we will write f_* for Rf_* and f^* for Lf^* , etc, and we will write $\otimes$ for $\otimes_{\mathcal{O}_X}^{\mathbf{L}}$, etc. The composition of the top arrows is adjoint to a map

$$g_*f_*(f^*(g^*K \otimes b(\mathcal{O}_Z)) \otimes a(\mathcal{O}_Y)) \rightarrow K$$

The left hand side is equal to $K \otimes g_*f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y))$ by Derived Categories of Schemes, Lemma 22.1 and inspection of the definitions shows the map comes from the map

$$g_*f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)) \xleftarrow{g_*\epsilon} g_*(b(\mathcal{O}_Z) \otimes f_*a(\mathcal{O}_Y)) \xrightarrow{g_*\alpha} g_*(b(\mathcal{O}_Z)) \xrightarrow{\beta} \mathcal{O}_Z$$

tensored with id_K . Here ϵ is the isomorphism from Derived Categories of Schemes, Lemma 22.1 and β comes from the counit map $g_*b \rightarrow \text{id}$. Similarly, the composition of the lower horizontal arrows is adjoint to id_K tensored with the composition

$$g_*f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)) \xrightarrow{g_*f_*\delta} g_*f_*(ab(\mathcal{O}_Z)) \xrightarrow{g_*\gamma} g_*(b(\mathcal{O}_Z)) \xrightarrow{\beta} \mathcal{O}_Z$$

where γ comes from the counit map $f_*a \rightarrow \text{id}$ and δ is the map whose adjoint is the composition

$$f_*(f^*b(\mathcal{O}_Z) \otimes a(\mathcal{O}_Y)) \xleftarrow{\epsilon} b(\mathcal{O}_Z) \otimes f_*a(\mathcal{O}_Y) \xrightarrow{\alpha} b(\mathcal{O}_Z)$$

By general properties of adjoint functors, adjoint maps, and counits (see Categories, Section 24) we have $\gamma \circ f_*\delta = \alpha \circ \epsilon^{-1}$ as desired. $\square$

9. Right adjoint of pushforward for closed immersions

0A74 Let $i : (Z, \mathcal{O}_Z) \rightarrow (X, \mathcal{O}_X)$ be a morphism of ringed spaces such that i is a homeomorphism onto a closed subset and such that $i^\sharp : \mathcal{O}_X \rightarrow i_*\mathcal{O}_Z$ is surjective. (For example a closed immersion of schemes.) Let $\mathcal{I} = \text{Ker}(i^\sharp)$. For a sheaf of $\mathcal{O}_X$ -modules $\mathcal{F}$ the sheaf

$$\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})$$

is a sheaf of $\mathcal{O}_X$ -modules annihilated by $\mathcal{I}$. Hence by Modules, Lemma 13.4 there is a sheaf of $\mathcal{O}_Z$ -modules, which we will denote $\mathcal{H}om(\mathcal{O}_Z, \mathcal{F})$, such that

$$i_*\mathcal{H}om(\mathcal{O}_Z, \mathcal{F}) = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})$$

as $\mathcal{O}_X$ -modules. We spell out what this means.

0A75 **Lemma 9.1.** *With notation as above. The functor $\mathcal{H}om(\mathcal{O}_Z, -)$ is a right adjoint to the functor $i_* : \text{Mod}(\mathcal{O}_Z) \rightarrow \text{Mod}(\mathcal{O}_X)$. For $V \subset Z$ open we have*

$$\Gamma(V, \mathcal{H}om(\mathcal{O}_Z, \mathcal{F})) = \{s \in \Gamma(U, \mathcal{F}) \mid \mathcal{I}s = 0\}$$

where $U \subset X$ is an open whose intersection with Z is V .

Proof. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Z$ -modules. Then

$$\text{Hom}_{\mathcal{O}_X}(i_*\mathcal{G}, \mathcal{F}) = \text{Hom}_{i_*\mathcal{O}_Z}(i_*\mathcal{G}, \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})) = \text{Hom}_{\mathcal{O}_Z}(\mathcal{G}, \mathcal{H}om(\mathcal{O}_Z, \mathcal{F}))$$

The first equality by Modules, Lemma 22.3 and the second by the fully faithfulness of i_* , see Modules, Lemma 13.4. The description of sections is left to the reader. $\square$

The functor

$$\text{Mod}(\mathcal{O}_X) \longrightarrow \text{Mod}(\mathcal{O}_Z), \quad \mathcal{F} \longmapsto \mathcal{H}om(\mathcal{O}_Z, \mathcal{F})$$

is left exact and has a derived extension

$$R\mathcal{H}om(\mathcal{O}_Z, -) : D(\mathcal{O}_X) \rightarrow D(\mathcal{O}_Z).$$

0A76 **Lemma 9.2.** *With notation as above. The functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ is the right adjoint of the functor $Ri_* : D(\mathcal{O}_Z) \rightarrow D(\mathcal{O}_X)$.*

Proof. This is a consequence of the fact that i_* and $\mathcal{H}om(\mathcal{O}_Z, -)$ are adjoint functors by Lemma 9.1. See Derived Categories, Lemma 30.3. $\square$

0A77 **Lemma 9.3.** *With notation as above. We have*

$$Ri_* R\mathcal{H}om(\mathcal{O}_Z, K) = R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, K)$$

in $D(\mathcal{O}_X)$ for all K in $D(\mathcal{O}_Z)$.

Proof. This is immediate from the construction of the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$. $\square$

0E2I **Lemma 9.4.** *With notation as above. For $M \in D(\mathcal{O}_Z)$ we have*

$$R\mathcal{H}om_{\mathcal{O}_X}(Ri_* M, K) = Ri_* R\mathcal{H}om_{\mathcal{O}_Z}(M, R\mathcal{H}om(\mathcal{O}_Z, K))$$

in $D(\mathcal{O}_Z)$ for all K in $D(\mathcal{O}_X)$.

Proof. This is immediate from the construction of the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ and the fact that if $\mathcal{K}^\bullet$ is a K-injective complex of $\mathcal{O}_X$ -modules, then $\mathcal{H}om(\mathcal{O}_Z, \mathcal{K}^\bullet)$ is a K-injective complex of $\mathcal{O}_Z$ -modules, see Derived Categories, Lemma 31.9. $\square$

0A78 **Lemma 9.5.** *Let $i : Z \rightarrow X$ be a pseudo-coherent closed immersion of schemes (any closed immersion if X is locally Noetherian). Then*

- (1) $R\mathcal{H}om(\mathcal{O}_Z, -)$ maps $D_{Q\text{Coh}}^+(\mathcal{O}_X)$ into $D_{Q\text{Coh}}^+(\mathcal{O}_Z)$, and
- (2) if $X = \text{Spec}(A)$ and $Z = \text{Spec}(B)$, then the diagram

$$\begin{array}{ccc} D^+(B) & \longrightarrow & D_{Q\text{Coh}}^+(\mathcal{O}_Z) \\ \uparrow R\mathcal{H}om(B, -) & & \uparrow R\mathcal{H}om(\mathcal{O}_Z, -) \\ D^+(A) & \longrightarrow & D_{Q\text{Coh}}^+(\mathcal{O}_X) \end{array}$$

is commutative.

Proof. To explain the parenthetical remark, if X is locally Noetherian, then i is pseudo-coherent by More on Morphisms, Lemma 60.9.

Let K be an object of $D_{Q\text{Coh}}^+(\mathcal{O}_X)$. To prove (1), by Morphisms, Lemma 4.1 it suffices to show that i_* applied to $H^n(R\mathcal{H}om(\mathcal{O}_Z, K))$ produces a quasi-coherent module on X . By Lemma 9.3 this means we have to show that $R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, K)$ is in $D_{Q\text{Coh}}(\mathcal{O}_X)$. Since i is pseudo-coherent the sheaf $\mathcal{O}_Z$ is a pseudo-coherent $\mathcal{O}_X$ -module. Hence the result follows from Derived Categories of Schemes, Lemma 10.8.

Assume $X = \text{Spec}(A)$ and $Z = \text{Spec}(B)$ as in (2). Let $I^\bullet$ be a bounded below complex of injective A -modules representing an object K of $D^+(A)$. Then we know that $R\text{Hom}(B, K) = \text{Hom}_A(B, I^\bullet)$ viewed as a complex of B -modules. Choose a quasi-isomorphism

$$\tilde{I}^\bullet \longrightarrow I^\bullet$$

where $I^\bullet$ is a bounded below complex of injective $\mathcal{O}_X$ -modules. It follows from the description of the functor $\mathcal{H}om(\mathcal{O}_Z, -)$ in Lemma 9.1 that there is a map

$$\text{Hom}_A(B, I^\bullet) \longrightarrow \Gamma(Z, \mathcal{H}om(\mathcal{O}_Z, I^\bullet))$$

Observe that $\mathcal{H}om(\mathcal{O}_Z, I^\bullet)$ represents $R\mathcal{H}om(\mathcal{O}_Z, \tilde{K})$. Applying the universal property of the $\sim$ functor we obtain a map

$$\widetilde{\text{Hom}_A(B, I^\bullet)} \longrightarrow R\mathcal{H}om(\mathcal{O}_Z, \tilde{K})$$

in $D(\mathcal{O}_Z)$. We may check that this map is an isomorphism in $D(\mathcal{O}_Z)$ after applying i_* . However, once we apply i_* we obtain the isomorphism of Derived Categories of Schemes, Lemma 10.8 via the identification of Lemma 9.3. $\square$

0A99 **Lemma 9.6.** *Let $i : Z \rightarrow X$ be a closed immersion of schemes. Assume X is a locally Noetherian. Then $R\mathcal{H}om(\mathcal{O}_Z, -)$ maps $D_{Q\text{Coh}}^+(\mathcal{O}_X)$ into $D_{Q\text{Coh}}^+(\mathcal{O}_Z)$.*

Proof. The question is local on X , hence we may assume that X is affine. Say $X = \text{Spec}(A)$ and $Z = \text{Spec}(B)$ with A Noetherian and $A \rightarrow B$ surjective. In this case, we can apply Lemma 9.5 to translate the question into algebra. The corresponding algebra result is a consequence of Dualizing Complexes, Lemma 13.4. $\square$

0A9X **Lemma 9.7.** *Let X be a quasi-compact and quasi-separated scheme. Let $i : Z \rightarrow X$ be a pseudo-coherent closed immersion (if X is Noetherian, then any closed immersion is pseudo-coherent). Let $a : D_{Q\text{Coh}}(\mathcal{O}_X) \rightarrow D_{Q\text{Coh}}(\mathcal{O}_Z)$ be the right adjoint to Ri_* . Then there is a functorial isomorphism*

$$a(K) = R\mathcal{H}om(\mathcal{O}_Z, K)$$

for $K \in D_{Q\text{Coh}}^+(\mathcal{O}_X)$.

Proof. (The parenthetical statement follows from More on Morphisms, Lemma 60.9.) By Lemma 9.2 the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ is a right adjoint to $Ri_* : D(\mathcal{O}_Z) \rightarrow D(\mathcal{O}_X)$. Moreover, by Lemma 9.5 and Lemma 3.5 both $R\mathcal{H}om(\mathcal{O}_Z, -)$ and a map $D_{Q\text{Coh}}^+(\mathcal{O}_X)$ into $D_{Q\text{Coh}}^+(\mathcal{O}_Z)$. Hence we obtain the isomorphism by uniqueness of adjoint functors. $\square$

0B6M **Example 9.8.** If $i : Z \rightarrow X$ is closed immersion of Noetherian schemes, then the diagram

$$\begin{array}{ccc} i_* a(K) & \xrightarrow{\quad \text{Tr}_{i,K} \quad} & K \\ \parallel & & \parallel \\ i_* R\mathcal{H}om(\mathcal{O}_Z, K) & \xlongequal{\quad} R\mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{O}_Z, K) \longrightarrow & K \end{array}$$

is commutative for $K \in D_{Q\text{Coh}}^+(\mathcal{O}_X)$. Here the horizontal equality sign is Lemma 9.3 and the lower horizontal arrow is induced by the map $\mathcal{O}_X \rightarrow i_* \mathcal{O}_Z$. The commutativity of the diagram is a consequence of Lemma 9.7.

10. Right adjoint of pushforward for closed immersions and base change

0E2J Consider a cartesian diagram of schemes

$$\begin{array}{ccc} Z' & \xrightarrow{\quad i' \quad} & X' \\ g \downarrow & & \downarrow f \\ Z & \xrightarrow{\quad i \quad} & X \end{array}$$

where i is a closed immersion. If Z and X' are tor independent over X , then there is a canonical base change map

0E2K (10.0.1) $Lg^* R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow R\mathcal{H}om(\mathcal{O}_{Z'}, Lf^* K)$

in $D(\mathcal{O}_{Z'})$ functorial for K in $D(\mathcal{O}_X)$. Namely, by adjointness of Lemma 9.2 such an arrow is the same thing as a map

$$Ri'_* Lg^* R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow Lf^* K$$

in $D(\mathcal{O}_{X'})$. By tor independence we have $Ri'_* \circ Lg^* = Lf^* \circ Ri_*$ (see Derived Categories of Schemes, Lemma 22.9). Thus this is the same thing as a map

$$Lf^* Ri_* R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow Lf^* K$$

For this we can use $Lf^*(can)$ where $can : Ri_* R\mathcal{H}om(\mathcal{O}_Z, K) \rightarrow K$ is the counit of the adjunction.

0E2L **Lemma 10.1.** *In the situation above, the map (10.0.1) is an isomorphism if and only if the base change map*

$$Lf^* R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, K) \longrightarrow R\mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{O}_{Z'}, Lf^* K)$$

of Cohomology, Remark 42.13 is an isomorphism.

Proof. The statement makes sense because $\mathcal{O}_{Z'} = Lf^* \mathcal{O}_Z$ by the assumed tor independence. Since i'_* is exact and faithful we see that it suffices to show the map (10.0.1) is an isomorphism after applying Ri'_* . Since $Ri'_* \circ Lg^* = Lf^* \circ Ri_*$ by the assumed tor independence and Derived Categories of Schemes, Lemma 22.9 we obtain a map

$$Lf^* Ri_* R\mathcal{H}om(\mathcal{O}_Z, K) \longrightarrow Ri'_* R\mathcal{H}om(\mathcal{O}_{Z'}, Lf^* K)$$

whose source and target are as in the statement of the lemma by Lemma 9.3. We omit the verification that this is the same map as the one constructed in Cohomology, Remark 42.13. $\square$

0E2M **Lemma 10.2.** *In the situation above, assume f is flat and i pseudo-coherent. Then (10.0.1) is an isomorphism for K in $D_{Q\text{Coh}}^+(\mathcal{O}_X)$.*

Proof. First proof. To prove this map is an isomorphism, we may work locally. Hence we may assume X, X', Z, Z' are affine, say corresponding to the rings A, A', B, B' . Then B and A' are tor independent over A . By Lemma 10.1 it suffices to check that

$$R\text{Hom}_A(B, K) \otimes_A^{\mathbf{L}} A' = R\text{Hom}_{A'}(B', K \otimes_A^{\mathbf{L}} A')$$

in $D(A')$ for all $K \in D^+(A)$. Here we use Derived Categories of Schemes, Lemma 10.8 and the fact that B , resp. B' is pseudo-coherent as an A -module, resp. A' -module to compare derived hom on the level of rings and schemes. The displayed equality follows from More on Algebra, Lemma 98.3 part (3). See also the discussion in Dualizing Complexes, Section 14.

Second proof⁴. Let $z' \in Z'$ with image $z \in Z$. First show that (10.0.1) on stalks at z' induces the map

$$R\text{Hom}(\mathcal{O}_{Z,z}, K_z) \otimes_{\mathcal{O}_{Z,x}}^{\mathbf{L}} \mathcal{O}_{Z',z'} \longrightarrow R\text{Hom}(\mathcal{O}_{Z',z'}, K_z \otimes_{\mathcal{O}_{X,z}}^{\mathbf{L}} \mathcal{O}_{X',z'})$$

from Dualizing Complexes, Equation (14.0.1). Namely, the constructions of these maps are identical. Then apply Dualizing Complexes, Lemma 14.2. $\square$

0E2N **Lemma 10.3.** *Let $i : Z \rightarrow X$ be a pseudo-coherent closed immersion of schemes. Let $M \in D_{Q\text{Coh}}(\mathcal{O}_X)$ locally have tor-amplitude in $[a, \infty)$. Let $K \in D_{Q\text{Coh}}^+(\mathcal{O}_X)$. Then there is a canonical isomorphism*

$$R\text{Hom}(\mathcal{O}_Z, K) \otimes_{\mathcal{O}_Z}^{\mathbf{L}} Li^* M = R\text{Hom}(\mathcal{O}_Z, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$$

in $D(\mathcal{O}_Z)$.

Proof. A map from LHS to RHS is the same thing as a map

$$Ri_* R\text{Hom}(\mathcal{O}_Z, K) \otimes_{\mathcal{O}_X}^{\mathbf{L}} M \longrightarrow K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M$$

by Lemmas 9.2 and 9.3. For this map we take the counit $Ri_* R\text{Hom}(\mathcal{O}_Z, K) \rightarrow K$ tensored with id_M . To see this map is an isomorphism under the hypotheses given, translate into algebra using Lemma 9.5 and then for example use More on Algebra, Lemma 98.3 part (3). Instead of using Lemma 9.5 you can look at stalks as in the second proof of Lemma 10.2. $\square$

11. Right adjoint of pushforward for finite morphisms

0AWZ If $i : Z \rightarrow X$ is a closed immersion of schemes, then there is a right adjoint $\text{Hom}(\mathcal{O}_Z, -)$ to the functor $i_* : \text{Mod}(\mathcal{O}_Z) \rightarrow \text{Mod}(\mathcal{O}_X)$ whose derived extension $R\text{Hom}(\mathcal{O}_Z, -)$ is the right adjoint to $Ri_* : D(\mathcal{O}_Z) \rightarrow D(\mathcal{O}_X)$. See Section 9. In the case of a finite morphism $f : Y \rightarrow X$ this strategy cannot work, as the functor $f_* : \text{Mod}(\mathcal{O}_Y) \rightarrow \text{Mod}(\mathcal{O}_X)$ is not exact in general and hence does not have a right adjoint. A replacement is to consider the exact functor $\text{Mod}(f_* \mathcal{O}_Y) \rightarrow \text{Mod}(\mathcal{O}_X)$ and consider the corresponding right adjoint and its derived extension.

Let $f : Y \rightarrow X$ be an affine morphism of schemes. For a sheaf of $\mathcal{O}_X$ -modules $\mathcal{F}$ the sheaf

$$\text{Hom}_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{F})$$

⁴This proof shows it suffices to assume K is in $D^+(\mathcal{O}_X)$.

is a sheaf of $f_*\mathcal{O}_Y$ -modules. We obtain a functor $\text{Mod}(\mathcal{O}_X) \rightarrow \text{Mod}(f_*\mathcal{O}_Y)$ which we will denote $\mathcal{H}om(f_*\mathcal{O}_Y, -)$.

0BUZ **Lemma 11.1.** *With notation as above. The functor $\mathcal{H}om(f_*\mathcal{O}_Y, -)$ is a right adjoint to the restriction functor $\text{Mod}(f_*\mathcal{O}_Y) \rightarrow \text{Mod}(\mathcal{O}_X)$. For an affine open $U \subset X$ we have*

$$\Gamma(U, \mathcal{H}om(f_*\mathcal{O}_Y, \mathcal{F})) = \text{Hom}_A(B, \mathcal{F}(U))$$

where $A = \mathcal{O}_X(U)$ and $B = \mathcal{O}_Y(f^{-1}(U))$.

Proof. Adjointness follows from Modules, Lemma 22.3. As f is affine we see that $f_*\mathcal{O}_Y$ is the quasi-coherent sheaf corresponding to B viewed as an A -module. Hence the description of sections over U follows from Schemes, Lemma 7.1. $\square$

The functor $\mathcal{H}om(f_*\mathcal{O}_Y, -)$ is left exact. Let

$$R\mathcal{H}om(f_*\mathcal{O}_Y, -) : D(\mathcal{O}_X) \longrightarrow D(f_*\mathcal{O}_Y)$$

be its derived extension.

0BV0 **Lemma 11.2.** *With notation as above. The functor $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ is the right adjoint of the functor $D(f_*\mathcal{O}_Y) \rightarrow D(\mathcal{O}_X)$.*

Proof. Follows from Lemma 11.1 and Derived Categories, Lemma 30.3. $\square$

0BV1 **Lemma 11.3.** *With notation as above. The composition*

$$D(\mathcal{O}_X) \xrightarrow{R\mathcal{H}om(f_*\mathcal{O}_Y, -)} D(f_*\mathcal{O}_Y) \rightarrow D(\mathcal{O}_X)$$

is the functor $K \mapsto R\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, K)$.

Proof. This is immediate from the construction. $\square$

0AX2 **Lemma 11.4.** *Let $f : Y \rightarrow X$ be a finite pseudo-coherent morphism of schemes (a finite morphism of Noetherian schemes is pseudo-coherent). The functor $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ maps $D_{QCoh}^+(\mathcal{O}_X)$ into $D_{QCoh}^+(f_*\mathcal{O}_Y)$. If X is quasi-compact and quasi-separated, then the diagram*

$$\begin{array}{ccc} D_{QCoh}^+(\mathcal{O}_X) & \xrightarrow{a} & D_{QCoh}^+(\mathcal{O}_Y) \\ & \searrow R\mathcal{H}om(f_*\mathcal{O}_Y, -) & \swarrow \Phi \\ & D_{QCoh}^+(f_*\mathcal{O}_Y) & \end{array}$$

is commutative, where a is the right adjoint of Lemma 3.1 for f and Φ is the equivalence of Derived Categories of Schemes, Lemma 5.4.

Proof. (The parenthetical remark follows from More on Morphisms, Lemma 60.9.) Since f is pseudo-coherent, the $\mathcal{O}_X$ -module $f_*\mathcal{O}_Y$ is pseudo-coherent, see More on Morphisms, Lemma 60.8. Thus $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ maps $D_{QCoh}^+(\mathcal{O}_X)$ into $D_{QCoh}^+(f_*\mathcal{O}_Y)$, see Derived Categories of Schemes, Lemma 10.8. Then $\Phi \circ a$ and $R\mathcal{H}om(f_*\mathcal{O}_Y, -)$ agree on $D_{QCoh}^+(\mathcal{O}_X)$ because these functors are both right adjoint to the restriction functor $D_{QCoh}^+(f_*\mathcal{O}_Y) \rightarrow D_{QCoh}^+(\mathcal{O}_X)$. To see this use Lemmas 3.5 and 11.2. $\square$

0AX3 **Remark 11.5.** If $f : Y \rightarrow X$ is a finite morphism of Noetherian schemes, then the diagram

$$\begin{array}{ccc} Rf_*a(K) & \xrightarrow{\text{Tr}_{f,K}} & K \\ \parallel & & \parallel \\ R\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, K) & \longrightarrow & K \end{array}$$

is commutative for $K \in D_{Q\text{Coh}}^+(\mathcal{O}_X)$. This follows from Lemma 11.4. The lower horizontal arrow is induced by the map $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ and the upper horizontal arrow is the trace map discussed in Section 7.

12. Right adjoint of pushforward for proper flat morphisms

0E4H For proper, flat, and finitely presented morphisms of quasi-compact and quasi-separated schemes the right adjoint of pushforward enjoys some remarkable properties.

0E4I **Lemma 12.1.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \rightarrow Y$ be a morphism of schemes which is proper, flat, and of finite presentation. Let a be the right adjoint for $Rf_* : D_{Q\text{Coh}}(\mathcal{O}_X) \rightarrow D_{Q\text{Coh}}(\mathcal{O}_Y)$ of Lemma 3.1. Then a commutes with direct sums.*

Proof. Let P be a perfect object of $D(\mathcal{O}_X)$. By Derived Categories of Schemes, Lemma 30.4 the complex Rf_*P is perfect on Y . Let K_i be a family of objects of $D_{Q\text{Coh}}(\mathcal{O}_Y)$. Then

$$\begin{aligned} \text{Hom}_{D(\mathcal{O}_X)}(P, a(\bigoplus K_i)) &= \text{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, \bigoplus K_i) \\ &= \bigoplus \text{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, K_i) \\ &= \bigoplus \text{Hom}_{D(\mathcal{O}_X)}(P, a(K_i)) \end{aligned}$$

because a perfect object is compact (Derived Categories of Schemes, Proposition 17.1). Since $D_{Q\text{Coh}}(\mathcal{O}_X)$ has a perfect generator (Derived Categories of Schemes, Theorem 15.3) we conclude that the map $\bigoplus a(K_i) \rightarrow a(\bigoplus K_i)$ is an isomorphism, i.e., a commutes with direct sums. $\square$

0E4J **Lemma 12.2.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \rightarrow Y$ be a morphism of schemes which is proper, flat, and of finite presentation. Let a be the right adjoint for $Rf_* : D_{Q\text{Coh}}(\mathcal{O}_X) \rightarrow D_{Q\text{Coh}}(\mathcal{O}_Y)$ of Lemma 3.1. Then*

- (1) *for every closed $T \subset Y$ if $Q \in D_{Q\text{Coh}}(Y)$ is supported on T , then $a(Q)$ is supported on $f^{-1}(T)$,*
- (2) *for every open $V \subset Y$ and any $K \in D_{Q\text{Coh}}(\mathcal{O}_Y)$ the map (4.1.1) is an isomorphism, and*

Proof. This follows from Lemmas 4.3, 4.4, and 12.1. $\square$

0E4K **Lemma 12.3.** *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \rightarrow Y$ be a morphism of schemes which is proper, flat, and of finite presentation. The map (8.0.1) is an isomorphism for every object K of $D_{Q\text{Coh}}(\mathcal{O}_Y)$.*

Proof. By Lemma 12.1 we know that a commutes with direct sums. Hence the collection of objects of $D_{Q\text{Coh}}(\mathcal{O}_Y)$ for which (8.0.1) is an isomorphism is a strictly

full, saturated, triangulated subcategory of $D_{Qcoh}(\mathcal{O}_Y)$ which is moreover preserved under taking direct sums. Since $D_{Qcoh}(\mathcal{O}_Y)$ is a module category (Derived Categories of Schemes, Theorem 18.2) generated by a single perfect object (Derived Categories of Schemes, Theorem 15.3) we can argue as in More on Algebra, Remark 59.11 to see that it suffices to prove (8.0.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 8.1. $\square$

The following lemma shows that the base change map (5.0.1) is an isomorphism for proper, flat morphisms of finite presentation. We will see in Example 15.2 that this does not remain true for perfect proper morphisms; in that case one has to make a tor independence condition.

0AAB **Lemma 12.4.** *Let $g : Y' \rightarrow Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $f : X \rightarrow Y$ be a proper, flat morphism of finite presentation. Then the base change map (5.0.1) is an isomorphism for all $K \in D_{Qcoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 12.2 formation of the functors a and a' commutes with restriction to opens of Y and Y' . Hence we may assume $Y' \rightarrow Y$ is a morphism of affine schemes, see Remark 6.1. In this case the statement follows from Lemma 6.2. $\square$

0B6S **Remark 12.5.** Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \rightarrow Y$ be a proper, flat morphism of finite presentation. Let a be the adjoint of Lemma 3.1 for f . In this situation, $\omega_{X/Y}^\bullet = a(\mathcal{O}_Y)$ is sometimes called the *relative dualizing complex*. By Lemma 12.3 there is a functorial isomorphism $a(K) = Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}^\bullet$ for $K \in D_{Qcoh}(\mathcal{O}_Y)$. Moreover, the trace map

$$\mathrm{Tr}_{f, \mathcal{O}_Y} : Rf_* \omega_{X/Y}^\bullet \rightarrow \mathcal{O}_Y$$

of Section 7 induces the trace map for all K in $D_{Qcoh}(\mathcal{O}_Y)$. More precisely the diagram

$$\begin{array}{ccc} Rf_* a(K) & \xrightarrow{\mathrm{Tr}_{f, K}} & K \\ \parallel & & \parallel \\ Rf_*(Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}^\bullet) & \xlongequal{\quad} & K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_* \omega_{X/Y}^\bullet \xrightarrow{\mathrm{id}_K \otimes \mathrm{Tr}_{f, \mathcal{O}_Y}} K \end{array}$$

where the equality on the lower right is Derived Categories of Schemes, Lemma 22.1. If $g : Y' \rightarrow Y$ is a morphism of quasi-compact and quasi-separated schemes and $X' = Y' \times_Y X$, then by Lemma 12.4 we have $\omega_{X'/Y'}^\bullet = L(g')^* \omega_{X/Y}^\bullet$ where $g' : X' \rightarrow X$ is the projection and by Lemma 7.1 the trace map

$$\mathrm{Tr}_{f', \mathcal{O}_{Y'}} : Rf'_* \omega_{X'/Y'}^\bullet \rightarrow \mathcal{O}_{Y'}$$

for $f' : X' \rightarrow Y'$ is the base change of $\mathrm{Tr}_{f, \mathcal{O}_Y}$ via the base change isomorphism.

0G81 **Remark 12.6.** Let $f : X \rightarrow Y$, $\omega_{X/Y}^\bullet$, and $\mathrm{Tr}_{f, \mathcal{O}_Y}$ be as in Remark 12.5. Let K and M be in $D_{Qcoh}(\mathcal{O}_X)$ with M pseudo-coherent (for example perfect). Suppose given a map $K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M \rightarrow \omega_{X/Y}^\bullet$ which corresponds to an isomorphism $K \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(M, \omega_{X/Y}^\bullet)$ via Cohomology, Equation (42.0.1). Then the relative cup product (Cohomology, Remark 28.7)

$$Rf_* K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_* M \rightarrow Rf_*(K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M) \rightarrow Rf_* \omega_{X/Y}^\bullet \xrightarrow{\mathrm{Tr}_{f, \mathcal{O}_Y}} \mathcal{O}_Y$$

determines an isomorphism $Rf_*K \rightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*M, \mathcal{O}_Y)$. Namely, since $\omega_{X/Y}^\bullet = a(\mathcal{O}_Y)$ the canonical map (3.5.1)

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(M, \omega_{X/Y}^\bullet) \rightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*M, \mathcal{O}_Y)$$

is an isomorphism by Lemma 3.6 and Remark 3.8 and the fact that M and Rf_*M are pseudo-coherent, see Derived Categories of Schemes, Lemma 30.5. To see that the relative cup product induces this isomorphism use the commutativity of the diagram in Cohomology, Remark 42.12.

Lemma 12.7. *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \rightarrow Y$ be a morphism of schemes which is proper, flat, and of finite presentation with relative dualizing complex $\omega_{X/Y}^\bullet$ (Remark 12.5). Then*

- (1) $\omega_{X/Y}^\bullet$ is a Y -perfect object of $D(\mathcal{O}_X)$,
- (2) $Rf_*\omega_{X/Y}^\bullet$ has vanishing cohomology sheaves in positive degrees,
- (3) $\mathcal{O}_X \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)$ is an isomorphism.

Proof. In view of the fact that formation of $\omega_{X/Y}^\bullet$ commutes with base change (see Remark 12.5), we may and do assume that Y is affine. For a perfect object E of $D(\mathcal{O}_X)$ we have

$$\begin{aligned} Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}^\bullet) &= Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega_{X/Y}^\bullet) \\ &= R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y) \\ &= (Rf_*E^\vee)^\vee \end{aligned}$$

For the first equality, see Cohomology, Lemma 50.5. For the second equality, see Lemma 3.6, Remark 3.8, and Derived Categories of Schemes, Lemma 30.4. The third equality is the definition of the dual. In particular these references also show that the outcome is a perfect object of $D(\mathcal{O}_Y)$. We conclude that $\omega_{X/Y}^\bullet$ is Y -perfect by More on Morphisms, Lemma 69.6. This proves (1).

Let M be an object of $D_{Qcoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned} \mathrm{Hom}_Y(M, Rf_*\omega_{X/Y}^\bullet) &= \mathrm{Hom}_X(Lf^*M, \omega_{X/Y}^\bullet) \\ &= \mathrm{Hom}_Y(Rf_*Lf^*M, \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(M \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*\mathcal{O}_X, \mathcal{O}_Y) \end{aligned}$$

The first equality holds by Cohomology, Lemma 28.1. The second equality by construction of a . The third equality by Derived Categories of Schemes, Lemma 22.1. Recall $Rf_*\mathcal{O}_X$ is perfect of tor amplitude in $[0, N]$ for some N , see Derived Categories of Schemes, Lemma 30.4. Thus we can represent $Rf_*\mathcal{O}_X$ by a complex of finite projective modules sitting in degrees $[0, N]$ (using More on Algebra, Lemma 74.2 and the fact that Y is affine). Hence if $M = \mathcal{O}_Y[-i]$ for some $i > 0$, then the last group is zero. Since Y is affine we conclude that $H^i(Rf_*\omega_{X/Y}^\bullet) = 0$ for $i > 0$. This proves (2).

Let E be a perfect object of $D_{QCoh}(\mathcal{O}_X)$. Then we have

$$\begin{aligned}
\mathrm{Hom}_X(E, R\mathcal{H}om_{\mathcal{O}_X}(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)) &= \mathrm{Hom}_X(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet) \\
&= \mathrm{Hom}_Y(Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}^\bullet), \mathcal{O}_Y) \\
&= \mathrm{Hom}_Y(Rf_*(R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega_{X/Y}^\bullet)), \mathcal{O}_Y) \\
&= \mathrm{Hom}_Y(R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y), \mathcal{O}_Y) \\
&= R\Gamma(Y, Rf_*E^\vee) \\
&= \mathrm{Hom}_X(E, \mathcal{O}_X)
\end{aligned}$$

The first equality holds by Cohomology, Lemma 42.2. The second equality is the definition of $\omega_{X/Y}^\bullet$. The third equality comes from the construction of the dual perfect complex $E^\vee$, see Cohomology, Lemma 50.5. The fourth equality follows from the equality $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, \omega_{X/Y}^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y)$ shown in the first paragraph of the proof. The fifth equality holds by double duality for perfect complexes (Cohomology, Lemma 50.5) and the fact that Rf_*E is perfect by Derived Categories of Schemes, Lemma 30.4. The last equality is Leray for f . This string of equalities essentially shows (3) holds by the Yoneda lemma. Namely, the object $R\mathcal{H}om(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)$ is in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 10.8. Taking $E = \mathcal{O}_X$ in the above we get a map $\alpha : \mathcal{O}_X \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)$ corresponding to $\mathrm{id}_{\mathcal{O}_X} \in \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X)$. Since all the isomorphisms above are functorial in E we see that the cone on α is an object C of $D_{QCoh}(\mathcal{O}_X)$ such that $\mathrm{Hom}(E, C) = 0$ for all perfect E . Since the perfect objects generate (Derived Categories of Schemes, Theorem 15.3) we conclude that α is an isomorphism. $\square$

0E2P **Lemma 12.8** (Rigidity). *Let Y be a quasi-compact and quasi-separated scheme. Let $f : X \rightarrow Y$ be a proper, flat morphism of finite presentation with relative dualizing complex $\omega_{X/Y}^\bullet$ (Remark 12.5). There is a canonical isomorphism*

$$0E2Q \quad (12.8.1) \quad \mathcal{O}_X = c(Lpr_1^* \omega_{X/Y}^\bullet) = c(Lpr_2^* \omega_{X/Y}^\bullet)$$

and a canonical isomorphism

$$0E2R \quad (12.8.2) \quad \omega_{X/Y}^\bullet = c\left(Lpr_1^* \omega_{X/Y}^\bullet \otimes_{\mathcal{O}_{X \times_Y X}}^{\mathbf{L}} Lpr_2^* \omega_{X/Y}^\bullet\right)$$

where c is the right adjoint of Lemma 3.1 for the diagonal $\Delta : X \rightarrow X \times_Y X$.

Proof. Let a be the right adjoint to Rf_* as in Lemma 3.1. Consider the cartesian square

$$\begin{array}{ccc}
X \times_Y X & \xrightarrow{q} & X \\
p \downarrow & & \downarrow f \\
X & \xrightarrow{f} & Y
\end{array}$$

Let b be the right adjoint for p as in Lemma 3.1. Then

$$\begin{aligned}
\omega_{X/Y}^\bullet &= c(b(\omega_{X/Y}^\bullet)) \\
&= c(Lp^* \omega_{X/Y}^\bullet \otimes_{\mathcal{O}_{X \times_Y X}}^{\mathbf{L}} b(\mathcal{O}_X)) \\
&= c(Lp^* \omega_{X/Y}^\bullet \otimes_{\mathcal{O}_{X \times_Y X}}^{\mathbf{L}} Lq^* a(\mathcal{O}_Y)) \\
&= c(Lp^* \omega_{X/Y}^\bullet \otimes_{\mathcal{O}_{X \times_Y X}}^{\mathbf{L}} Lq^* \omega_{X/Y}^\bullet)
\end{aligned}$$

as in (12.8.2). Explanation as follows:

- (1) The first equality holds as $\text{id} = c \circ b$ because $\text{id}_X = p \circ \Delta$.
- (2) The second equality holds by Lemma 12.3.
- (3) The third holds by Lemma 12.4 and the fact that $\mathcal{O}_X = Lf^*\mathcal{O}_Y$.
- (4) The fourth holds because $\omega_{X/Y}^\bullet = a(\mathcal{O}_Y)$.

Equation (12.8.1) is proved in exactly the same way. $\square$

0BRU **Remark 12.9.** Lemma 12.8 means our relative dualizing complex is *rigid* in a sense analogous to the notion introduced in [vdB97]. Namely, since the functor on the right of (12.8.2) is “quadratic” in $\omega_{X/Y}^\bullet$ and the functor on the left of (12.8.2) is “linear” this “pins down” the complex $\omega_{X/Y}^\bullet$ to some extent. There is an approach to duality theory using “rigid” (relative) dualizing complexes, see for example [Nee11], [Yek10], and [YZ09]. We will return to this in Section 28.

13. Right adjoint of pushforward for perfect proper morphisms

0AA9 The correct generality for this section would be to consider perfect proper morphisms of quasi-compact and quasi-separated schemes, see [LN07].

0A9R **Lemma 13.1.** *Let $f : X \rightarrow Y$ be a perfect proper morphism of Noetherian schemes. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ of Lemma 3.1. Then a commutes with direct sums.*

Proof. Let P be a perfect object of $D(\mathcal{O}_X)$. By More on Morphisms, Lemma 61.13 the complex Rf_*P is perfect on Y . Let K_i be a family of objects of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned} \text{Hom}_{D(\mathcal{O}_X)}(P, a(\bigoplus K_i)) &= \text{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, \bigoplus K_i) \\ &= \bigoplus \text{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, K_i) \\ &= \bigoplus \text{Hom}_{D(\mathcal{O}_X)}(P, a(K_i)) \end{aligned}$$

because a perfect object is compact (Derived Categories of Schemes, Proposition 17.1). Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Derived Categories of Schemes, Theorem 15.3) we conclude that the map $\bigoplus a(K_i) \rightarrow a(\bigoplus K_i)$ is an isomorphism, i.e., a commutes with direct sums. $\square$

0AAA **Lemma 13.2.** *Let $f : X \rightarrow Y$ be a perfect proper morphism of Noetherian schemes. Let a be the right adjoint for $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ of Lemma 3.1. Then*

- (1) *for every closed $T \subset Y$ if $Q \in D_{QCoh}(Y)$ is supported on T , then $a(Q)$ is supported on $f^{-1}(T)$,*
- (2) *for every open $V \subset Y$ and any $K \in D_{QCoh}(\mathcal{O}_Y)$ the map (4.1.1) is an isomorphism, and*

Proof. This follows from Lemmas 4.3, 4.4, and 13.1. $\square$

0A9U **Lemma 13.3.** *Let $f : X \rightarrow Y$ be a perfect proper morphism of Noetherian schemes. The map (8.0.1) is an isomorphism for every object K of $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 13.1 we know that a commutes with direct sums. Hence the collection of objects of $D_{QCoh}(\mathcal{O}_Y)$ for which (8.0.1) is an isomorphism is a strictly full, saturated, triangulated subcategory of $D_{QCoh}(\mathcal{O}_Y)$ which is moreover preserved under taking direct sums. Since $D_{QCoh}(\mathcal{O}_Y)$ is a module category (Derived

Categories of Schemes, Theorem 18.2) generated by a single perfect object (Derived Categories of Schemes, Theorem 15.3) we can argue as in More on Algebra, Remark 59.11 to see that it suffices to prove (8.0.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 8.1. $\square$

0BZG **Lemma 13.4.** *Let $f : X \rightarrow Y$ be a perfect proper morphism of Noetherian schemes. Let $g : Y' \rightarrow Y$ be a morphism with Y' Noetherian. If X and Y' are tor independent over Y , then the base change map (5.0.1) is an isomorphism for all $K \in D_{Qcoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 13.2 formation of the functors a and a' commutes with restriction to opens of Y and Y' . Hence we may assume $Y' \rightarrow Y$ is a morphism of affine schemes, see Remark 6.1. In this case the statement follows from Lemma 6.2. $\square$

14. Right adjoint of pushforward for effective Cartier divisors

0B4A Let X be a scheme and let $i : D \rightarrow X$ be the inclusion of an effective Cartier divisor. Denote $\mathcal{N} = i^*\mathcal{O}_X(D)$ the normal sheaf of i , see Morphisms, Section 31 and Divisors, Section 13. Recall that $R\mathcal{H}om(\mathcal{O}_D, -)$ denotes the right adjoint to $i_* : D(\mathcal{O}_D) \rightarrow D(\mathcal{O}_X)$ and has the property $i_*R\mathcal{H}om(\mathcal{O}_D, -) = R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, -)$, see Section 9.

0B4B **Lemma 14.1.** *As above, let X be a scheme and let $D \subset X$ be an effective Cartier divisor. There is a canonical isomorphism $R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) = \mathcal{N}[-1]$ in $D(\mathcal{O}_D)$.*

Proof. Equivalently, we are saying that $R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)$ has a unique nonzero cohomology sheaf in degree 1 and that this sheaf is isomorphic to $\mathcal{N}$. Since i_* is exact and fully faithful, it suffices to prove that $i_*R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)$ is isomorphic to $i_*\mathcal{N}[-1]$. We have $i_*R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X)$ by Lemma 9.3. We have a resolution

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \rightarrow i_*\mathcal{O}_D \rightarrow 0$$

where $\mathcal{I}$ is the ideal sheaf of D which we can use to compute. Since $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$ and $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_X) = \mathcal{O}_X(D)$ by a local computation, we see that

$$R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) = (\mathcal{O}_X \rightarrow \mathcal{O}_X(D))$$

where on the right hand side we have $\mathcal{O}_X$ in degree 0 and $\mathcal{O}_X(D)$ in degree 1. The result follows from the short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(D) \rightarrow i_*\mathcal{N} \rightarrow 0$$

coming from the fact that D is the zero scheme of the canonical section of $\mathcal{O}_X(D)$ and from the fact that $\mathcal{N} = i^*\mathcal{O}_X(D)$. $\square$

For every object K of $D(\mathcal{O}_X)$ there is a canonical map

0B4C (14.1.1)
$$Li^*K \otimes_{\mathcal{O}_D}^{\mathbf{L}} R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) \longrightarrow R\mathcal{H}om(\mathcal{O}_D, K)$$

in $D(\mathcal{O}_D)$ functorial in K and compatible with distinguished triangles. Namely, this map is adjoint to a map

$$i_*(Li^*K \otimes_{\mathcal{O}_D}^{\mathbf{L}} R\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X)) = K \otimes_{\mathcal{O}_X}^{\mathbf{L}} R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) \longrightarrow K$$

where the equality is Cohomology, Lemma 54.4 and the arrow comes from the canonical map $R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) \rightarrow \mathcal{O}_X$ induced by $\mathcal{O}_X \rightarrow i_*\mathcal{O}_D$.

If $K \in D_{Q\text{Coh}}(\mathcal{O}_X)$, then (14.1.1) is equal to (8.0.1) via the identification $a(K) = R\mathcal{H}om(\mathcal{O}_D, K)$ of Lemma 9.7. If $K \in D_{Q\text{Coh}}(\mathcal{O}_X)$ and X is Noetherian, then the following lemma is a special case of Lemma 13.3.

0AA4 **Lemma 14.2.** *As above, let X be a scheme and let $D \subset X$ be an effective Cartier divisor. Then (14.1.1) combined with Lemma 14.1 defines an isomorphism*

$$Li^*K \otimes_{\mathcal{O}_D}^{\mathbf{L}} \mathcal{N}[-1] \longrightarrow R\mathcal{H}om(\mathcal{O}_D, K)$$

functorial in K in $D(\mathcal{O}_X)$.

Proof. Since i_* is exact and fully faithful on modules, to prove the map is an isomorphism, it suffices to show that it is an isomorphism after applying i_* . We will use the short exact sequences $0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \rightarrow i_*\mathcal{O}_D \rightarrow 0$ and $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(D) \rightarrow i_*\mathcal{N} \rightarrow 0$ used in the proof of Lemma 14.1 without further mention. By Cohomology, Lemma 54.4 which was used to define the map (14.1.1) the left hand side becomes

$$K \otimes_{\mathcal{O}_X}^{\mathbf{L}} i_*\mathcal{N}[-1] = K \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\mathcal{O}_X \rightarrow \mathcal{O}_X(D))$$

The right hand side becomes

$$\begin{aligned} R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_D, K) &= R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \rightarrow \mathcal{O}_X), K) \\ &= R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \rightarrow \mathcal{O}_X), \mathcal{O}_X) \otimes_{\mathcal{O}_X}^{\mathbf{L}} K \end{aligned}$$

the final equality by Cohomology, Lemma 50.5. Since the map comes from the isomorphism

$$R\mathcal{H}om_{\mathcal{O}_X}((\mathcal{I} \rightarrow \mathcal{O}_X), \mathcal{O}_X) = (\mathcal{O}_X \rightarrow \mathcal{O}_X(D))$$

the lemma is clear. $\square$

15. Right adjoint of pushforward in examples

0BQV In this section we compute the right adjoint to pushforward in some examples. The isomorphisms are canonical but only in the weakest possible sense, i.e., we do not prove or claim that these isomorphisms are compatible with various operations such as base change and compositions of morphisms. There is a huge literature on these types of issues; the reader can start with the material in [Har66], [Con00] (these citations use a different starting point for duality but address the issue of constructing canonical representatives for relative dualizing complexes) and then continue looking at works by Joseph Lipman and collaborators.

0A9W **Lemma 15.1.** *Let Y be a quasi-compact and quasi-separated scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_Y$ -module of rank $n + 1$ with determinant $\mathcal{L} = \wedge^{n+1}(\mathcal{E})$. Let $f : X = \mathbf{P}(\mathcal{E}) \rightarrow Y$ be the projection. Let a be the right adjoint for $Rf_* : D_{Q\text{Coh}}(\mathcal{O}_X) \rightarrow D_{Q\text{Coh}}(\mathcal{O}_Y)$ of Lemma 3.1. Then there is an isomorphism*

$$c : f^*\mathcal{L}(-n-1)[n] \longrightarrow a(\mathcal{O}_Y)$$

In particular, if $\mathcal{E} = \mathcal{O}_Y^{\oplus n+1}$, then $X = \mathbf{P}_Y^n$ and we obtain $a(\mathcal{O}_Y) = \mathcal{O}_X(-n-1)[n]$.

Proof. In (the proof of) Cohomology of Schemes, Lemma 8.4 we constructed a canonical isomorphism

$$R^n f_*(f^*\mathcal{L}(-n-1)) \longrightarrow \mathcal{O}_Y$$

Moreover, $Rf_*(f^*\mathcal{L}(-n-1)[n]) = R^n f_*(f^*\mathcal{L}(-n-1))$, i.e., the other higher direct images are zero. Thus we find an isomorphism

$$Rf_*(f^*\mathcal{L}(-n-1)[n]) \longrightarrow \mathcal{O}_Y$$

This isomorphism determines c as in the statement of the lemma because a is the right adjoint of Rf_* . By Lemma 4.4 construction of the a is local on the base. In particular, to check that c is an isomorphism, we may work locally on Y . In other words, we may assume Y is affine and $\mathcal{E} = \mathcal{O}_Y^{\oplus n+1}$. In this case the sheaves $\mathcal{O}_X, \mathcal{O}_X(-1), \dots, \mathcal{O}_X(-n)$ generate $D_{Qcoh}(X)$, see Derived Categories of Schemes, Lemma 16.3. Hence it suffices to show that $c : \mathcal{O}_X(-n-1)[n] \rightarrow a(\mathcal{O}_Y)$ is transformed into an isomorphism under the functors

$$F_{i,p}(-) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), (-)[p])$$

for $i \in \{-n, \dots, 0\}$ and $p \in \mathbf{Z}$. For $F_{0,p}$ this holds by construction of the arrow c ! For $i \in \{-n, \dots, -1\}$ we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), \mathcal{O}_X(-n-1)[n+p]) = H^p(X, \mathcal{O}_X(-n-1-i)) = 0$$

by the computation of cohomology of projective space (Cohomology of Schemes, Lemma 8.1) and we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(i), a(\mathcal{O}_Y)[p]) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*\mathcal{O}_X(i), \mathcal{O}_Y[p]) = 0$$

because $Rf_*\mathcal{O}_X(i) = 0$ by the same lemma. Hence the source and the target of $F_{i,p}(c)$ vanish and $F_{i,p}(c)$ is necessarily an isomorphism. This finishes the proof. $\square$

0AAC Example 15.2. The base change map (5.0.1) is not an isomorphism if f is perfect proper and g is perfect. Let k be a field. Let $Y = \mathbf{A}_k^2$ and let $f : X \rightarrow Y$ be the blowup of Y in the origin. Denote $E \subset X$ the exceptional divisor. Then we can factor f as

$$X \xrightarrow{i} \mathbf{P}_Y^1 \xrightarrow{p} Y$$

This gives a factorization $a = c \circ b$ where a , b , and c are the right adjoints of Lemma 3.1 of Rf_* , Rp_* , and Ri_* . Denote $\mathcal{O}(n)$ the Serre twist of the structure sheaf on $\mathbf{P}_Y^1$ and denote $\mathcal{O}_X(n)$ its restriction to X . Note that $X \subset \mathbf{P}_Y^1$ is cut out by a degree one equation, hence $\mathcal{O}(X) = \mathcal{O}(1)$. By Lemma 15.1 we have $b(\mathcal{O}_Y) = \mathcal{O}(-2)[1]$. By Lemma 9.7 we have

$$a(\mathcal{O}_Y) = c(b(\mathcal{O}_Y)) = c(\mathcal{O}(-2)[1]) = R\mathcal{H}om(\mathcal{O}_X, \mathcal{O}(-2)[1]) = \mathcal{O}_X(-1)$$

Last equality by Lemma 14.2. Let $Y' = \mathrm{Spec}(k)$ be the origin in Y . The restriction of $a(\mathcal{O}_Y)$ to $X' = E = \mathbf{P}_k^1$ is an invertible sheaf of degree -1 placed in cohomological degree 0. But on the other hand, $a'(\mathcal{O}_{\mathrm{Spec}(k)}) = \mathcal{O}_E(-2)[1]$ which is an invertible sheaf of degree -2 placed in cohomological degree -1 , so different. In this example the hypothesis of Tor independence in Lemma 6.2 is violated.

0BQW Lemma 15.3. Let Y be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals. Set $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$ and $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$. There is a canonical isomorphism $c : \mathcal{N} \rightarrow \mathcal{E}xt_{\mathcal{O}_Y}^1(\mathcal{O}_X, \mathcal{O}_X)$.

Proof. Consider the canonical short exact sequence

$$\text{0BQX (15.3.1)} \quad 0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_Y/\mathcal{I}^2 \rightarrow \mathcal{O}_X \rightarrow 0$$

Let $U \subset X$ be open and let $s \in \mathcal{N}(U)$. Then we can pushout (15.3.1) via s to get an extension E_s of $\mathcal{O}_X|_U$ by $\mathcal{O}_X|_U$. This in turn defines a section $c(s)$ of $\mathcal{E}xt_{\mathcal{O}_Y}^1(\mathcal{O}_X, \mathcal{O}_X)$ over U . See Cohomology, Lemma 42.1 and Derived Categories, Lemma 27.6. Conversely, given an extension

$$0 \rightarrow \mathcal{O}_X|_U \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X|_U \rightarrow 0$$

of $\mathcal{O}_U$ -modules, we can find an open covering $U = \bigcup U_i$ and sections $e_i \in \mathcal{E}(U_i)$ mapping to $1 \in \mathcal{O}_X(U_i)$. Then e_i defines a map $\mathcal{O}_Y|_{U_i} \rightarrow \mathcal{E}|_{U_i}$ whose kernel contains $\mathcal{I}^2$. In this way we see that $\mathcal{E}|_{U_i}$ comes from a pushout as above. This shows that c is surjective. We omit the proof of injectivity. $\square$

0BQY **Lemma 15.4.** *Let Y be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals. Set $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$. If $\mathcal{I}$ is Koszul-regular (Divisors, Definition 20.2) then composition on $R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ defines isomorphisms*

$$\wedge^i(\mathcal{E}xt_{\mathcal{O}_Y}^1(\mathcal{O}_X, \mathcal{O}_X)) \longrightarrow \mathcal{E}xt_{\mathcal{O}_Y}^i(\mathcal{O}_X, \mathcal{O}_X)$$

for all i .

Proof. By composition we mean the map

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

of Cohomology, Lemma 42.5. This induces multiplication maps

$$\mathcal{E}xt_{\mathcal{O}_Y}^a(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{E}xt_{\mathcal{O}_Y}^b(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{E}xt_{\mathcal{O}_Y}^{a+b}(\mathcal{O}_X, \mathcal{O}_X)$$

Please compare with More on Algebra, Equation (63.0.1). The statement of the lemma means that the induced map

$$\mathcal{E}xt_{\mathcal{O}_Y}^1(\mathcal{O}_X, \mathcal{O}_X) \otimes \dots \otimes \mathcal{E}xt_{\mathcal{O}_Y}^1(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{E}xt_{\mathcal{O}_Y}^i(\mathcal{O}_X, \mathcal{O}_X)$$

factors through the wedge product and then induces an isomorphism. To see this is true we may work locally on Y . Hence we may assume that we have global sections $f_1, \dots, f_r$ of $\mathcal{O}_Y$ which generate $\mathcal{I}$ and which form a Koszul regular sequence. Denote

$$\mathcal{A} = \mathcal{O}_Y \langle \xi_1, \dots, \xi_r \rangle$$

the sheaf of strictly commutative differential graded $\mathcal{O}_Y$ -algebras which is a (divided power) polynomial algebra on $\xi_1, \dots, \xi_r$ in degree -1 over $\mathcal{O}_Y$ with differential d given by the rule $d\xi_i = f_i$. Let us denote $\mathcal{A}^\bullet$ the underlying complex of $\mathcal{O}_Y$ -modules which is the Koszul complex mentioned above. Thus the canonical map $\mathcal{A}^\bullet \rightarrow \mathcal{O}_X$ is a quasi-isomorphism. We obtain quasi-isomorphisms

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \rightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{A}^\bullet) \rightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_X)$$

by Cohomology, Lemma 46.9. The differentials of the latter complex are zero, and hence

$$\mathcal{E}xt_{\mathcal{O}_Y}^i(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-i}, \mathcal{O}_X)$$

For $j \in \{1, \dots, r\}$ let $\delta_j : \mathcal{A} \rightarrow \mathcal{A}$ be the derivation of degree 1 with $\delta_j(\xi_i) = \delta_{ij}$ (Kronecker delta). A computation shows that $\delta_j \circ d = -d \circ \delta_j$ which shows that we get a morphism of complexes

$$\delta_j : \mathcal{A}^\bullet \rightarrow \mathcal{A}^\bullet[1].$$

Whence δ_j defines a section of the corresponding $\mathcal{E}xt$ -sheaf. Another computation shows that $\delta_1, \dots, \delta_r$ map to a basis for $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-1}, \mathcal{O}_X)$ over $\mathcal{O}_X$. Since it is clear that $\delta_j \circ \delta_j = 0$ and $\delta_j \circ \delta_{j'} = -\delta_{j'} \circ \delta_j$ as endomorphisms of $\mathcal{A}$ and hence in the $\mathcal{E}xt$ -sheaves we obtain the statement that our map above factors through the exterior power. To see we get the desired isomorphism the reader checks that the elements

$$\delta_{j_1} \circ \dots \circ \delta_{j_i}$$

for $j_1 < \dots < j_i$ map to a basis of the sheaf $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}^{-i}, \mathcal{O}_X)$ over $\mathcal{O}_X$. $\square$

0BQZ **Lemma 15.5.** *Let Y be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals. Set $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$ and $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$. If $\mathcal{I}$ is Koszul-regular (Divisors, Definition 20.2) then*

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) = \wedge^r \mathcal{N}[-r]$$

where $r : Y \rightarrow \{1, 2, 3, \dots\}$ sends y to the minimal number of generators of $\mathcal{I}$ needed in a neighbourhood of y .

Proof. We can use Lemmas 15.3 and 15.4 to see that we have isomorphisms $\wedge^i \mathcal{N} \rightarrow \mathcal{E}xt_{\mathcal{O}_Y}^i(\mathcal{O}_X, \mathcal{O}_X)$ for $i \geq 0$. Thus it suffices to show that the map $\mathcal{O}_Y \rightarrow \mathcal{O}_X$ induces an isomorphism

$$\mathcal{E}xt_{\mathcal{O}_Y}^r(\mathcal{O}_X, \mathcal{O}_Y) \longrightarrow \mathcal{E}xt_{\mathcal{O}_Y}^r(\mathcal{O}_X, \mathcal{O}_X)$$

and that $\mathcal{E}xt_{\mathcal{O}_Y}^i(\mathcal{O}_X, \mathcal{O}_Y)$ is zero for $i \neq r$. These statements are local on Y . Thus we may assume that we have global sections $f_1, \dots, f_r$ of $\mathcal{O}_Y$ which generate $\mathcal{I}$ and which form a Koszul regular sequence. Let $\mathcal{A}^\bullet$ be the Koszul complex on $f_1, \dots, f_r$ as introduced in the proof of Lemma 15.4. Then

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) = \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y)$$

by Cohomology, Lemma 46.9. Denote $1 : \mathcal{A}^\bullet \rightarrow \mathcal{O}_Y$ the map of differential graded $\mathcal{O}_Y$ -algebras given by the identity map of $\mathcal{A}^0 = \mathcal{O}_Y \rightarrow \mathcal{O}_Y$ in degree 0. With δ_j as in the proof of Lemma 15.4 we get an isomorphism of graded $\mathcal{O}_Y$ -modules

$$\mathcal{O}_Y \langle \delta_1, \dots, \delta_r \rangle \longrightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y)$$

by mapping $\delta_{j_1} \dots \delta_{j_i}$ to $1 \circ \delta_{j_1} \circ \dots \circ \delta_{j_i}$ in degree i . Via this isomorphism the differential on the right hand side induces a differential d on the left hand side. By our sign rules we have $d(1) = -\sum f_j \delta_j$. Since $\delta_j : \mathcal{A}^\bullet \rightarrow \mathcal{A}^\bullet[1]$ is a morphism of complexes, it follows that

$$d(\delta_{j_1} \dots \delta_{j_i}) = (-\sum f_j \delta_j) \delta_{j_1} \dots \delta_{j_i}$$

Observe that we have $d = \sum f_j \delta_j$ on the differential graded algebra $\mathcal{A}$. Therefore the map defined by the rule

$$1 \circ \delta_{j_1} \dots \delta_{j_i} \longmapsto (\delta_{j_1} \circ \dots \circ \delta_{j_i})(\xi_1 \dots \xi_r)$$

will define an isomorphism of complexes

$$\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y) \longrightarrow \mathcal{A}^\bullet[-r]$$

if r is odd and commuting with differentials up to sign if r is even. In any case these complexes have isomorphic cohomology, which shows the desired vanishing. The isomorphism on cohomology in degree r under the map

$$\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_Y) \longrightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{O}_X)$$

also follows in a straightforward manner from this. (We observe that our choice of conventions regarding Koszul complexes does intervene in the definition of the isomorphism $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_Y) = \wedge^r \mathcal{N}[-r]$.) $\square$

0BR0 **Lemma 15.6.** *Let Y be a quasi-compact and quasi-separated scheme. Let $i : X \rightarrow Y$ be a Koszul-regular closed immersion. Let a be the right adjoint of $Ri_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ of Lemma 3.1. Then there is an isomorphism*

$$\wedge^r \mathcal{N}[-r] \longrightarrow a(\mathcal{O}_Y)$$

where $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X)$ is the normal sheaf of i (Morphisms, Section 31) and r is its rank viewed as a locally constant function on X .

Proof. Recall, from Lemmas 9.7 and 9.3, that $a(\mathcal{O}_Y)$ is an object of $D_{Q\text{Coh}}(\mathcal{O}_X)$ whose pushforward to Y is $R\mathcal{H}om_{\mathcal{O}_Y}(i_*\mathcal{O}_X, \mathcal{O}_Y)$. Thus the result follows from Lemma 15.5. $\square$

0BRT **Lemma 15.7.** *Let S be a Noetherian scheme. Let $f : X \rightarrow S$ be a smooth proper morphism of relative dimension d . Let a be the right adjoint of $Rf_* : D_{Q\text{Coh}}(\mathcal{O}_X) \rightarrow D_{Q\text{Coh}}(\mathcal{O}_S)$ as in Lemma 3.1. Then there is an isomorphism*

$$\wedge^d \Omega_{X/S}[d] \longrightarrow a(\mathcal{O}_S)$$

in $D(\mathcal{O}_X)$.

Proof. Set $\omega_{X/S}^\bullet = a(\mathcal{O}_S)$ as in Remark 12.5. Let c be the right adjoint of Lemma 3.1 for $\Delta : X \rightarrow X \times_S X$. Because Δ is the diagonal of a smooth morphism it is a Koszul-regular immersion, see Divisors, Lemma 22.11. In particular, Δ is a perfect proper morphism (More on Morphisms, Lemma 61.7) and we obtain

$$\begin{aligned} \mathcal{O}_X &= c(L\text{pr}_1^* \omega_{X/S}^\bullet) \\ &= L\Delta^*(L\text{pr}_1^* \omega_{X/S}^\bullet) \otimes_{\mathcal{O}_X}^{\mathbf{L}} c(\mathcal{O}_{X \times_S X}) \\ &= \omega_{X/S}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} c(\mathcal{O}_{X \times_S X}) \\ &= \omega_{X/S}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \wedge^d(\mathcal{N}_\Delta)[-d] \end{aligned}$$

The first equality is (12.8.1) because $\omega_{X/S}^\bullet = a(\mathcal{O}_S)$. The second equality by Lemma 13.3. The third equality because $\text{pr}_1 \circ \Delta = \text{id}_X$. The fourth equality by Lemma 15.6. Observe that $\wedge^d(\mathcal{N}_\Delta)$ is an invertible $\mathcal{O}_X$ -module. Hence $\wedge^d(\mathcal{N}_\Delta)[-d]$ is an invertible object of $D(\mathcal{O}_X)$ and we conclude that $a(\mathcal{O}_S) = \omega_{X/S}^\bullet = \wedge^d(\mathcal{C}_\Delta)[d]$. Since the conormal sheaf $\mathcal{C}_\Delta$ of Δ is $\Omega_{X/S}$ by Morphisms, Lemma 32.7 the proof is complete. $\square$

16. Upper shriek functors

0A9Y In this section, we construct the functors $f^!$ for morphisms between schemes which are of finite type and separated over a fixed Noetherian base using compactifications. As is customary in coherent duality, there are a number of diagrams that have to be shown to be commutative. We suggest the reader, after reading the construction, skips the verification of the lemmas and continues to the next section where we discuss properties of the upper shriek functors.

0F42 **Situation 16.1.** Here S is a Noetherian scheme and FTS_S is the category whose

- (1) objects are schemes X over S such that the structure morphism $X \rightarrow S$ is both separated and of finite type, and
- (2) morphisms $f : X \rightarrow Y$ between objects are morphisms of schemes over S .

In Situation 16.1 given a morphism $f : X \rightarrow Y$ in FTS_S , we will define an exact functor

$$f^! : D_{Q\text{Coh}}^+(\mathcal{O}_Y) \rightarrow D_{Q\text{Coh}}^+(\mathcal{O}_X)$$

of triangulated categories. Namely, we choose a compactification $X \rightarrow \overline{X}$ over Y which is possible by More on Flatness, Theorem 33.8 and Lemma 32.2. Denote $\bar{f} : \overline{X} \rightarrow Y$ the structure morphism. Let $\bar{a} : D_{Q\text{Coh}}(\mathcal{O}_Y) \rightarrow D_{Q\text{Coh}}(\mathcal{O}_{\overline{X}})$ be the right adjoint of $R\bar{f}_*$ constructed in Lemma 3.1. Then we set

$$f^! K = \bar{a}(K)|_X$$

for $K \in D_{QCoH}^+(\mathcal{O}_Y)$. The result is an object of $D_{QCoH}^+(\mathcal{O}_X)$ by Lemma 3.5.

0AA0 **Lemma 16.2.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . The functor $f^!$ is, up to canonical isomorphism, independent of the choice of the compactification.*

Proof. The category of compactifications of X over Y is defined in More on Flatness, Section 32. By More on Flatness, Theorem 33.8 and Lemma 32.2 it is nonempty. To every choice of a compactification

$$j : X \rightarrow \bar{X}, \quad \bar{f} : \bar{X} \rightarrow Y$$

the construction above associates the functor $j^* \circ \bar{a} : D_{QCoH}^+(\mathcal{O}_Y) \rightarrow D_{QCoH}^+(\mathcal{O}_X)$ where $\bar{a}$ is the right adjoint of $R\bar{f}_*$ constructed in Lemma 3.1.

Suppose given a morphism $g : \bar{X}_1 \rightarrow \bar{X}_2$ between compactifications $j_i : X \rightarrow \bar{X}_i$ over Y such that $g^{-1}(j_2(X)) = j_1(X)$ ⁵. Let $\bar{c}$ be the right adjoint of Lemma 3.1 for g . Then $\bar{c} \circ \bar{a}_2 = \bar{a}_1$ because these functors are adjoint to $R\bar{f}_{2,*} \circ Rg_* = R(\bar{f}_2 \circ g)_*$. By (4.1.1) we have a canonical transformation

$$j_1^* \circ \bar{c} \longrightarrow j_2^*$$

of functors $D_{QCoH}^+(\mathcal{O}_{\bar{X}_2}) \rightarrow D_{QCoH}^+(\mathcal{O}_X)$ which is an isomorphism by Lemma 4.4. The composition

$$j_1^* \circ \bar{a}_1 \longrightarrow j_1^* \circ \bar{c} \circ \bar{a}_2 \longrightarrow j_2^* \circ \bar{a}_2$$

is an isomorphism of functors which we will denote by α_g .

Consider two compactifications $j_i : X \rightarrow \bar{X}_i$, $i = 1, 2$ of X over Y . By More on Flatness, Lemma 32.1 part (b) we can find a compactification $j : X \rightarrow \bar{X}$ with dense image and morphisms $g_i : \bar{X} \rightarrow \bar{X}_i$ of compactifications. By More on Flatness, Lemma 32.1 part (c) we have $g_i^{-1}(j_i(X)) = j(X)$. Hence we get isomorphisms

$$\alpha_{g_i} : j^* \circ \bar{a} \longrightarrow j_i^* \circ \bar{a}_i$$

by the previous paragraph. We obtain an isomorphism

$$\alpha_{g_2} \circ \alpha_{g_1}^{-1} : j_1^* \circ \bar{a}_1 \rightarrow j_2^* \circ \bar{a}_2$$

To finish the proof we have to show that these isomorphisms are well defined. We claim it suffices to show the composition of isomorphisms constructed in the previous paragraph is another (for a precise statement see the next paragraph). We suggest the reader check this is true on a napkin, but we will also completely spell it out in the rest of this paragraph. Namely, consider a second choice of a compactification $j' : X \rightarrow \bar{X}'$ with dense image and morphisms of compactifications $g'_i : \bar{X}' \rightarrow \bar{X}_i$. By More on Flatness, Lemma 32.1 we can find a compactification $j'' : X \rightarrow \bar{X}''$ with dense image and morphisms of compactifications $h : \bar{X}'' \rightarrow \bar{X}$ and $h' : \bar{X}'' \rightarrow \bar{X}'$. We may even assume $g_1 \circ h = g'_1 \circ h'$ and $g_2 \circ h = g'_2 \circ h'$. The result of the next paragraph gives

$$\alpha_{g_i} \circ \alpha_h = \alpha_{g_i \circ h} = \alpha_{g'_i \circ h'} = \alpha_{g'_i} \circ \alpha_{h'}$$

for $i = 1, 2$. Since these are all isomorphisms of functors we conclude that $\alpha_{g_2} \circ \alpha_{g_1}^{-1} = \alpha_{g'_2} \circ \alpha_{g'_1}^{-1}$ as desired.

⁵This may fail with our definition of compactification. See More on Flatness, Section 32.

Suppose given compactifications $j_i : X \rightarrow \bar{X}_i$ for $i = 1, 2, 3$. Suppose given morphisms $g : \bar{X}_1 \rightarrow \bar{X}_2$ and $h : \bar{X}_2 \rightarrow \bar{X}_3$ of compactifications such that $g^{-1}(j_2(X)) = j_1(X)$ and $h^{-1}(j_3(X)) = j_2(X)$. Let $\bar{a}_i$ be as above. The claim above means that

$$\alpha_g \circ \alpha_h = \alpha_{g \circ h} : j_1^* \circ \bar{a}_1 \rightarrow j_3^* \circ \bar{a}_3$$

Let $\bar{c}$, resp. $\bar{d}$ be the right adjoint of Lemma 3.1 for g , resp. h . Then $\bar{c} \circ \bar{a}_2 = \bar{a}_1$ and $\bar{d} \circ \bar{a}_3 = \bar{a}_2$ and there are canonical transformations

$$j_1^* \circ \bar{c} \rightarrow j_2^* \quad \text{and} \quad j_2^* \circ \bar{d} \rightarrow j_3^*$$

of functors $D_{Q\text{Coh}}^+(\mathcal{O}_{\bar{X}_2}) \rightarrow D_{Q\text{Coh}}^+(\mathcal{O}_X)$ and $D_{Q\text{Coh}}^+(\mathcal{O}_{\bar{X}_3}) \rightarrow D_{Q\text{Coh}}^+(\mathcal{O}_X)$ for the same reasons as above. Denote $\bar{e}$ the right adjoint of Lemma 3.1 for $h \circ g$. There is a canonical transformation

$$j_1^* \circ \bar{e} \rightarrow j_3^*$$

of functors $D_{Q\text{Coh}}^+(\mathcal{O}_{\bar{X}_3}) \rightarrow D_{Q\text{Coh}}^+(\mathcal{O}_X)$ given by (4.1.1). Spelling things out we have to show that the composition

$$\alpha_h \circ \alpha_g : j_1^* \circ \bar{a}_1 \rightarrow j_1^* \circ \bar{c} \circ \bar{a}_2 \rightarrow j_2^* \circ \bar{a}_2 \rightarrow j_2^* \circ \bar{d} \circ \bar{a}_3 \rightarrow j_3^* \circ \bar{a}_3$$

is the same as the composition

$$\alpha_{h \circ g} : j_1^* \circ \bar{a}_1 \rightarrow j_1^* \circ \bar{e} \circ \bar{a}_3 \rightarrow j_3^* \circ \bar{a}_3$$

We split this into two parts. The first is to show that the diagram

$$\begin{array}{ccc} \bar{a}_1 & \xrightarrow{\quad} & \bar{c} \circ \bar{a}_2 \\ \downarrow & & \downarrow \\ \bar{e} \circ \bar{a}_3 & \xrightarrow{\quad} & \bar{c} \circ \bar{d} \circ \bar{a}_3 \end{array}$$

commutes where the lower horizontal arrow comes from the identification $\bar{e} = \bar{c} \circ \bar{d}$. This is true because the corresponding diagram of total direct image functors

$$\begin{array}{ccc} R\bar{f}_{1,*} & \xrightarrow{\quad} & Rg_* \circ R\bar{f}_{2,*} \\ \downarrow & & \downarrow \\ R(h \circ g)_* \circ R\bar{f}_{3,*} & \xrightarrow{\quad} & Rg_* \circ Rh_* \circ R\bar{f}_{3,*} \end{array}$$

is commutative (insert future reference here). The second part is to show that the composition

$$j_1^* \circ \bar{c} \circ \bar{d} \rightarrow j_2^* \circ \bar{d} \rightarrow j_3^*$$

is equal to the map

$$j_1^* \circ \bar{e} \rightarrow j_3^*$$

via the identification $\bar{e} = \bar{c} \circ \bar{d}$. This was proven in Lemma 5.1 (note that in the current case the morphisms f', g' of that lemma are equal to id_X). $\square$

0ATX **Lemma 16.3.** *In Situation 16.1 let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be composable morphisms of FTS_S. Then there is a canonical isomorphism $(g \circ f)^! \rightarrow f^! \circ g^!$.*

Proof. Choose a compactification $i : Y \rightarrow \bar{Y}$ of Y over Z . Choose a compactification $X \rightarrow \bar{X}$ of X over $\bar{Y}$. This uses More on Flatness, Theorem 33.8 and Lemma 32.2 twice. Let $\bar{a}$ be the right adjoint of Lemma 3.1 for $\bar{X} \rightarrow \bar{Y}$ and let $\bar{b}$ be the right adjoint of Lemma 3.1 for $\bar{Y} \rightarrow Z$. Then $\bar{a} \circ \bar{b}$ is the right adjoint of Lemma 3.1 for the composition $\bar{X} \rightarrow Z$. Hence $g^! = i^* \circ \bar{b}$ and $(g \circ f)^! = (X \rightarrow \bar{X})^* \circ \bar{a} \circ \bar{b}$. Let U be the inverse image of Y in $\bar{X}$ so that we get the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & U & \xrightarrow{\quad} & \bar{X} \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ Y & \xrightarrow{\quad} & \bar{Y} & & \\ \downarrow & \swarrow & \downarrow & & \\ Z & & & & \end{array}$$

Let $\bar{a}'$ be the right adjoint of Lemma 3.1 for $U \rightarrow Y$. Then $f^! = j^* \circ \bar{a}'$. We obtain

$$\gamma : (j')^* \circ \bar{a} \rightarrow \bar{a}' \circ i^*$$

by (4.1.1) and we can use it to define

$$(g \circ f)^! = (j' \circ j)^* \circ \bar{a} \circ \bar{b} = j^* \circ (j')^* \circ \bar{a} \circ \bar{b} \rightarrow j^* \circ \bar{a}' \circ i^* \circ \bar{b} = f^! \circ g^!$$

which is an isomorphism on objects of $D_{Q\text{Coh}}^+(\mathcal{O}_Z)$ by Lemma 4.4. To finish the proof we show that this isomorphism is independent of choices made.

Suppose we have two diagrams

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & U_1 & \xrightarrow{\quad} & \bar{X}_1 \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ Y & \xrightarrow{\quad} & \bar{Y}_1 & & \\ \downarrow & \swarrow & \downarrow & & \\ Z & & & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} X & \xrightarrow{\quad} & U_2 & \xrightarrow{\quad} & \bar{X}_2 \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ Y & \xrightarrow{\quad} & \bar{Y}_2 & & \\ \downarrow & \swarrow & \downarrow & & \\ Z & & & & \end{array}$$

We can first choose a compactification $i : Y \rightarrow \bar{Y}$ with dense image of Y over Z which dominates both $\bar{Y}_1$ and $\bar{Y}_2$, see More on Flatness, Lemma 32.1. By More on Flatness, Lemma 32.3 and Categories, Lemmas 27.13 and 27.14 we can choose a compactification $X \rightarrow \bar{X}$ with dense image of X over $\bar{Y}$ with morphisms $\bar{X} \rightarrow \bar{X}_1$ and $\bar{X} \rightarrow \bar{X}_2$ and such that the composition $\bar{X} \rightarrow \bar{Y} \rightarrow \bar{Y}_1$ is equal to the composition $\bar{X} \rightarrow \bar{X}_1 \rightarrow \bar{Y}_1$ and such that the composition $\bar{X} \rightarrow \bar{Y} \rightarrow \bar{Y}_2$ is equal to the composition $\bar{X} \rightarrow \bar{X}_2 \rightarrow \bar{Y}_2$. Thus we see that it suffices to compare the

maps determined by our diagrams when we have a commutative diagram as follows

$$\begin{array}{ccccc}
 X & \xrightarrow{j_1} & U_1 & \xrightarrow{j'_1} & \bar{X}_1 \\
 \parallel & \searrow & \downarrow & \searrow & \downarrow \\
 X & \xrightarrow{j_2} & U_2 & \xrightarrow{j'_2} & \bar{X}_2 \\
 \downarrow & \swarrow & \swarrow & \swarrow & \downarrow \\
 Y & \xrightarrow{i_1} & \bar{Y}_1 & & \\
 \parallel & \searrow & \downarrow & \searrow & \\
 Y & \xrightarrow{i_2} & \bar{Y}_2 & & \\
 \downarrow & \swarrow & & & \\
 Z & & & &
 \end{array}$$

and moreover the compactifications $X \rightarrow \bar{X}_1$ and $Y \rightarrow \bar{Y}_2$ have dense image. We use $\bar{a}_i$, $\bar{a}'_i$, $\bar{c}$, and $\bar{c}'$ for the right adjoint of Lemma 3.1 for $\bar{X}_i \rightarrow \bar{Y}_i$, $U_i \rightarrow Y$, $\bar{X}_1 \rightarrow \bar{X}_2$, and $U_1 \rightarrow U_2$. Each of the squares

$$\begin{array}{ccccc}
 X \longrightarrow U_1 & U_2 \longrightarrow \bar{X}_2 & U_1 \longrightarrow \bar{X}_1 & Y \longrightarrow \bar{Y}_1 & X \longrightarrow \bar{X}_1 \\
 \downarrow \quad \quad \downarrow & \downarrow \quad \quad \downarrow & \downarrow \quad \quad \downarrow & \downarrow \quad \quad \downarrow & \downarrow \quad \quad \downarrow \\
 X \longrightarrow U_2 & Y \longrightarrow \bar{Y}_2 & Y \longrightarrow \bar{Y}_1 & Y \longrightarrow \bar{Y}_2 & X \longrightarrow \bar{X}_2
 \end{array}$$

is cartesian (see More on Flatness, Lemma 32.1 part (c) for A, D, E and recall that U_i is the inverse image of Y by $\bar{X}_i \rightarrow \bar{Y}_i$ for B, C) and hence gives rise to a base change map (4.1.1) as follows

$$\begin{array}{lll}
 \gamma_A : j_1^* \circ \bar{c}' \rightarrow j_2^* & \gamma_B : (j'_2)^* \circ \bar{a}_2 \rightarrow \bar{a}'_2 \circ i_2^* & \gamma_C : (j'_1)^* \circ \bar{a}_1 \rightarrow \bar{a}'_1 \circ i_1^* \\
 \gamma_D : i_1^* \circ \bar{d} \rightarrow i_2^* & \gamma_E : (j'_1 \circ j_1)^* \circ \bar{c} \rightarrow (j'_2 \circ j_2)^* &
 \end{array}$$

Denote $f_1^! = j_1^* \circ \bar{a}'_1$, $f_2^! = j_2^* \circ \bar{a}'_2$, $g_1^! = i_1^* \circ \bar{b}_1$, $g_2^! = i_2^* \circ \bar{b}_2$, $(g \circ f)_1^! = (j'_1 \circ j_1)^* \circ \bar{a}_1 \circ \bar{b}_1$, and $(g \circ f)_2^! = (j'_2 \circ j_2)^* \circ \bar{a}_2 \circ \bar{b}_2$. The construction given in the first paragraph of the proof and in Lemma 16.2 uses

- (1) γ_C for the map $(g \circ f)_1^! \rightarrow f_1^! \circ g_1^!$,
- (2) γ_B for the map $(g \circ f)_2^! \rightarrow f_2^! \circ g_2^!$,
- (3) γ_A for the map $f_1^! \rightarrow f_2^!$,
- (4) γ_D for the map $g_1^! \rightarrow g_2^!$, and
- (5) γ_E for the map $(g \circ f)_1^! \rightarrow (g \circ f)_2^!$.

We have to show that the diagram

$$\begin{array}{ccc}
 (g \circ f)_1^! & \xrightarrow{\gamma_E} & (g \circ f)_2^! \\
 \gamma_C \downarrow & & \downarrow \gamma_B \\
 f_1^! \circ g_1^! & \xrightarrow{\gamma_A \circ \gamma_D} & f_2^! \circ g_2^!
 \end{array}$$

is commutative. We will use Lemmas 5.1 and 5.2 and with (abuse of) notation as in Remark 5.3 (in particular dropping $\star$ products with identity transformations from

the notation). We can write $\gamma_E = \gamma_A \circ \gamma_F$ where

$$\begin{array}{ccc} U_1 & \longrightarrow & \overline{X}_1 \\ \downarrow & F & \downarrow \\ U_2 & \longrightarrow & \overline{X}_2 \end{array}$$

Thus we see that

$$\gamma_B \circ \gamma_E = \gamma_B \circ \gamma_A \circ \gamma_F = \gamma_A \circ \gamma_B \circ \gamma_F$$

the last equality because the two squares A and B only intersect in one point (similar to the last argument in Remark 5.3). Thus it suffices to prove that $\gamma_D \circ \gamma_C = \gamma_B \circ \gamma_F$. Since both of these are equal to the map (4.1.1) for the square

$$\begin{array}{ccc} U_1 & \longrightarrow & \overline{X}_1 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \overline{Y}_2 \end{array}$$

we conclude. $\square$

0ATY **Lemma 16.4.** *In Situation 16.1 the constructions of Lemmas 16.2 and 16.3 define a pseudo functor from the category FTS_S into the 2-category of categories (see Categories, Definition 29.5).*

Proof. To show this we have to prove given morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$, $h : Z \rightarrow T$ that

$$\begin{array}{ccc} (h \circ g \circ f)^! & \xrightarrow{\gamma_{A+B}} & f^! \circ (h \circ g)^! \\ \gamma_{B+C} \downarrow & & \downarrow \gamma_C \\ (g \circ f)^! \circ h^! & \xrightarrow{\gamma_A} & f^! \circ g^! \circ h^! \end{array}$$

is commutative (for the meaning of the γ 's, see below). To do this we choose a compactification $\overline{Z}$ of Z over T , then a compactification $\overline{Y}$ of Y over $\overline{Z}$, and then a compactification $\overline{X}$ of X over $\overline{Y}$. This uses More on Flatness, Theorem 33.8 and Lemma 32.2. Let $W \subset \overline{Y}$ be the inverse image of Z under $\overline{Y} \rightarrow \overline{Z}$ and let $U \subset V \subset \overline{X}$ be the inverse images of $Y \subset W$ under $\overline{X} \rightarrow \overline{Y}$. This produces the following diagram

$$\begin{array}{ccccccc} X & \longrightarrow & U & \longrightarrow & V & \longrightarrow & \overline{X} \\ f \downarrow & & \downarrow & A & \downarrow & B & \downarrow \\ Y & \longrightarrow & Y & \longrightarrow & W & \longrightarrow & \overline{Y} \\ g \downarrow & & \downarrow & & \downarrow & C & \downarrow \\ Z & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & \overline{Z} \\ h \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & T & \longrightarrow & T & \longrightarrow & T \end{array}$$

Without introducing tons of notation but arguing exactly as in the proof of Lemma 16.3 we see that the maps in the first displayed diagram use the maps (4.1.1) for the rectangles $A+B$, $B+C$, A , and C as indicated. Since by Lemmas 5.1 and 5.2

we have $\gamma_{A+B} = \gamma_A \circ \gamma_B$ and $\gamma_{B+C} = \gamma_C \circ \gamma_B$ we conclude that the desired equality holds provided $\gamma_A \circ \gamma_C = \gamma_C \circ \gamma_A$. This is true because the two squares A and C only intersect in one point (similar to the last argument in Remark 5.3). $\square$

0B6T **Lemma 16.5.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . There are canonical maps*

$$\mu_{f,K} : Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} f^!\mathcal{O}_Y \longrightarrow f^!K$$

functorial in K in $D_{Qcoh}^+(\mathcal{O}_Y)$. If $g : Y \rightarrow Z$ is another morphism of FTS_S , then the diagram

$$\begin{array}{ccccc} Lf^*(Lg^*K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} g^!\mathcal{O}_Z) \otimes_{\mathcal{O}_X}^{\mathbf{L}} f^!\mathcal{O}_Y & \xrightarrow{\mu_f} & f^!(Lg^*K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} g^!\mathcal{O}_Z) & \xrightarrow{f^!\mu_g} & f^!g^!K \\ \parallel & & & & \parallel \\ Lf^*Lg^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*g^!\mathcal{O}_Z \otimes_{\mathcal{O}_X}^{\mathbf{L}} f^!\mathcal{O}_Y & \xrightarrow{\mu_f} & Lf^*Lg^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} f^!g^!\mathcal{O}_Z & \xrightarrow{\mu_{g \circ f}} & f^!g^!K \end{array}$$

commutes for all $K \in D_{Qcoh}^+(\mathcal{O}_Z)$.

Proof. If f is proper, then $f^! = a$ and we can use (8.0.1) and if g is also proper, then Lemma 8.4 proves the commutativity of the diagram (in greater generality).

Let us define the map $\mu_{f,K}$. Choose a compactification $j : X \rightarrow \overline{X}$ of X over Y . Since $f^!$ is defined as $j^* \circ \overline{a}$ we obtain $\mu_{f,K}$ as the restriction of the map (8.0.1)

$$L\overline{f}^*K \otimes_{\mathcal{O}_{\overline{X}}}^{\mathbf{L}} \overline{a}(\mathcal{O}_Y) \longrightarrow \overline{a}(K)$$

to X . To see this is independent of the choice of the compactification we argue as in the proof of Lemma 16.2. We urge the reader to read the proof of that lemma first.

Assume given a morphism $g : \overline{X}_1 \rightarrow \overline{X}_2$ between compactifications $j_i : X \rightarrow \overline{X}_i$ over Y such that $g^{-1}(j_2(X)) = j_1(X)$. Denote $\overline{c}$ the right adjoint for pushforward of Lemma 3.1 for the morphism g . The maps

$$L\overline{f}_1^*K \otimes_{\mathcal{O}_{\overline{X}_1}}^{\mathbf{L}} \overline{a}_1(\mathcal{O}_Y) \longrightarrow \overline{a}_1(K) \quad \text{and} \quad L\overline{f}_2^*K \otimes_{\mathcal{O}_{\overline{X}_2}}^{\mathbf{L}} \overline{a}_2(\mathcal{O}_Y) \longrightarrow \overline{a}_2(K)$$

fit into the commutative diagram

$$\begin{array}{ccccc} Lg^*(L\overline{f}_2^*K \otimes_{\mathcal{O}_{\overline{X}_2}}^{\mathbf{L}} \overline{a}_2(\mathcal{O}_Y)) \otimes_{\mathcal{O}_{\overline{X}_1}}^{\mathbf{L}} \overline{c}(\mathcal{O}_{\overline{X}_2}) & \xrightarrow{\sigma} & \overline{c}(L\overline{f}_2^*K \otimes_{\mathcal{O}_{\overline{X}_2}}^{\mathbf{L}} \overline{a}_2(\mathcal{O}_Y)) & \longrightarrow & \overline{c}(\overline{a}_2(K)) \\ \parallel & & & & \parallel \\ L\overline{f}_1^*K \otimes_{\mathcal{O}_{\overline{X}_1}}^{\mathbf{L}} Lg^*\overline{a}_2(\mathcal{O}_Y) \otimes_{\mathcal{O}_{\overline{X}_1}}^{\mathbf{L}} \overline{c}(\mathcal{O}_{\overline{X}_2}) & \xrightarrow{1 \otimes \tau} & L\overline{f}_1^*K \otimes_{\mathcal{O}_{\overline{X}_1}}^{\mathbf{L}} \overline{a}_1(\mathcal{O}_Y) & \longrightarrow & \overline{a}_1(K) \end{array}$$

by Lemma 8.4. By Lemma 8.3 the maps σ and τ restrict to an isomorphism over X . In fact, we can say more. Recall that in the proof of Lemma 16.2 we used the map (4.1.1) $\gamma : j_1^* \circ \overline{c} \rightarrow j_2^*$ to construct our isomorphism $\alpha_g : j_1^* \circ \overline{a}_1 \rightarrow j_2^* \circ \overline{a}_2$. Pulling back to map σ by j_1 we obtain the identity map on $j_2^*(L\overline{f}_2^*K \otimes_{\mathcal{O}_{\overline{X}_2}}^{\mathbf{L}} \overline{a}_2(\mathcal{O}_Y))$ if we identify $j_1^*\overline{c}(\mathcal{O}_{\overline{X}_2})$ with $\mathcal{O}_X$ via $j_1^* \circ \overline{c} \rightarrow j_2^*$, see Lemma 8.2. Similarly, the map $\tau : Lg^*\overline{a}_2(\mathcal{O}_Y) \otimes_{\mathcal{O}_{\overline{X}_1}}^{\mathbf{L}} \overline{c}(\mathcal{O}_{\overline{X}_2}) \rightarrow \overline{a}_1(\mathcal{O}_Y) = \overline{c}(\overline{a}_2(\mathcal{O}_Y))$ pulls back to the identity

map on $j_2^* \bar{a}_2(\mathcal{O}_Y)$. We conclude that pulling back by j_1 and applying γ wherever we can we obtain a commutative diagram

$$\begin{array}{ccc}
 j_2^* \left(L\bar{f}_2^* K \otimes^{\mathbf{L}} \bar{a}_2(\mathcal{O}_Y) \right) & \longrightarrow & j_2^* \bar{a}_2(K) \\
 \downarrow & & \nwarrow \alpha_g \\
 j_1^* L\bar{f}_1^* K \otimes^{\mathbf{L}} j_2^* \bar{a}_2(\mathcal{O}_Y) & \xleftarrow{1 \otimes \alpha_g} j_1^* (L\bar{f}_1^* K \otimes^{\mathbf{L}} \bar{a}_1(\mathcal{O}_Y)) \longrightarrow & j_1^* \bar{a}_1(K)
 \end{array}$$

The commutativity of this diagram exactly tells us that the map $\mu_{f,K}$ constructed using the compactification $\bar{X}_1$ is the same as the map $\mu_{f,K}$ constructed using the compactification $\bar{X}_2$ via the identification α_g used in the proof of Lemma 16.2. Some categorical arguments exactly as in the proof of Lemma 16.2 now show that $\mu_{f,K}$ is well defined (small detail omitted).

Having said this, the commutativity of the diagram in the statement of our lemma follows from the construction of the isomorphism $(g \circ f)^! \rightarrow f^! \circ g^!$ (first part of the proof of Lemma 16.3 using $\bar{X} \rightarrow \bar{Y} \rightarrow Z$) and the result of Lemma 8.4 for $\bar{X} \rightarrow \bar{Y} \rightarrow Z$. $\square$

17. Properties of upper shriek functors

0ATZ Here are some properties of the upper shriek functors.

0AU0 **Lemma 17.1.** *In Situation 16.1 let Y be an object of FTS_S and let $j : X \rightarrow Y$ be an open immersion. Then there is a canonical isomorphism $j^! = j^*$ of functors.*

For an étale morphism $f : X \rightarrow Y$ of FTS_S we also have $f^* \cong f^!$, see Lemma 18.2.

Proof. In this case we may choose $\bar{X} = Y$ as our compactification. Then the right adjoint of Lemma 3.1 for $\text{id} : Y \rightarrow Y$ is the identity functor and hence $j^! = j^*$ by definition. $\square$

0G4J **Lemma 17.2.** *In Situation 16.1 let*

$$\begin{array}{ccc}
 U & \xrightarrow{j} & X \\
 g \downarrow & & \downarrow f \\
 V & \xrightarrow{j'} & Y
 \end{array}$$

be a commutative diagram of FTS_S where j and j' are open immersions. Then $j^ \circ f^! = g^! \circ (j')^*$ as functors $D_{Q\text{Coh}}^+(\mathcal{O}_Y) \rightarrow D^+(\mathcal{O}_U)$.*

Proof. Let $h = f \circ j = j' \circ g$. By Lemma 16.3 we have $h^! = j^! \circ f^! = g^! \circ (j')^!$. By Lemma 17.1 we have $j^! = j^*$ and $(j')^! = (j')^*$. $\square$

0AA1 **Lemma 17.3.** *In Situation 16.1 let Y be an object of FTS_S and let $f : X = \mathbf{A}_Y^1 \rightarrow Y$ be the projection. Then there is a (noncanonical) isomorphism $f^!(-) \cong Lf^*(-)[1]$ of functors.*

Proof. Since $X = \mathbf{A}_Y^1 \subset \mathbf{P}_Y^1$ and since $\mathcal{O}_{\mathbf{P}_Y^1}(-2)|_X \cong \mathcal{O}_X$ this follows from Lemmas 15.1 and 13.3. $\square$

0AA2 **Lemma 17.4.** *In Situation 16.1 let Y be an object of FTS_S and let $i : X \rightarrow Y$ be a closed immersion. Then there is a canonical isomorphism $i^!(-) = R\text{Hom}(\mathcal{O}_X, -)$ of functors.*

Proof. This is a restatement of Lemma 9.7. $\square$

0BV2 **Remark 17.5** (Local description upper shriek). In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . Using the lemmas above we can compute $f^!$ locally as follows. Suppose that we are given affine opens

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ g \downarrow & & \downarrow f \\ V & \xrightarrow{i} & Y \end{array}$$

Since $j^! \circ f^! = g^! \circ i^!$ (Lemma 16.3) and since $j^!$ and $i^!$ are given by restriction (Lemma 17.1) we see that

$$(f^!E)|_U = g^!(E|_V)$$

for any $E \in D_{Coh}^+(\mathcal{O}_X)$. Write $U = \text{Spec}(A)$ and $V = \text{Spec}(R)$ and let $\varphi : R \rightarrow A$ be the finite type ring map corresponding to g . Choose a presentation $A = P/I$ where $P = R[x_1, \dots, x_n]$ is a polynomial algebra in n variables over R . Choose an object $K \in D^+(R)$ corresponding to $E|_V$ (Derived Categories of Schemes, Lemma 3.5). Then we claim that $f^!E|_U$ corresponds to

$$\varphi^!(K) = R\text{Hom}(A, K \otimes_R^{\mathbf{L}} P)[n]$$

where $R\text{Hom}(A, -) : D(P) \rightarrow D(A)$ is the functor of Dualizing Complexes, Section 13 and where $\varphi^! : D(R) \rightarrow D(A)$ is the functor of Dualizing Complexes, Section 24. Namely, the choice of presentation gives a factorization

$$U \rightarrow \mathbf{A}_V^n \rightarrow \mathbf{A}_V^{n-1} \rightarrow \dots \rightarrow \mathbf{A}_V^1 \rightarrow V$$

Applying Lemma 17.3 exactly n times we see that $(\mathbf{A}_V^n \rightarrow V)^!(E|_V)$ corresponds to $K \otimes_R^{\mathbf{L}} P[n]$. By Lemmas 9.5 and 17.4 the last step corresponds to applying $R\text{Hom}(A, -)$.

0AU1 **Lemma 17.6.** In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . Then $f^!$ maps $D_{Coh}^+(\mathcal{O}_Y)$ into $D_{Coh}^+(\mathcal{O}_X)$.

Proof. The question is local on X hence we may assume that X and Y are affine schemes. In this case we can factor $f : X \rightarrow Y$ as

$$X \xrightarrow{i} \mathbf{A}_Y^n \rightarrow \mathbf{A}_Y^{n-1} \rightarrow \dots \rightarrow \mathbf{A}_Y^1 \rightarrow Y$$

where i is a closed immersion. The lemma follows from By Lemmas 17.3 and 9.6 and Dualizing Complexes, Lemma 15.10 and induction. $\square$

0AA3 **Lemma 17.7.** In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . If K is a dualizing complex for Y , then $f^!K$ is a dualizing complex for X .

Proof. The question is local on X hence we may assume that X and Y are affine schemes. In this case we can factor $f : X \rightarrow Y$ as

$$X \xrightarrow{i} \mathbf{A}_Y^n \rightarrow \mathbf{A}_Y^{n-1} \rightarrow \dots \rightarrow \mathbf{A}_Y^1 \rightarrow Y$$

where i is a closed immersion. By Lemma 17.3 and Dualizing Complexes, Lemma 15.10 and induction we see that the $p^!K$ is a dualizing complex on $\mathbf{A}_Y^n$ where $p : \mathbf{A}_Y^n \rightarrow Y$ is the projection. Similarly, by Dualizing Complexes, Lemma 15.9 and Lemmas 9.5 and 17.4 we see that $i^!$ transforms dualizing complexes into dualizing complexes. $\square$

0AU2 **Lemma 17.8.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S. Let K be a dualizing complex on Y . Set $D_Y(M) = R\mathcal{H}om_{\mathcal{O}_Y}(M, K)$ for $M \in D_{Coh}(\mathcal{O}_Y)$ and $D_X(E) = R\mathcal{H}om_{\mathcal{O}_X}(E, f^!K)$ for $E \in D_{Coh}(\mathcal{O}_X)$. Then there is a canonical isomorphism*

$$f^!M \longrightarrow D_X(Lf^*D_Y(M))$$

for $M \in D_{Coh}^+(\mathcal{O}_Y)$.

Proof. Choose compactification $j : X \subset \bar{X}$ of X over Y (More on Flatness, Theorem 33.8 and Lemma 32.2). Let a be the right adjoint of Lemma 3.1 for $\bar{X} \rightarrow Y$. Set $D_{\bar{X}}(E) = R\mathcal{H}om_{\mathcal{O}_{\bar{X}}}(E, a(K))$ for $E \in D_{Coh}(\mathcal{O}_{\bar{X}})$. Since formation of $R\mathcal{H}om$ commutes with restriction to opens and since $f^! = j^* \circ a$ we see that it suffices to prove that there is a canonical isomorphism

$$a(M) \longrightarrow D_{\bar{X}}(L\bar{f}^*D_Y(M))$$

for $M \in D_{Coh}(\mathcal{O}_Y)$. For $F \in D_{QCoh}(\mathcal{O}_X)$ we have

$$\begin{aligned} \mathrm{Hom}_{\bar{X}}(F, D_{\bar{X}}(L\bar{f}^*D_Y(M))) &= \mathrm{Hom}_{\bar{X}}(F \otimes_{\mathcal{O}_X}^{\mathbf{L}} L\bar{f}^*D_Y(M), a(K)) \\ &= \mathrm{Hom}_Y(R\bar{f}_*(F \otimes_{\mathcal{O}_X}^{\mathbf{L}} L\bar{f}^*D_Y(M)), K) \\ &= \mathrm{Hom}_Y(R\bar{f}_*(F) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} D_Y(M), K) \\ &= \mathrm{Hom}_Y(R\bar{f}_*(F), D_Y(D_Y(M))) \\ &= \mathrm{Hom}_Y(R\bar{f}_*(F), M) \\ &= \mathrm{Hom}_{\bar{X}}(F, a(M)) \end{aligned}$$

The first equality by Cohomology, Lemma 42.2. The second by definition of a . The third by Derived Categories of Schemes, Lemma 22.1. The fourth equality by Cohomology, Lemma 42.2 and the definition of D_Y . The fifth equality by Lemma 2.5. The final equality by definition of a . Hence we see that $a(M) = D_{\bar{X}}(L\bar{f}^*D_Y(M))$ by Yoneda's lemma. $\square$

0B6U **Lemma 17.9.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S. Assume f is perfect (e.g., flat). Then*

- (a) $f^!\mathcal{O}_Y$ is in $D_{Coh}^b(\mathcal{O}_X)$,
- (b) $f^!\mathcal{O}_Y$ has finite tor dimension in $D(f^{-1}\mathcal{O}_Y)$,
- (c) $\mathcal{O}_X \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(f^!\mathcal{O}_Y, f^!\mathcal{O}_Y)$ is an isomorphism,
- (d) $f^!$ maps $D_{Coh}^b(\mathcal{O}_Y)$ into $D_{Coh}^b(\mathcal{O}_X)$,
- (e) the map $\mu_{f,K} : Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} f^!\mathcal{O}_Y \rightarrow f^!K$ of Lemma 16.5 is an isomorphism for all $K \in D_{QCoh}^+(\mathcal{O}_Y)$.

Proof. (A flat morphism of finite presentation is perfect, see More on Morphisms, Lemma 61.5.) Assertions (a), (b), and (c) are local on X . Thus we may assume X and Y are affine. Then Remark 17.5 turns (a), (b), and (c) into (1)(a), (1)(b), and (1)(c) of Dualizing Complexes, Lemma 25.2. (Use Derived Categories of Schemes, Lemmas 10.5, 10.1, 10.3 10.8 to match the assertions.)

To prove (d) and (e) we begin with a series of preliminary remarks.

- (1) We already know that $f^!$ sends $D_{Coh}^+(\mathcal{O}_Y)$ into $D_{Coh}^+(\mathcal{O}_X)$, see Lemma 17.6.
- (2) If f is an open immersion, then $f^! = f^*$ and (d) and (e) are true because we can take $\bar{X} = Y$ in the construction of $f^!$ and μ_f , see Lemma 17.1.
- (3) If f is a perfect proper morphism, then (e) is true by Lemma 13.3.

- (4) If there exists an open covering $X = \bigcup U_i$ and (d) is true for $U_i \rightarrow Y$, then (d) is true for $X \rightarrow Y$. Same for (e). This holds because the construction of $f^!$ and μ_f commutes with passing to open subschemes.
- (5) If $g : Y \rightarrow Z$ is a second perfect morphism in FTS_S and (e) holds for f and g , then $f^!g^!\mathcal{O}_Z = Lf^*g^!\mathcal{O}_Z \otimes_{\mathcal{O}_X}^{\mathbf{L}} f^!\mathcal{O}_Y$ and (e) holds for $g \circ f$ by the commutative diagram of Lemma 16.5.
- (6) If (d) and (e) hold for both f and g , then (d) and (e) hold for $g \circ f$. Namely, then $f^!g^!\mathcal{O}_Z$ is bounded above (by the previous point) and $L(g \circ f)^*$ has finite cohomological dimension and (d) follows from (e) which we saw above.

From these points we see it suffices to prove (d) and (e) in case X is affine. Choose an immersion $X \rightarrow \mathbf{A}_Y^n$ (Morphisms, Lemma 39.2) which we factor as $X \rightarrow U \rightarrow \mathbf{A}_Y^n \rightarrow Y$ where $X \rightarrow U$ is a closed immersion and $U \subset \mathbf{A}_Y^n$ is open. Note that $X \rightarrow U$ is a perfect closed immersion by More on Morphisms, Lemma 61.8. Thus it suffices to prove the lemma for a perfect closed immersion and for the projection $\mathbf{A}_Y^n \rightarrow Y$.

Let $f : X \rightarrow Y$ be a perfect closed immersion. We already know (d) holds. Let $K \in D_{Coh}^b(\mathcal{O}_Y)$. Then $f^!K = R\mathcal{H}om(\mathcal{O}_X, K)$ (Lemma 17.4) and $f_*f^!K = R\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K)$. Since f is perfect, the complex $f_*\mathcal{O}_X$ is perfect and hence $R\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K)$ is bounded above. This proves that (d) holds. Some details omitted.

Let $f : \mathbf{A}_Y^n \rightarrow Y$ be the projection. Then (d) holds by repeated application of Lemma 17.3. Finally, (e) is true because it holds for $\mathbf{P}_Y^n \rightarrow Y$ (flat and proper) and because $\mathbf{A}_Y^n \subset \mathbf{P}_Y^n$ is an open. $\square$

0E9T **Lemma 17.10.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . If f is flat, then $f^!\mathcal{O}_Y$ is a Y -perfect object of $D(\mathcal{O}_X)$ and $\mathcal{O}_X \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(f^!\mathcal{O}_Y, f^!\mathcal{O}_Y)$ is an isomorphism.*

Proof. Both assertions are local on X . Thus we may assume X and Y are affine. Then Remark 17.5 turns the lemma into an algebra lemma, namely Dualizing Complexes, Lemma 25.2. (Use Derived Categories of Schemes, Lemma 35.3 to match the languages.) $\square$

0B6V **Lemma 17.11.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . Assume $f : X \rightarrow Y$ is a local complete intersection morphism. Then*

- (1) $f^!\mathcal{O}_Y$ is an invertible object of $D(\mathcal{O}_X)$, and
- (2) $f^!$ maps perfect complexes to perfect complexes.

Proof. Recall that a local complete intersection morphism is perfect, see More on Morphisms, Lemma 62.4. By Lemma 17.9 it suffices to show that $f^!\mathcal{O}_Y$ is an invertible object in $D(\mathcal{O}_X)$. This question is local on X and Y . Hence we may assume that $X \rightarrow Y$ factors as $X \rightarrow \mathbf{A}_Y^n \rightarrow Y$ where the first arrow is a Koszul regular immersion. See More on Morphisms, Section 62. The result holds for $\mathbf{A}_Y^n \rightarrow Y$ by Lemma 17.3. Thus it suffices to prove the lemma when f is a Koszul regular immersion. Working locally once again we reduce to the case $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$, where $A = B/(f_1, \dots, f_r)$ for some regular sequence $f_1, \dots, f_r \in B$ (use that for Noetherian local rings the notion of Koszul regular and regular are the same, see More on Algebra, Lemma 30.7). Thus $X \rightarrow Y$

is a composition

$$X = X_r \rightarrow X_{r-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 = Y$$

where each arrow is the inclusion of an effective Cartier divisor. In this way we reduce to the case of an inclusion of an effective Cartier divisor $i : D \rightarrow X$. In this case $i^! \mathcal{O}_X = \mathcal{N}[1]$ by Lemma 14.1 and the proof is complete. $\square$

18. Base change for upper shriek

0BZX In Situation 16.1 let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram in FTS_S such that X and Y' are Tor independent over Y . Our setup is currently not sufficient to construct a base change map $L(g')^* \circ f^! \rightarrow (f')^! \circ Lg^*$ in this generality. The reason is that in general it will not be possible to choose a compactification $j : X \rightarrow \overline{X}$ over Y such that $\overline{X}$ and Y' are tor independent over Y and hence our construction of the base change map in Section 5 does not apply⁶.

A partial remedy will be found in Section 28. Namely, if the morphism f is flat, then there is a good notion of a relative dualizing complex and using Lemmas 28.9, 28.6, and 17.9 we may construct a canonical base change isomorphism. If we ever need to use this, we will add precise statements and proofs later in this chapter.

0E9U **Lemma 18.1.** *In Situation 16.1 let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of FTS_S with g flat. Then there is an isomorphism $L(g')^ \circ f^! \rightarrow (f')^! \circ Lg^*$ on $D_{Qcoh}^+(\mathcal{O}_Y)$.*

Proof. Namely, because g is flat, for every choice of compactification $j : X \rightarrow \overline{X}$ of X over Y the scheme $\overline{X}$ is Tor independent of Y' . Denote $j' : X' \rightarrow \overline{X}'$ the base change of j and $\overline{g}' : \overline{X}' \rightarrow \overline{X}$ the projection. We define the base change map as the composition

$$L(g')^* \circ f^! = L(g')^* \circ j^* \circ a = (j')^* \circ L(\overline{g}')^* \circ a \longrightarrow (j')^* \circ a' \circ Lg^* = (f')^! \circ Lg^*$$

where the middle arrow is the base change map (5.0.1) and a and a' are the right adjoints to pushforward of Lemma 3.1 for $\overline{X} \rightarrow Y$ and $\overline{X}' \rightarrow Y'$. This construction is independent of the choice of compactification (we will formulate a precise lemma and prove it, if we ever need this result).

⁶The reader who is well versed with derived algebraic geometry will realize this is not a “real” problem. Namely, taking $\overline{X}'$ to be the derived fibre product of $\overline{X}$ and Y' over Y , one can argue exactly as in the proof of Lemma 18.1 to define this map. After all, the Tor independence of X and Y' guarantees that X' will be an open subscheme of the derived scheme $\overline{X}'$.

To finish the proof it suffices to show that the base change map $L(g')^* \circ a \rightarrow a' \circ Lg^*$ is an isomorphism on $D_{Qcoh}^+(\mathcal{O}_Y)$. By Lemma 4.4 formation of a and a' commutes with restriction to affine opens of Y and Y' . Thus by Remark 6.1 we may assume that Y and Y' are affine. Thus the result by Lemma 6.2. $\square$

0FWI **Lemma 18.2.** *In Situation 16.1 let $f : X \rightarrow Y$ be an étale morphism of FTS_S. Then $f^! \cong f^*$ as functors on $D_{Qcoh}^+(\mathcal{O}_Y)$.*

Proof. We are going to use that an étale morphism is flat, syntomic, and a local complete intersection morphism (Morphisms, Lemma 36.10 and 36.12 and More on Morphisms, Lemma 62.8). By Lemma 17.9 it suffices to show $f^! \mathcal{O}_Y = \mathcal{O}_X$. By Lemma 17.11 we know that $f^! \mathcal{O}_Y$ is an invertible module. Consider the commutative diagram

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_2} & X \\ p_1 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

and the diagonal $\Delta : X \rightarrow X \times_Y X$. Since Δ is an open immersion (by Morphisms, Lemmas 35.13 and 36.5), by Lemma 17.1 we have $\Delta^! = \Delta^*$. By Lemma 16.3 we have $\Delta^! \circ p_1^! \circ f^! = f^!$. By Lemma 18.1 applied to the diagram we have $p_1^! \mathcal{O}_X = p_2^* f^! \mathcal{O}_Y$. Hence we conclude

$$f^! \mathcal{O}_Y = \Delta^! p_1^! f^! \mathcal{O}_Y = \Delta^* (p_1^* f^! \mathcal{O}_Y \otimes p_1^! \mathcal{O}_X) = \Delta^* (p_2^* f^! \mathcal{O}_Y \otimes p_1^* f^! \mathcal{O}_Y) = (f^! \mathcal{O}_Y)^{\otimes 2}$$

where in the second step we have used Lemma 17.9 once more. Thus $f^! \mathcal{O}_Y = \mathcal{O}_X$ as desired. $\square$

In the rest of this section, we formulate some easy to prove results which would be consequences of a good theory of the base change map.

0BZY **Lemma 18.3** (Makeshift base change). *In Situation 16.1 let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of FTS_S. Let $E \in D_{Qcoh}^+(\mathcal{O}_Y)$ be an object such that $Lg^ E$ is in $D^+(\mathcal{O}_Y)$. If f is flat, then $L(g')^* f^! E$ and $(f')^! Lg^* E$ restrict to isomorphic objects of $D(\mathcal{O}_{U'})$ for $U' \subset X'$ affine open mapping into affine opens of Y , Y' , and X .*

Proof. By our assumptions we immediately reduce to the case where X, Y, Y' , and X' are affine. Say $Y = \text{Spec}(R)$, $Y' = \text{Spec}(R')$, $X = \text{Spec}(A)$, and $X' = \text{Spec}(A')$. Then $A' = A \otimes_R R'$. Let E correspond to $K \in D^+(R)$. Denoting $\varphi : R \rightarrow A$ and $\varphi' : R' \rightarrow A'$ the given maps we see from Remark 17.5 that $L(g')^* f^! E$ and $(f')^! Lg^* E$ correspond to $\varphi^!(K) \otimes_A^{\mathbf{L}} A'$ and $(\varphi')^!(K \otimes_R^{\mathbf{L}} R')$ where $\varphi^!$ and $(\varphi')^!$ are the functors from Dualizing Complexes, Section 24. The result follows from Dualizing Complexes, Lemma 24.6. $\square$

0BZZ **Lemma 18.4.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S. Assume f is flat. Set $\omega_{X/Y}^\bullet = f^! \mathcal{O}_Y$ in $D_{Coh}^b(X)$. Let $y \in Y$ and $h : X_y \rightarrow X$ the projection. Then $Lh^* \omega_{X/Y}^\bullet$ is a dualizing complex on X_y .*

Proof. The complex $\omega_{X/Y}^\bullet$ is in D_{Coh}^b by Lemma 17.9. Being a dualizing complex is a local property. Hence by Lemma 18.3 it suffices to show that $(X_y \rightarrow y)^! \mathcal{O}_y$ is a dualizing complex on X_y . This follows from Lemma 17.7. $\square$

19. A duality theory

0AU3 In this section we spell out what kind of a duality theory our very general results above give for finite type separated schemes over a fixed Noetherian base scheme.

Recall that a dualizing complex on a Noetherian scheme X , is an object of $D(\mathcal{O}_X)$ which affine locally gives a dualizing complex for the corresponding rings, see Definition 2.2.

Given a Noetherian scheme S denote FTS_S the category of schemes which are of finite type and separated over S . Then:

- (1) the functors $f^!$ turn D_{QCoh}^+ into a pseudo functor on FTS_S ,
- (2) if $f : X \rightarrow Y$ is a proper morphism in FTS_S , then $f^!$ is the restriction of the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ to $D_{QCoh}^+(\mathcal{O}_Y)$ and there is a canonical isomorphism

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, f^! M) \rightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* K, M)$$

for all $K \in D_{Coh}^-(\mathcal{O}_X)$ and $M \in D_{QCoh}^+(\mathcal{O}_Y)$,

- (3) if an object X of FTS_S has a dualizing complex $\omega_X^\bullet$, then the functor $D_X = R\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X^\bullet)$ defines an involution of $D_{Coh}(\mathcal{O}_X)$ switching $D_{Coh}^+(\mathcal{O}_X)$ and $D_{Coh}^-(\mathcal{O}_X)$ and fixing $D_{Coh}^b(\mathcal{O}_X)$,
- (4) if $f : X \rightarrow Y$ is a morphism of FTS_S and $\omega_Y^\bullet$ is a dualizing complex on Y , then
 - (a) $\omega_X^\bullet = f^! \omega_Y^\bullet$ is a dualizing complex for X ,
 - (b) $f^! M = D_X(Lf^* D_Y(M))$ canonically for $M \in D_{QCoh}^+(\mathcal{O}_Y)$, and
 - (c) if in addition f is proper then

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* K, \omega_Y^\bullet)$$

for K in $D_{Coh}^-(\mathcal{O}_X)$,

- (5) if $f : X \rightarrow Y$ is a closed immersion in FTS_S , then $f^!(-) = R\mathcal{H}om(\mathcal{O}_X, -)$,
- (6) if $f : Y \rightarrow X$ is a finite morphism in FTS_S , then $f_* f^!(-) = R\mathcal{H}om_{\mathcal{O}_X}(f_* \mathcal{O}_Y, -)$,
- (7) if $f : X \rightarrow Y$ is the inclusion of an effective Cartier divisor into an object of FTS_S , then $f^!(-) = Lf^*(-) \otimes_{\mathcal{O}_X} f^* \mathcal{O}_Y(X)[-1]$,
- (8) if $f : X \rightarrow Y$ is a Koszul regular immersion of codimension c into an object of FTS_S , then $f^!(-) \cong Lf^*(-) \otimes_{\mathcal{O}_X} \wedge^c \mathcal{N}[-c]$, and
- (9) if $f : X \rightarrow Y$ is a smooth proper morphism of relative dimension d in FTS_S , then $f^!(-) \cong Lf^*(-) \otimes_{\mathcal{O}_X} \Omega_{X/Y}^d[d]$.

This follows from Lemmas 2.5, 3.6, 9.7, 11.4, 14.2, 15.6, 15.7, 16.3, 16.4, 17.4, 17.7, 17.8, and 17.9 and Example 3.9. We have obtained our functors by a very abstract procedure which finally rests on invoking an existence theorem (Derived Categories, Proposition 38.2). This means we have, in general, no explicit description of the functors $f^!$. This can sometimes be a problem. But in fact, it is often enough to know the existence of a dualizing complex and the duality isomorphism to pin down $f^!$.

20. Glueing dualizing complexes

0AU5 We will now use glueing of dualizing complexes to get a theory which works for all finite type schemes over S given a pair $(S, \omega_S^\bullet)$ as in Situation 20.1. This is similar to [Har66, Remark on page 310].

0AU4 **Situation 20.1.** Here S is a Noetherian scheme and $\omega_S^\bullet$ is a dualizing complex.

In Situation 20.1 let X be a scheme of finite type over S . Let $\mathcal{U} : X = \bigcup_{i=1, \dots, n} U_i$ be a finite open covering of X by objects of FTS_S , see Situation 16.1. All this means is that the morphisms $U_i \rightarrow S$ are separated (as they are already of finite type). Every affine scheme of finite type over S is an object of FTS_S by Schemes, Lemma 21.13 hence such open coverings certainly exist. Then for each $i, j, k \in \{1, \dots, n\}$ the morphisms $p_i : U_i \rightarrow S$, $p_{ij} : U_i \cap U_j \rightarrow S$, and $p_{ijk} : U_i \cap U_j \cap U_k \rightarrow S$ are separated and each of these schemes is an object of FTS_S . From such an open covering we obtain

- (1) $\omega_i^\bullet = p_i^! \omega_S^\bullet$ a dualizing complex on U_i , see Section 19,
- (2) for each i, j a canonical isomorphism $\varphi_{ij} : \omega_i^\bullet|_{U_i \cap U_j} \rightarrow \omega_j^\bullet|_{U_i \cap U_j}$, and

0AU6 (3) for each i, j, k we have

$$\varphi_{ik}|_{U_i \cap U_j \cap U_k} = \varphi_{jk}|_{U_i \cap U_j \cap U_k} \circ \varphi_{ij}|_{U_i \cap U_j \cap U_k}$$

in $D(\mathcal{O}_{U_i \cap U_j \cap U_k})$.

Here, in (2) we use that $(U_i \cap U_j \rightarrow U_i)^\dagger$ is given by restriction (Lemma 17.1) and that we have canonical isomorphisms

$$(U_i \cap U_j \rightarrow U_i)^\dagger \circ p_i^! = p_{ij}^! = (U_i \cap U_j \rightarrow U_j)^\dagger \circ p_j^!$$

by Lemma 16.3 and to get (3) we use that the upper shriek functors form a pseudo functor by Lemma 16.4.

In the situation just described a *dualizing complex normalized relative to $\omega_S^\bullet$* and $\mathcal{U}$ is a pair (K, α_i) where $K \in D(\mathcal{O}_X)$ and $\alpha_i : K|_{U_i} \rightarrow \omega_i^\bullet$ are isomorphisms such that φ_{ij} is given by $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$. Since being a dualizing complex on a scheme is a local property we see that dualizing complexes normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$ are indeed dualizing complexes.

0AU7 **Lemma 20.2.** *In Situation 20.1 let X be a scheme of finite type over S and let $\mathcal{U}$ be a finite open covering of X by schemes separated over S . If there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$, then it is unique up to unique isomorphism.*

Proof. If (K, α_i) and (K', α'_i) are two, then we consider $L = R\mathcal{H}om_{\mathcal{O}_X}(K, K')$. By Lemma 2.6 and its proof, this is an invertible object of $D(\mathcal{O}_X)$. Using α_i and α'_i we obtain an isomorphism

$$\alpha_i^t \otimes \alpha'_i : L|_{U_i} \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(\omega_i^\bullet, \omega_i^\bullet) = \mathcal{O}_{U_i}[0]$$

This already implies that $L = H^0(L)[0]$ in $D(\mathcal{O}_X)$. Moreover, $H^0(L)$ is an invertible sheaf with given trivializations on the opens U_i of X . Finally, the condition that $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$ and $\alpha'_j|_{U_i \cap U_j} \circ (\alpha'_i)^{-1}|_{U_i \cap U_j}$ both give φ_{ij} implies that the transition maps are 1 and we get an isomorphism $H^0(L) = \mathcal{O}_X$. $\square$

0AU8 **Lemma 20.3.** *In Situation 20.1 let X be a scheme of finite type over S and let $\mathcal{U}, \mathcal{V}$ be two finite open coverings of X by schemes separated over S . If there exists*

a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$, then there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{V}$ and these complexes are canonically isomorphic.

Proof. It suffices to prove this when $\mathcal{U}$ is given by the opens $U_1, \dots, U_n$ and $\mathcal{V}$ by the opens $U_1, \dots, U_{n+m}$. In fact, we may and do even assume $m = 1$. To go from a dualizing complex (K, α_i) normalized relative to $\omega_S^\bullet$ and $\mathcal{V}$ to a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$ is achieved by forgetting about α_i for $i = n + 1$. Conversely, let (K, α_i) be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$. To finish the proof we need to construct a map $\alpha_{n+1} : K|_{U_{n+1}} \rightarrow \omega_{n+1}^\bullet$ satisfying the desired conditions. To do this we observe that $U_{n+1} = \bigcup U_i \cap U_{n+1}$ is an open covering. It is clear that $(K|_{U_{n+1}}, \alpha_i|_{U_i \cap U_{n+1}})$ is a dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $U_{n+1} = \bigcup U_i \cap U_{n+1}$. On the other hand, by condition (3) the pair $(\omega_{n+1}^\bullet|_{U_{n+1}}, \varphi_{n+1i})$ is another dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $U_{n+1} = \bigcup U_i \cap U_{n+1}$. By Lemma 20.2 we obtain a unique isomorphism

$$\alpha_{n+1} : K|_{U_{n+1}} \longrightarrow \omega_{n+1}^\bullet$$

compatible with the given local isomorphisms. It is a pleasant exercise to show that this means it satisfies the required property. $\square$

0AU9 **Lemma 20.4.** *In Situation 20.1 let X be a scheme of finite type over S and let $\mathcal{U}$ be a finite open covering of X by schemes separated over S . Then there exists a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$.*

Proof. Say $\mathcal{U} : X = \bigcup_{i=1, \dots, n} U_i$. We prove the lemma by induction on n . The base case $n = 1$ is immediate. Assume $n > 1$. Set $X' = U_1 \cup \dots \cup U_{n-1}$ and let $(K', \{\alpha'_i\}_{i=1, \dots, n-1})$ be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}' : X' = \bigcup_{i=1, \dots, n-1} U_i$. It is clear that $(K'|_{X' \cap U_n}, \alpha'_i|_{U_i \cap U_n})$ is a dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $X' \cap U_n = \bigcup_{i=1, \dots, n-1} U_i \cap U_n$. On the other hand, by condition (3) the pair $(\omega_n^\bullet|_{X' \cap U_n}, \varphi_{ni})$ is another dualizing complex normalized relative to $\omega_S^\bullet$ and the covering $X' \cap U_n = \bigcup_{i=1, \dots, n-1} U_i \cap U_n$. By Lemma 20.2 we obtain a unique isomorphism

$$\epsilon : K'|_{X' \cap U_n} \longrightarrow \omega_n^\bullet|_{X' \cap U_n}$$

compatible with the given local isomorphisms. By Cohomology, Lemma 45.1 we obtain $K \in D(\mathcal{O}_X)$ together with isomorphisms $\beta : K|_{X'} \rightarrow K'$ and $\gamma : K|_{U_n} \rightarrow \omega_n^\bullet$ such that $\epsilon = \gamma|_{X' \cap U_n} \circ \beta|_{X' \cap U_n}^{-1}$. Then we define

$$\alpha_i = \alpha'_i \circ \beta|_{U_i}, i = 1, \dots, n-1, \text{ and } \alpha_n = \gamma$$

We still need to verify that φ_{ij} is given by $\alpha_j|_{U_i \cap U_j} \circ \alpha_i^{-1}|_{U_i \cap U_j}$. For $i, j \leq n-1$ this follows from the corresponding condition for α'_i . For $i = j = n$ it is clear as well. If $i < j = n$, then we get

$$\alpha_n|_{U_i \cap U_n} \circ \alpha_i^{-1}|_{U_i \cap U_n} = \gamma|_{U_i \cap U_n} \circ \beta^{-1}|_{U_i \cap U_n} \circ (\alpha'_i)^{-1}|_{U_i \cap U_n} = \epsilon|_{U_i \cap U_n} \circ (\alpha'_i)^{-1}|_{U_i \cap U_n}$$

This is equal to α_{in} exactly because ϵ is the unique map compatible with the maps α'_i and α_{ni} . $\square$

Let $(S, \omega_S^\bullet)$ be as in Situation 20.1. The upshot of the lemmas above is that given any scheme X of finite type over S , there is a pair (K, α_U) given up to unique isomorphism, consisting of an object $K \in D(\mathcal{O}_X)$ and isomorphisms $\alpha_U : K|_U \rightarrow$

$\omega_U^\bullet$ for every open subscheme $U \subset X$ which is separated over S . Here $\omega_U^\bullet = (U \rightarrow S)^! \omega_S^\bullet$ is a dualizing complex on U , see Section 19. Moreover, if $\mathcal{U} : X = \bigcup U_i$ is a finite open covering by opens which are separated over S , then (K, α_{U_i}) is a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$. Namely, uniqueness up to unique isomorphism by Lemma 20.2, existence for one open covering by Lemma 20.4, and the fact that K then works for all open coverings is Lemma 20.3.

0AUA **Definition 20.5.** Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex on S . Let X be a scheme of finite type over S . The complex K constructed above is called the *dualizing complex normalized relative to $\omega_S^\bullet$* and is denoted $\omega_X^\bullet$.

As the terminology suggest, a dualizing complex normalized relative to $\omega_S^\bullet$ is not just an object of the derived category of X but comes equipped with the local isomorphisms described above. This does not conflict with setting $\omega_X^\bullet = p^! \omega_S^\bullet$ where $p : X \rightarrow S$ is the structure morphism if X is separated over S . More generally we have the following sanity check.

0AUB **Lemma 20.6.** Let $(S, \omega_S^\bullet)$ be as in Situation 20.1. Let $f : X \rightarrow Y$ be a morphism of finite type schemes over S . Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Then $\omega_X^\bullet$ is a dualizing complex normalized relative to $\omega_Y^\bullet$.

Proof. This is just a matter of bookkeeping. Choose a finite affine open covering $\mathcal{V} : Y = \bigcup V_j$. For each j choose a finite affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$. Set $\mathcal{U} : X = \bigcup U_{ji}$. The schemes V_j and U_{ji} are separated over S , hence we have the upper shriek functors for $q_j : V_j \rightarrow S$, $p_{ji} : U_{ji} \rightarrow S$ and $f_{ji} : U_{ji} \rightarrow V_j$ and $f'_{ji} : U_{ji} \rightarrow Y$. Let (L, β_j) be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{V}$. Let (K, γ_{ji}) be a dualizing complex normalized relative to $\omega_S^\bullet$ and $\mathcal{U}$. (In other words, $L = \omega_Y^\bullet$ and $K = \omega_X^\bullet$.) We can define

$$\alpha_{ji} : K|_{U_{ji}} \xrightarrow{\gamma_{ji}} p_{ji}^! \omega_S^\bullet = f_{ji}^! q_j^! \omega_S^\bullet \xrightarrow{f_{ji}^! \beta_j^{-1}} f_{ji}^! (L|_{V_j}) = (f'_{ji})^! (L)$$

To finish the proof we have to show that $\alpha_{ji}|_{U_{ji} \cap U_{j'i'}} \circ \alpha_{j'i'}^{-1}|_{U_{ji} \cap U_{j'i'}}$ is the canonical isomorphism $(f'_{ji})^! (L)|_{U_{ji} \cap U_{j'i'}} \rightarrow (f'_{j'i'})^! (L)|_{U_{ji} \cap U_{j'i'}}$. This is formal and we omit the details. $\square$

0AUC **Lemma 20.7.** Let $(S, \omega_S^\bullet)$ be as in Situation 20.1. Let $j : X \rightarrow Y$ be an open immersion of schemes of finite type over S . Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Then there is a canonical isomorphism $\omega_X^\bullet = \omega_Y^\bullet|_X$.

Proof. Immediate from the construction of normalized dualizing complexes given just above Definition 20.5. $\square$

0AUD **Lemma 20.8.** Let $(S, \omega_S^\bullet)$ be as in Situation 20.1. Let $f : X \rightarrow Y$ be a proper morphism of schemes of finite type over S . Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Let a be the right adjoint of Lemma 3.1 for f . Then there is a canonical isomorphism $a(\omega_Y^\bullet) = \omega_X^\bullet$.

Proof. Let $p : X \rightarrow S$ and $q : Y \rightarrow S$ be the structure morphisms. If X and Y are separated over S , then this follows from the fact that $\omega_X^\bullet = p^! \omega_S^\bullet$, $\omega_Y^\bullet = q^! \omega_S^\bullet$, $f^! = a$, and $f^! \circ q^! = p^!$ (Lemma 16.3). In the general case we first use Lemma 20.6 to reduce to the case $Y = S$. In this case X and Y are separated over S and we've just seen the result. $\square$

Let $(S, \omega_S^\bullet)$ be as in Situation 20.1. For a scheme X of finite type over S denote $\omega_X^\bullet$ the dualizing complex for X normalized relative to $\omega_S^\bullet$. Define $D_X(-) = R\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X^\bullet)$ as in Lemma 2.5. Let $f : X \rightarrow Y$ be a morphism of finite type schemes over S . Define

$$f_{new}^! = D_X \circ Lf^* \circ D_Y : D_{Coh}^+(\mathcal{O}_Y) \rightarrow D_{Coh}^+(\mathcal{O}_X)$$

If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are composable morphisms between schemes of finite type over S , define

$$\begin{aligned} (g \circ f)_{new}^! &= D_X \circ L(g \circ f)^* \circ D_Z \\ &= D_X \circ Lf^* \circ Lg^* \circ D_Z \\ &\rightarrow D_X \circ Lf^* \circ D_Y \circ D_Y \circ Lg^* \circ D_Z \\ &= f_{new}^! \circ g_{new}^! \end{aligned}$$

where the arrow is defined in Lemma 2.5. We collect the results together in the following lemma.

0AUE **Lemma 20.9.** *Let $(S, \omega_S^\bullet)$ be as in Situation 20.1. With $f_{new}^!$ and $\omega_X^\bullet$ defined for all (morphisms of) schemes of finite type over S as above:*

- (1) *the functors $f_{new}^!$ and the arrows $(g \circ f)_{new}^! \rightarrow f_{new}^! \circ g_{new}^!$ turn D_{Coh}^+ into a pseudo functor from the category of schemes of finite type over S into the 2-category of categories,*
- (2) *$\omega_X^\bullet = (X \rightarrow S)_{new}^! \omega_S^\bullet$,*
- (3) *the functor D_X defines an involution of $D_{Coh}(\mathcal{O}_X)$ switching $D_{Coh}^+(\mathcal{O}_X)$ and $D_{Coh}^-(\mathcal{O}_X)$ and fixing $D_{Coh}^b(\mathcal{O}_X)$,*
- (4) *$\omega_X^\bullet = f_{new}^! \omega_Y^\bullet$ for $f : X \rightarrow Y$ a morphism of finite type schemes over S ,*
- (5) *$f_{new}^! M = D_X(Lf^* D_Y(M))$ for $M \in D_{Coh}^+(\mathcal{O}_Y)$, and*
- (6) *if in addition f is proper, then $f_{new}^!$ is isomorphic to the restriction of the right adjoint of $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ to $D_{Coh}^+(\mathcal{O}_Y)$ and there is a canonical isomorphism*

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, f_{new}^! M) \rightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* K, M)$$

for $K \in D_{Coh}^-(\mathcal{O}_X)$ and $M \in D_{Coh}^+(\mathcal{O}_Y)$, and

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* K, \omega_Y^\bullet)$$

for $K \in D_{Coh}^-(\mathcal{O}_X)$ and

If X is separated over S , then $\omega_X^\bullet$ is canonically isomorphic to $(X \rightarrow S)_{new}^! \omega_S^\bullet$ and if f is a morphism between schemes separated over S , then there is a canonical isomorphism⁷ $f_{new}^! K = f^! K$ for K in D_{Coh}^+ .

Proof. Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$, $h : Z \rightarrow T$ be morphisms of schemes of finite type over S . We have to show that

$$\begin{array}{ccc} (h \circ g \circ f)_{new}^! & \longrightarrow & f_{new}^! \circ (h \circ g)_{new}^! \\ \downarrow & & \downarrow \\ (g \circ f)_{new}^! \circ h_{new}^! & \longrightarrow & f_{new}^! \circ g_{new}^! \circ h_{new}^! \end{array}$$

⁷We haven't checked that these are compatible with the isomorphisms $(g \circ f)^! \rightarrow f^! \circ g^!$ and $(g \circ f)_{new}^! \rightarrow f_{new}^! \circ g_{new}^!$. We will do this here if we need this later.

is commutative. Let $\eta_Y : \text{id} \rightarrow D_Y^2$ and $\eta_Z : \text{id} \rightarrow D_Z^2$ be the canonical isomorphisms of Lemma 2.5. Then, using Categories, Lemma 28.2, a computation (omitted) shows that both arrows $(h \circ g \circ f)_{new}^! \rightarrow f_{new}^! \circ g_{new}^! \circ h_{new}^!$ are given by

$$1 \star \eta_Y \star 1 \star \eta_Z \star 1 : D_X \circ Lf^* \circ Lg^* \circ Lh^* \circ D_T \longrightarrow D_X \circ Lf^* \circ D_Y^2 \circ Lg^* \circ D_Z^2 \circ Lh^* \circ D_T$$

This proves (1). Part (2) is immediate from the definition of $(X \rightarrow S)_{new}^!$ and the fact that $D_S(\omega_S^\bullet) = \mathcal{O}_S$. Part (3) is Lemma 2.5. Part (4) follows by the same argument as part (2). Part (5) is the definition of $f_{new}^!$.

Proof of (6). Let a be the right adjoint of Lemma 3.1 for the proper morphism $f : X \rightarrow Y$ of schemes of finite type over S . The issue is that we do not know X or Y is separated over S (and in general this won't be true) hence we cannot immediately apply Lemma 17.8 to f over S . To get around this we use the canonical identification $\omega_X^\bullet = a(\omega_Y^\bullet)$ of Lemma 20.8. Hence $f_{new}^!$ is the restriction of a to $D_{Coh}^+(\mathcal{O}_Y)$ by Lemma 17.8 applied to $f : X \rightarrow Y$ over the base scheme Y ! The displayed equalities hold by Example 3.9.

The final assertions follow from the construction of normalized dualizing complexes and the already used Lemma 17.8. $\square$

0BV3 **Remark 20.10.** Let S be a Noetherian scheme which has a dualizing complex. Let $f : X \rightarrow Y$ be a morphism of schemes of finite type over S . Then the functor

$$f_{new}^! : D_{Coh}^+(\mathcal{O}_Y) \rightarrow D_{Coh}^+(\mathcal{O}_X)$$

is independent of the choice of the dualizing complex $\omega_S^\bullet$ up to canonical isomorphism. We sketch the proof. Any second dualizing complex is of the form $\omega_S^\bullet \otimes_{\mathcal{O}_S}^L \mathcal{L}$ where $\mathcal{L}$ is an invertible object of $D(\mathcal{O}_S)$, see Lemma 2.6. For any separated morphism $p : U \rightarrow S$ of finite type we have $p^!(\omega_S^\bullet \otimes_{\mathcal{O}_S}^L \mathcal{L}) = p^!(\omega_S^\bullet) \otimes_{\mathcal{O}_U}^L Lp^* \mathcal{L}$ by Lemma 8.1. Hence, if $\omega_X^\bullet$ and $\omega_Y^\bullet$ are the dualizing complexes normalized relative to $\omega_S^\bullet$ we see that $\omega_X^\bullet \otimes_{\mathcal{O}_X}^L La^* \mathcal{L}$ and $\omega_Y^\bullet \otimes_{\mathcal{O}_Y}^L Lb^* \mathcal{L}$ are the dualizing complexes normalized relative to $\omega_S^\bullet \otimes_{\mathcal{O}_S}^L \mathcal{L}$ (where $a : X \rightarrow S$ and $b : Y \rightarrow S$ are the structure morphisms). Then the result follows as

$$\begin{aligned} & R\mathcal{H}om_{\mathcal{O}_X}(Lf^* R\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet \otimes_{\mathcal{O}_Y}^L Lb^* \mathcal{L}), \omega_X^\bullet \otimes_{\mathcal{O}_X}^L La^* \mathcal{L}) \\ &= R\mathcal{H}om_{\mathcal{O}_X}(Lf^* R(\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet) \otimes_{\mathcal{O}_Y}^L Lb^* \mathcal{L}), \omega_X^\bullet \otimes_{\mathcal{O}_X}^L La^* \mathcal{L}) \\ &= R\mathcal{H}om_{\mathcal{O}_X}(Lf^* R\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet) \otimes_{\mathcal{O}_X}^L La^* \mathcal{L}, \omega_X^\bullet \otimes_{\mathcal{O}_X}^L La^* \mathcal{L}) \\ &= R\mathcal{H}om_{\mathcal{O}_X}(Lf^* R\mathcal{H}om_{\mathcal{O}_Y}(K, \omega_Y^\bullet), \omega_X^\bullet) \end{aligned}$$

for $K \in D_{Coh}^+(\mathcal{O}_Y)$. The last equality because $La^* \mathcal{L}$ is invertible in $D(\mathcal{O}_X)$.

0B6X **Example 20.11.** Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let $f : X \rightarrow Y$ be a proper morphism of finite type schemes over S . Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. In this situation we have $a(\omega_Y^\bullet) = \omega_X^\bullet$ (Lemma 20.8) and hence the trace map (Section 7) is a canonical arrow

$$\text{Tr}_f : Rf_* \omega_X^\bullet \longrightarrow \omega_Y^\bullet$$

which produces the isomorphisms (Lemma 20.9)

$$\text{Hom}_X(L, \omega_X^\bullet) = \text{Hom}_Y(Rf_* L, \omega_Y^\bullet)$$

and

$$Rf_* R\mathcal{H}om_{\mathcal{O}_X}(L, \omega_X^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* L, \omega_Y^\bullet)$$

for L in $D_{Qcoh}(\mathcal{O}_X)$.

0AX4 **Remark 20.12.** Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let $f : X \rightarrow Y$ be a finite morphism between schemes of finite type over S . Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be dualizing complexes normalized relative to $\omega_S^\bullet$. Then we have

$$f_*\omega_X^\bullet = R\mathcal{H}om(f_*\mathcal{O}_X, \omega_Y^\bullet)$$

in $D_{Qcoh}^+(f_*\mathcal{O}_X)$ by Lemmas 11.4 and 20.8 and the trace map of Example 20.11 is the map

$$\mathrm{Tr}_f : Rf_*\omega_X^\bullet = f_*\omega_X^\bullet = R\mathcal{H}om(f_*\mathcal{O}_X, \omega_Y^\bullet) \longrightarrow \omega_Y^\bullet$$

which often goes under the name “evaluation at 1”.

0B6W **Remark 20.13.** Let $f : X \rightarrow Y$ be a flat proper morphism of finite type schemes over a pair $(S, \omega_S^\bullet)$ as in Situation 20.1. The relative dualizing complex (Remark 12.5) is $\omega_{X/Y}^\bullet = a(\mathcal{O}_Y)$. By Lemma 20.8 we have the first canonical isomorphism in

$$\omega_X^\bullet = a(\omega_Y^\bullet) = Lf^*\omega_Y^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}^\bullet$$

in $D(\mathcal{O}_X)$. The second canonical isomorphism follows from the discussion in Remark 12.5.

21. Dimension functions

0BV4 We need a bit more information about how the dimension functions change when passing to a scheme of finite type over another.

0AWL **Lemma 21.1.** *Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let X be a scheme of finite type over S and let $\omega_X^\bullet$ be the dualizing complex normalized relative to $\omega_S^\bullet$. If $x \in X$ is a closed point lying over a closed point s of S , then $\omega_{X,x}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$ provided that $\omega_{S,s}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{S,s}$.*

Proof. We may replace X by an affine neighbourhood of x , hence we may and do assume that $f : X \rightarrow S$ is separated. Then $\omega_X^\bullet = f^!\omega_S^\bullet$. We have to show that $R\mathcal{H}om_{\mathcal{O}_{X,x}}(\kappa(x), \omega_{X,x}^\bullet)$ is sitting in degree 0. Let $i_x : x \rightarrow X$ denote the inclusion morphism which is a closed immersion as x is a closed point. Hence $R\mathcal{H}om_{\mathcal{O}_{X,x}}(\kappa(x), \omega_{X,x}^\bullet)$ represents $i_x^!\omega_X^\bullet$ by Lemma 17.4. Consider the commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{\quad} & X \\ \pi \downarrow & & \downarrow f \\ s & \xrightarrow{\quad} & S \end{array}$$

By Morphisms, Lemma 20.3 the extension $\kappa(x)/\kappa(s)$ is finite and hence π is a finite morphism. We conclude that

$$i_x^!\omega_X^\bullet = i_x^!f^!\omega_S^\bullet = \pi^!i_s^!\omega_S^\bullet$$

Thus if $\omega_{S,s}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{S,s}$, then $i_s^!\omega_S^\bullet = \kappa(s)[0]$ by the same reasoning as above. We have

$$R\pi_*(\pi^!(\kappa(s)[0])) = R\mathcal{H}om_{\mathcal{O}_s}(R\pi_*(\kappa(x)[0]), \kappa(s)[0]) = \widetilde{\mathrm{Hom}_{\kappa(s)}(\kappa(x), \kappa(s))}$$

The first equality by Example 3.9 applied with $L = \kappa(x)[0]$. The second equality holds because π_* is exact. Thus $\pi^!(\kappa(s)[0])$ is supported in degree 0 and we win. $\square$

0AWM **Lemma 21.2.** *Let S be a Noetherian scheme and let $\omega_S^\bullet$ be a dualizing complex. Let $f : X \rightarrow S$ be of finite type and let $\omega_X^\bullet$ be the dualizing complex normalized relative to $\omega_S^\bullet$. For all $x \in X$ we have*

$$\delta_X(x) - \delta_S(f(x)) = \text{trdeg}_{\kappa(f(x))}(\kappa(x))$$

where δ_S , resp. δ_X is the dimension function of $\omega_S^\bullet$, resp. $\omega_X^\bullet$, see Lemma 2.7.

Proof. We may replace X by an affine neighbourhood of x . Hence we may and do assume there is a compactification $X \subset \bar{X}$ over S . Then we may replace X by $\bar{X}$ and assume that X is proper over S . We may also assume X is connected by replacing X by the connected component of X containing x . Next, recall that both δ_X and the function $x \mapsto \delta_S(f(x)) + \text{trdeg}_{\kappa(f(x))}(\kappa(x))$ are dimension functions on X , see Morphisms, Lemma 52.3 (and the fact that S is universally catenary by Lemma 2.7). By Topology, Lemma 20.3 we see that the difference is locally constant, hence constant as X is connected. Thus it suffices to prove equality in any point of X . By Properties, Lemma 5.9 the scheme X has a closed point x . Since $X \rightarrow S$ is proper the image s of x is closed in S . Thus we may apply Lemma 21.1 to conclude. $\square$

0BV5 **Lemma 21.3.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . Let $x \in X$ with image $y \in Y$. Then*

$$H^i(f^! \mathcal{O}_Y)_x \neq 0 \Rightarrow -\dim_x(X_y) \leq i.$$

Proof. Since the statement is local on X we may assume X and Y are affine schemes. Write $X = \text{Spec}(A)$ and $Y = \text{Spec}(R)$. Then $f^! \mathcal{O}_Y$ corresponds to the relative dualizing complex $\omega_{A/R}^\bullet$ of Dualizing Complexes, Section 25 by Remark 17.5. Thus the lemma follows from Dualizing Complexes, Lemma 25.7. $\square$

0BV6 **Lemma 21.4.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . Let $x \in X$ with image $y \in Y$. If f is flat, then*

$$H^i(f^! \mathcal{O}_Y)_x \neq 0 \Rightarrow -\dim_x(X_y) \leq i \leq 0.$$

In fact, if all fibres of f have dimension $\leq d$, then $f^! \mathcal{O}_Y$ has tor-amplitude in $[-d, 0]$ as an object of $D(X, f^{-1} \mathcal{O}_Y)$.

Proof. Arguing exactly as in the proof of Lemma 21.3 this follows from Dualizing Complexes, Lemma 25.8. $\square$

0E9V **Lemma 21.5.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . Let $x \in X$ with image $y \in Y$. Assume*

- (1) $\mathcal{O}_{Y,y}$ is Cohen-Macaulay, and
- (2) $\text{trdeg}_{\kappa(f(\xi))}(\kappa(\xi)) \leq r$ for any generic point ξ of an irreducible component of X containing x .

Then

$$H^i(f^! \mathcal{O}_Y)_x \neq 0 \Rightarrow -r \leq i$$

and the stalk $H^{-r}(f^! \mathcal{O}_Y)_x$ is (S_2) as an $\mathcal{O}_{X,x}$ -module.

Proof. After replacing X by an open neighbourhood of x , we may assume every irreducible component of X passes through x . Then arguing exactly as in the proof of Lemma 21.3 this follows from Dualizing Complexes, Lemma 25.9. $\square$

0BV7 **Lemma 21.6.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . If f is flat and quasi-finite, then*

$$f^! \mathcal{O}_Y = \omega_{X/Y}[0]$$

for some coherent $\mathcal{O}_X$ -module $\omega_{X/Y}$ flat over Y .

Proof. Consequence of Lemma 21.4 and the fact that the cohomology sheaves of $f^! \mathcal{O}_Y$ are coherent by Lemma 17.6. $\square$

0BV8 **Lemma 21.7.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . If f is Cohen-Macaulay (More on Morphisms, Definition 22.1), then*

$$f^! \mathcal{O}_Y = \omega_{X/Y}[d]$$

for some coherent $\mathcal{O}_X$ -module $\omega_{X/Y}$ flat over Y where d is the locally constant function on X which gives the relative dimension of X over Y .

Proof. The relative dimension d is well defined and locally constant by Morphisms, Lemma 29.4. The cohomology sheaves of $f^! \mathcal{O}_Y$ are coherent by Lemma 17.6. We will get flatness of $\omega_{X/Y}$ from Lemma 21.4 if we can show the other cohomology sheaves of $f^! \mathcal{O}_Y$ are zero.

The question is local on X , hence we may assume X and Y are affine and the morphism has relative dimension d . If $d = 0$, then the result follows directly from Lemma 21.6. If $d > 0$, then we may assume there is a factorization

$$X \xrightarrow{g} \mathbf{A}_Y^d \xrightarrow{p} Y$$

with g quasi-finite and flat, see More on Morphisms, Lemma 22.8. Then $f^! = g^! \circ p^!$. By Lemma 17.3 we see that $p^! \mathcal{O}_Y \cong \mathcal{O}_{\mathbf{A}_Y^d}[-d]$. We conclude by the case $d = 0$. $\square$

0BV9 **Remark 21.8.** Let S be a Noetherian scheme endowed with a dualizing complex $\omega_S^\bullet$. In this case Lemmas 21.3, 21.4, 21.6, and 21.7 are true for any morphism $f : X \rightarrow Y$ of finite type schemes over S but with $f^!$ replaced by $f_{new}^!$. This is clear because in each case the proof reduces immediately to the affine case and then $f^! = f_{new}^!$ by Lemma 20.9.

22. Dualizing modules

0AWH This section is a continuation of Dualizing Complexes, Section 19.

Let X be a Noetherian scheme and let $\omega_X^\bullet$ be a dualizing complex. Let $n \in \mathbf{Z}$ be the smallest integer such that $H^n(\omega_X^\bullet)$ is nonzero. In other words, $-n$ is the maximal value of the dimension function associated to $\omega_X^\bullet$ (Lemma 2.7). Sometimes $H^n(\omega_X^\bullet)$ is called a *dualizing module* or *dualizing sheaf* for X and then it is often denoted by ω_X . We will say “let ω_X be a dualizing module” to indicate the above.

Care has to be taken when using dualizing modules ω_X on Noetherian schemes X :

- (1) the integer n may change when passing from X to an open U of X and then it won't be true that $\omega_X|_U = \omega_U$,
- (2) the dualizing complex isn't unique; the dualizing module is only unique up to tensoring by an invertible module.

The second problem will often be irrelevant because we will work with X of finite type over a base change S which is endowed with a fixed dualizing complex $\omega_S^\bullet$ and $\omega_X^\bullet$ will be the dualizing complex normalized relative to $\omega_S^\bullet$. The first problem will not occur if X is equidimensional, more precisely, if the dimension function associated to $\omega_X^\bullet$ (Lemma 2.7) maps every generic point of X to the same integer.

0AWI **Example 22.1.** Say $S = \operatorname{Spec}(A)$ with $(A, \mathfrak{m}, \kappa)$ a local Noetherian ring, and $\omega_S^\bullet$ corresponds to a normalized dualizing complex $\omega_A^\bullet$. Then if $f : X \rightarrow S$ is proper over S and $\omega_X^\bullet = f^! \omega_S^\bullet$ the coherent sheaf

$$\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$$

is a dualizing module and is often called the dualizing module of X (with S and $\omega_S^\bullet$ being understood). We will see that this has good properties.

0AWJ **Example 22.2.** Say X is an equidimensional scheme of finite type over a field k . Then it is customary to take $\omega_X^\bullet$ the dualizing complex normalized relative to $k[0]$ and to refer to

$$\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$$

as the dualizing module of X . If X is separated over k , then $\omega_X^\bullet = f^! \mathcal{O}_{\operatorname{Spec}(k)}$ where $f : X \rightarrow \operatorname{Spec}(k)$ is the structure morphism by Lemma 20.9. If X is proper over k , then this is a special case of Example 22.1.

0AWK **Lemma 22.3.** *Let X be a connected Noetherian scheme and let ω_X be a dualizing module on X . The support of ω_X is the union of the irreducible components of maximal dimension with respect to any dimension function and ω_X is a coherent $\mathcal{O}_X$ -module having property (S_2) .*

Proof. By our conventions discussed above there exists a dualizing complex $\omega_X^\bullet$ such that ω_X is the leftmost nonvanishing cohomology sheaf. Since X is connected, any two dimension functions differ by a constant (Topology, Lemma 20.3). Hence we may use the dimension function associated to $\omega_X^\bullet$ (Lemma 2.7). With these remarks in place, the lemma now follows from Dualizing Complexes, Lemma 17.5 and the definitions (in particular Cohomology of Schemes, Definition 11.1). $\square$

0AWN **Lemma 22.4.** *Let X/A with $\omega_X^\bullet$ and ω_X be as in Example 22.1. Then*

- (1) $H^i(\omega_X^\bullet) \neq 0 \Rightarrow i \in \{-\dim(X), \dots, 0\}$,
- (2) *the dimension of the support of $H^i(\omega_X^\bullet)$ is at most $-i$,*
- (3) *$\operatorname{Supp}(\omega_X)$ is the union of the components of dimension $\dim(X)$, and*
- (4) *ω_X has property (S_2) .*

Proof. Let δ_X and δ_S be the dimension functions associated to $\omega_X^\bullet$ and $\omega_S^\bullet$ as in Lemma 21.2. As X is proper over A , every closed subscheme of X contains a closed point x which maps to the closed point $s \in S$ and $\delta_X(x) = \delta_S(s) = 0$. Hence $\delta_X(\xi) = \dim(\overline{\{\xi\}})$ for any point $\xi \in X$. Hence we can check each of the statements of the lemma by looking at what happens over $\operatorname{Spec}(\mathcal{O}_{X,x})$ in which case the result follows from Dualizing Complexes, Lemmas 16.5 and 17.5. Some details omitted. The last two statements can also be deduced from Lemma 22.3. $\square$

0AWP **Lemma 22.5.** *Let X/A with dualizing module ω_X be as in Example 22.1. Let $d = \dim(X_s)$ be the dimension of the closed fibre. If $\dim(X) = d + \dim(A)$, then the dualizing module ω_X represents the functor*

$$\mathcal{F} \longmapsto \operatorname{Hom}_A(H^d(X, \mathcal{F}), \omega_A)$$

on the category of coherent $\mathcal{O}_X$ -modules.

Proof. We have

$$\begin{aligned}
 \mathrm{Hom}_X(\mathcal{F}, \omega_X) &= \mathrm{Ext}_X^{-\dim(X)}(\mathcal{F}, \omega_X^\bullet) \\
 &= \mathrm{Hom}_X(\mathcal{F}[\dim(X)], \omega_X^\bullet) \\
 &= \mathrm{Hom}_X(\mathcal{F}[\dim(X)], f^!(\omega_A^\bullet)) \\
 &= \mathrm{Hom}_S(Rf_*\mathcal{F}[\dim(X)], \omega_A^\bullet) \\
 &= \mathrm{Hom}_A(H^d(X, \mathcal{F}), \omega_A)
 \end{aligned}$$

The first equality because $H^i(\omega_X^\bullet) = 0$ for $i < -\dim(X)$, see Lemma 22.4 and Derived Categories, Lemma 27.3. The second equality follows from the definition of Ext groups. The third equality is our choice of $\omega_X^\bullet$. The fourth equality holds because $f^!$ is the right adjoint of Lemma 3.1 for f , see Section 19. The final equality holds because $R^i f_*\mathcal{F}$ is zero for $i > d$ (Cohomology of Schemes, Lemma 20.9) and $H^j(\omega_A^\bullet)$ is zero for $j < -\dim(A)$. $\square$

23. Cohen-Macaulay schemes

0AWQ This section is the continuation of Dualizing Complexes, Section 20. Duality takes a particularly simple form for Cohen-Macaulay schemes.

0AWT **Lemma 23.1.** *Let X be a locally Noetherian scheme with dualizing complex $\omega_X^\bullet$.*

- (1) *X is Cohen-Macaulay $\Leftrightarrow \omega_X^\bullet$ locally has a unique nonzero cohomology sheaf,*
- (2) *$\mathcal{O}_{X,x}$ is Cohen-Macaulay $\Leftrightarrow \omega_{X,x}^\bullet$ has a unique nonzero cohomology,*
- (3) *$U = \{x \in X \mid \mathcal{O}_{X,x} \text{ is Cohen-Macaulay}\}$ is open and Cohen-Macaulay.*

If X is connected and Cohen-Macaulay, then there is an integer n and a coherent Cohen-Macaulay $\mathcal{O}_X$ -module ω_X such that $\omega_X^\bullet = \omega_X[-n]$.

Proof. By definition and Dualizing Complexes, Lemma 15.6 for every $x \in X$ the complex $\omega_{X,x}^\bullet$ is a dualizing complex over $\mathcal{O}_{X,x}$. By Dualizing Complexes, Lemma 20.2 we see that (2) holds.

To see (3) assume that $\mathcal{O}_{X,x}$ is Cohen-Macaulay. Let n_x be the unique integer such that $H^{n_x}(\omega_{X,x}^\bullet)$ is nonzero. For an affine neighbourhood $V \subset X$ of x we have $\omega_X^\bullet|_V$ is in $D_{Coh}^b(\mathcal{O}_V)$ hence there are finitely many nonzero coherent modules $H^i(\omega_X^\bullet)|_V$. Thus after shrinking V we may assume only H^{n_x} is nonzero, see Modules, Lemma 9.5. In this way we see that $\mathcal{O}_{X,v}$ is Cohen-Macaulay for every $v \in V$. This proves that U is open as well as a Cohen-Macaulay scheme.

Proof of (1). The implication $\Leftarrow$ follows from (2). The implication $\Rightarrow$ follows from the discussion in the previous paragraph, where we showed that if $\mathcal{O}_{X,x}$ is Cohen-Macaulay, then in a neighbourhood of x the complex $\omega_X^\bullet$ has only one nonzero cohomology sheaf.

Assume X is connected and Cohen-Macaulay. The above shows that the map $x \mapsto n_x$ is locally constant. Since X is connected it is constant, say equal to n . Setting $\omega_X = H^n(\omega_X^\bullet)$ we see that the lemma holds because ω_X is Cohen-Macaulay by Dualizing Complexes, Lemma 20.2 (and Cohomology of Schemes, Definition 11.4). $\square$

0AWU **Lemma 23.2.** *Let X be a locally Noetherian scheme. If there exists a coherent sheaf ω_X such that $\omega_X[0]$ is a dualizing complex on X , then X is a Cohen-Macaulay scheme.*

Proof. This follows immediately from Dualizing Complexes, Lemma 20.3 and our definitions. $\square$

0C0Z **Lemma 23.3.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . Let $x \in X$. If f is flat, then the following are equivalent*

- (1) f is Cohen-Macaulay at x ,
- (2) $f^!\mathcal{O}_Y$ has a unique nonzero cohomology sheaf in a neighbourhood of x .

Proof. One direction of the lemma follows from Lemma 21.7. To prove the converse, we may assume $f^!\mathcal{O}_Y$ has a unique nonzero cohomology sheaf. Let $y = f(x)$. Let $\xi_1, \dots, \xi_n \in X_y$ be the generic points of the fibre X_y specializing to x . Let $d_1, \dots, d_n$ be the dimensions of the corresponding irreducible components of X_y . The morphism $f : X \rightarrow Y$ is Cohen-Macaulay at η_i by More on Morphisms, Lemma 22.7. Hence by Lemma 21.7 we see that $d_1 = \dots = d_n$. If d denotes the common value, then $d = \dim_x(X_y)$. After shrinking X we may assume all fibres have dimension at most d (Morphisms, Lemma 28.4). Then the only nonzero cohomology sheaf $\omega = H^{-d}(f^!\mathcal{O}_Y)$ is flat over Y by Lemma 21.4. Hence, if $h : X_y \rightarrow X$ denotes the canonical morphism, then $Lh^*(f^!\mathcal{O}_Y) = Lh^*(\omega[d]) = (h^*\omega)[d]$ by Derived Categories of Schemes, Lemma 22.8. Thus $h^*\omega[d]$ is the dualizing complex of X_y by Lemma 18.4. Hence X_y is Cohen-Macaulay by Lemma 23.1. This proves f is Cohen-Macaulay at x as desired. $\square$

0C10 **Remark 23.4.** In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . Assume f is a Cohen-Macaulay morphism of relative dimension d . Let $\omega_{X/Y} = H^{-d}(f^!\mathcal{O}_Y)$ be the unique nonzero cohomology sheaf of $f^!\mathcal{O}_Y$, see Lemma 21.7. Then there is a canonical isomorphism

$$f^!K = Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}[d]$$

for $K \in D_{Qcoh}^+(\mathcal{O}_Y)$, see Lemma 17.9. In particular, if S has a dualizing complex $\omega_S^\bullet$, $\omega_Y^\bullet = (Y \rightarrow S)^!\omega_S^\bullet$, and $\omega_X^\bullet = (X \rightarrow S)^!\omega_S^\bullet$ then we have

$$\omega_X^\bullet = Lf^*\omega_Y^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \omega_{X/Y}[d]$$

Thus if further X and Y are connected and Cohen-Macaulay and if ω_Y and ω_X denote the unique nonzero cohomology sheaves of $\omega_Y^\bullet$ and $\omega_X^\bullet$, then we have

$$\omega_X = f^*\omega_Y \otimes_{\mathcal{O}_X} \omega_{X/Y}.$$

Similar results hold for X and Y arbitrary finite type schemes over S (i.e., not necessarily separated over S) with dualizing complexes normalized with respect to $\omega_S^\bullet$ as in Section 20.

24. Gorenstein schemes

0AWV This section is the continuation of Dualizing Complexes, Section 21.

0AWW **Definition 24.1.** Let X be a scheme. We say X is *Gorenstein* if X is locally Noetherian and $\mathcal{O}_{X,x}$ is Gorenstein for all $x \in X$.

This definition makes sense because a Noetherian ring is said to be Gorenstein if and only if all of its local rings are Gorenstein, see Dualizing Complexes, Definition 21.1.

0C00 **Lemma 24.2.** *A Gorenstein scheme is Cohen-Macaulay.*

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 21.2. $\square$

0DWG **Lemma 24.3.** *A regular scheme is Gorenstein.*

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 21.3. $\square$

0BFQ **Lemma 24.4.** *Let X be a locally Noetherian scheme.*

- (1) *If X has a dualizing complex $\omega_X^\bullet$, then*
 - (a) *X is Gorenstein $\Leftrightarrow \omega_X^\bullet$ is an invertible object of $D(\mathcal{O}_X)$,*
 - (b) *$\mathcal{O}_{X,x}$ is Gorenstein $\Leftrightarrow \omega_{X,x}^\bullet$ is an invertible object of $D(\mathcal{O}_{X,x})$,*
 - (c) *$U = \{x \in X \mid \mathcal{O}_{X,x} \text{ is Gorenstein}\}$ is an open Gorenstein subscheme.*
- (2) *If X is Gorenstein, then X has a dualizing complex if and only if $\mathcal{O}_X[0]$ is a dualizing complex.*

Proof. Looking affine locally this follows from the corresponding result in algebra, namely Dualizing Complexes, Lemma 21.4. $\square$

0BVA **Lemma 24.5.** *If $f : Y \rightarrow X$ is a local complete intersection morphism with X a Gorenstein scheme, then Y is Gorenstein.*

Proof. By More on Morphisms, Lemma 62.5 it suffices to prove the corresponding statement about ring maps. This is Dualizing Complexes, Lemma 21.7. $\square$

0C01 **Lemma 24.6.** *The property $\mathcal{P}(S) = \text{“}S \text{ is Gorenstein”}$ is local in the syntomic topology.*

Proof. Let $\{S_i \rightarrow S\}$ be a syntomic covering. The scheme S is locally Noetherian if and only if each S_i is Noetherian, see Descent, Lemma 16.1. Thus we may now assume S and S_i are locally Noetherian. If S is Gorenstein, then each S_i is Gorenstein by Lemma 24.5. Conversely, if each S_i is Gorenstein, then for each point $s \in S$ we can pick i and $t \in S_i$ mapping to s . Then $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{S_i,t}$ is a flat local ring homomorphism with $\mathcal{O}_{S_i,t}$ Gorenstein. Hence $\mathcal{O}_{S,s}$ is Gorenstein by Dualizing Complexes, Lemma 21.8. $\square$

25. Gorenstein morphisms

0C02 This section is one in a series. The corresponding sections for normal morphisms, regular morphisms, and Cohen-Macaulay morphisms can be found in More on Morphisms, Sections 20, 21, and 22.

The following lemma says that it does not make sense to define geometrically Gorenstein schemes, since these would be the same as Gorenstein schemes.

0C03 **Lemma 25.1.** *Let X be a locally Noetherian scheme over the field k . Let k'/k be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x . Then we have*

$$\mathcal{O}_{X,x} \text{ is Gorenstein} \Leftrightarrow \mathcal{O}_{X_{k'},x'} \text{ is Gorenstein}$$

If X is locally of finite type over k , the same holds for any field extension k'/k .

Proof. In both cases the ring map $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X_{k'},x'}$ is a faithfully flat local homomorphism of Noetherian local rings. Thus if $\mathcal{O}_{X_{k'},x'}$ is Gorenstein, then so is $\mathcal{O}_{X,x}$ by Dualizing Complexes, Lemma 21.8. To go up, we use Dualizing Complexes, Lemma 21.8 as well. Thus we have to show that

$$\mathcal{O}_{X_{k'},x'}/\mathfrak{m}_x \mathcal{O}_{X_{k'},x'} = \kappa(x) \otimes_k k'$$

is Gorenstein. Note that in the first case $k \rightarrow k'$ is finitely generated and in the second case $k \rightarrow \kappa(x)$ is finitely generated. Hence this follows as property (A) holds for Gorenstein, see Dualizing Complexes, Lemma 23.1. $\square$

The lemma above guarantees that the following is the correct definition of Gorenstein morphisms.

0C04 **Definition 25.2.** Let $f : X \rightarrow Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

- (1) Let $x \in X$, and $y = f(x)$. We say that f is *Gorenstein at x* if f is flat at x , and the local ring of the scheme X_y at x is Gorenstein.
- (2) We say f is a *Gorenstein morphism* if f is Gorenstein at every point of X .

Here is a translation.

0C05 **Lemma 25.3.** Let $f : X \rightarrow Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent

- (1) f is Gorenstein, and
- (2) f is flat and its fibres are Gorenstein schemes.

Proof. This follows directly from the definitions. $\square$

0C06 **Lemma 25.4.** A Gorenstein morphism is Cohen-Macaulay.

Proof. Follows from Lemma 24.2 and the definitions. $\square$

0C15 **Lemma 25.5.** A syntomic morphism is Gorenstein. Equivalently a flat local complete intersection morphism is Gorenstein.

Proof. Recall that a syntomic morphism is flat and its fibres are local complete intersections over fields, see Morphisms, Lemma 30.11. Since a local complete intersection over a field is a Gorenstein scheme by Lemma 24.5 we conclude. The properties “syntomic” and “flat and local complete intersection morphism” are equivalent by More on Morphisms, Lemma 62.8. $\square$

0C11 **Lemma 25.6.** Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms. Assume that the fibres X_y , Y_z and X_z of f , g , and $g \circ f$ are locally Noetherian.

- (1) If f is Gorenstein at x and g is Gorenstein at $f(x)$, then $g \circ f$ is Gorenstein at x .
- (2) If f and g are Gorenstein, then $g \circ f$ is Gorenstein.
- (3) If $g \circ f$ is Gorenstein at x and f is flat at x , then f is Gorenstein at x and g is Gorenstein at $f(x)$.
- (4) If $g \circ f$ is Gorenstein and f is flat, then f is Gorenstein and g is Gorenstein at every point in the image of f .

Proof. After translating into algebra this follows from Dualizing Complexes, Lemma 21.8. $\square$

0C12 **Lemma 25.7.** *Let $f : X \rightarrow Y$ be a flat morphism of locally Noetherian schemes. If X is Gorenstein, then f is Gorenstein and $\mathcal{O}_{Y,f(x)}$ is Gorenstein for all $x \in X$.*

Proof. After translating into algebra this follows from Dualizing Complexes, Lemma 21.8. $\square$

0C07 **Lemma 25.8.** *Let $f : X \rightarrow Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes. Let $Y' \rightarrow Y$ be locally of finite type. Let $f' : X' \rightarrow Y'$ be the base change of f . Let $x' \in X'$ be a point with image $x \in X$.*

- (1) *If f is Gorenstein at x , then $f' : X' \rightarrow Y'$ is Gorenstein at x' .*
- (2) *If f is flat at x and f' is Gorenstein at x' , then f is Gorenstein at x .*
- (3) *If $Y' \rightarrow Y$ is flat at $f'(x')$ and f' is Gorenstein at x' , then f is Gorenstein at x .*

Proof. Note that the assumption on $Y' \rightarrow Y$ implies that for $y' \in Y'$ mapping to $y \in Y$ the field extension $\kappa(y')/\kappa(y)$ is finitely generated. Hence also all the fibres $X'_{y'} = (X_y)_{\kappa(y')}$ are locally Noetherian, see Varieties, Lemma 11.1. Thus the lemma makes sense. Set $y' = f'(x')$ and $y = f(x)$. Hence we get the following commutative diagram of local rings

$$\begin{array}{ccc} \mathcal{O}_{X',x'} & \longleftarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow \\ \mathcal{O}_{Y',y'} & \longleftarrow & \mathcal{O}_{Y,y} \end{array}$$

where the upper left corner is a localization of the tensor product of the upper right and lower left corners over the lower right corner.

Assume f is Gorenstein at x . The flatness of $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ implies the flatness of $\mathcal{O}_{Y',y'} \rightarrow \mathcal{O}_{X',x'}$, see Algebra, Lemma 100.1. The fact that $\mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}$ is Gorenstein implies that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'} \mathcal{O}_{X',x'}$ is Gorenstein, see Lemma 25.1. Hence we see that f' is Gorenstein at x' .

Assume f is flat at x and f' is Gorenstein at x' . The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'} \mathcal{O}_{X',x'}$ is Gorenstein implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}$ is Gorenstein, see Lemma 25.1. Hence we see that f is Gorenstein at x .

Assume $Y' \rightarrow Y$ is flat at y' and f' is Gorenstein at x' . The flatness of $\mathcal{O}_{Y',y'} \rightarrow \mathcal{O}_{X',x'}$ and $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{Y',y'}$ implies the flatness of $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$, see Algebra, Lemma 100.1. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'} \mathcal{O}_{X',x'}$ is Gorenstein implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}$ is Gorenstein, see Lemma 25.1. Hence we see that f is Gorenstein at x . $\square$

0E0Q **Lemma 25.9.** *Let $f : X \rightarrow Y$ be a morphism of schemes which is flat and locally of finite type. Then formation of the set $\{x \in X \mid f \text{ is Gorenstein at } x\}$ commutes with arbitrary base change.*

Proof. The assumption implies any fibre of f is locally of finite type over a field and hence locally Noetherian and the same is true for any base change. Thus the statement makes sense. Looking at fibres we reduce to the following problem: let X be a scheme locally of finite type over a field k , let K/k be a field extension, and let $x_K \in X_K$ be a point with image $x \in X$. Problem: show that $\mathcal{O}_{X_K, x_K}$ is Gorenstein if and only if $\mathcal{O}_{X,x}$ is Gorenstein.

The problem can be solved using a bit of algebra as follows. Choose an affine open $\text{Spec}(A) \subset X$ containing x . Say x corresponds to $\mathfrak{p} \subset A$. With $A_K = A \otimes_k K$ we see that $\text{Spec}(A_K) \subset X_K$ contains x_K . Say x_K corresponds to $\mathfrak{p}_K \subset A_K$. Let $\omega_A^\bullet$ be a dualizing complex for A . By Dualizing Complexes, Lemma 25.3 $\omega_A^\bullet \otimes_A A_K$ is a dualizing complex for A_K . Now we are done because $A_{\mathfrak{p}} \rightarrow (A_K)_{\mathfrak{p}_K}$ is a flat local homomorphism of Noetherian rings and hence $(\omega_A^\bullet)_{\mathfrak{p}}$ is an invertible object of $D(A_{\mathfrak{p}})$ if and only if $(\omega_A^\bullet)_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} (A_K)_{\mathfrak{p}_K}$ is an invertible object of $D((A_K)_{\mathfrak{p}_K})$. Some details omitted; hint: look at cohomology modules. $\square$

0C08 **Lemma 25.10.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTSs. Let $x \in X$. If f is flat, then the following are equivalent*

- (1) f is Gorenstein at x ,
- (2) $f^! \mathcal{O}_Y$ is isomorphic to an invertible object in a neighbourhood of x .

In particular, the set of points where f is Gorenstein is open in X .

Proof. Set $\omega^\bullet = f^! \mathcal{O}_Y$. By Lemma 18.4 this is a bounded complex with coherent cohomology sheaves whose derived restriction $Lh^* \omega^\bullet$ to the fibre X_y is a dualizing complex on X_y . Denote $i : x \rightarrow X_y$ the inclusion of a point. Then the following are equivalent

- (1) f is Gorenstein at x ,
- (2) $\mathcal{O}_{X_y, x}$ is Gorenstein,
- (3) $Lh^* \omega^\bullet$ is invertible in a neighbourhood of x ,
- (4) $Li^* Lh^* \omega^\bullet$ has exactly one nonzero cohomology of dimension 1 over $\kappa(x)$,
- (5) $L(h \circ i)^* \omega^\bullet$ has exactly one nonzero cohomology of dimension 1 over $\kappa(x)$,
- (6) $\omega^\bullet$ is invertible in a neighbourhood of x .

The equivalence of (1) and (2) is by definition (as f is flat). The equivalence of (2) and (3) follows from Lemma 24.4. The equivalence of (3) and (4) follows from More on Algebra, Lemma 77.1. The equivalence of (4) and (5) holds because $Li^* Lh^* = L(h \circ i)^*$. The equivalence of (5) and (6) holds by More on Algebra, Lemma 77.1. Thus the lemma is clear. $\square$

0C09 **Lemma 25.11.** *Let $f : X \rightarrow S$ be a morphism of schemes which is flat and locally of finite presentation. Let $x \in X$ with image $s \in S$. Set $d = \dim_x(X_s)$. The following are equivalent*

- (1) f is Gorenstein at x ,
- (2) there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite morphism $U \rightarrow \mathbf{A}_S^d$ over S which is Gorenstein at x ,
- (3) there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite Gorenstein morphism $U \rightarrow \mathbf{A}_S^d$ over S ,
- (4) for any S -morphism $g : U \rightarrow \mathbf{A}_S^d$ of an open neighbourhood $U \subset X$ of x we have: g is quasi-finite at $x \Rightarrow g$ is Gorenstein at x .

In particular, the set of points where f is Gorenstein is open in X .

Proof. Choose affine open $U = \text{Spec}(A) \subset X$ with $x \in U$ and $V = \text{Spec}(R) \subset S$ with $f(U) \subset V$. Then $R \rightarrow A$ is a flat ring map of finite presentation. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x . After replacing A by a principal localization we may assume there exists a quasi-finite map $R[x_1, \dots, x_d] \rightarrow A$, see Algebra, Lemma

125.2. Thus there exists at least one pair (U, g) consisting of an open neighbourhood $U \subset X$ of x and a locally⁸ quasi-finite morphism $g : U \rightarrow \mathbf{A}_S^d$.

Having said this, the lemma translates into the following algebra problem (translation omitted). Given $R \rightarrow A$ flat and of finite presentation, a prime $\mathfrak{p} \subset A$ and $\varphi : R[x_1, \dots, x_d] \rightarrow A$ quasi-finite at $\mathfrak{p}$ the following are equivalent

- (a) $\text{Spec}(\varphi)$ is Gorenstein at $\mathfrak{p}$, and
- (b) $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is Gorenstein at $\mathfrak{p}$.
- (c) $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is Gorenstein in an open neighbourhood of $\mathfrak{p}$.

In each case $R[x_1, \dots, x_n] \rightarrow A$ is flat at $\mathfrak{p}$ hence by openness of flatness (Algebra, Theorem 129.4), we may assume $R[x_1, \dots, x_n] \rightarrow A$ is flat (replace A by a suitable principal localization). By Algebra, Lemma 168.1 there exists $R_0 \subset R$ and $R_0[x_1, \dots, x_n] \rightarrow A_0$ such that R_0 is of finite type over $\mathbf{Z}$ and $R_0 \rightarrow A_0$ is of finite type and $R_0[x_1, \dots, x_n] \rightarrow A_0$ is flat. Note that the set of points where a flat finite type morphism is Gorenstein commutes with base change by Lemma 25.8. In this way we reduce to the case where R is Noetherian.

Thus we may assume X and S affine and that we have a factorization of f of the form

$$X \xrightarrow{g} \mathbf{A}_S^n \xrightarrow{p} S$$

with g flat and quasi-finite and S Noetherian. Then X and $\mathbf{A}_S^n$ are separated over S and we have

$$f^! \mathcal{O}_S = g^! p^! \mathcal{O}_S = g^! \mathcal{O}_{\mathbf{A}_S^n}[n]$$

by known properties of upper shriek functors (Lemmas 16.3 and 17.3). Hence the equivalence of (a), (b), and (c) by Lemma 25.10. $\square$

0C0A **Lemma 25.12.** *The property $\mathcal{P}(f)$ = “the fibres of f are locally Noetherian and f is Gorenstein” is local in the fppf topology on the target and local in the syntomic topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f)$ where $\mathcal{P}_1(f)$ = “ f is flat”, and $\mathcal{P}_2(f)$ = “the fibres of f are locally Noetherian and Gorenstein”. We know that $\mathcal{P}_1$ is local in the fppf topology on the source and the target, see Descent, Lemmas 23.15 and 27.1. Thus we have to deal with $\mathcal{P}_2$.

Let $f : X \rightarrow Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \rightarrow Y\}_{i \in I}$ be an fppf covering of Y . Denote $f_i : X_i \rightarrow Y_i$ the base change of f by φ_i . Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i, y_i} = \text{Spec}(\kappa(y_i)) \times_{\text{Spec}(\kappa(y))} X_y.$$

and that $\kappa(y_i)/\kappa(y)$ is a finitely generated field extension. Hence if X_y is locally Noetherian, then X_{i, y_i} is locally Noetherian, see Varieties, Lemma 11.1. And if in addition X_y is Gorenstein, then X_{i, y_i} is Gorenstein, see Lemma 25.1. Thus $\mathcal{P}_2$ is fppf local on the target.

Let $\{X_i \rightarrow X\}$ be a syntomic covering of X . Let $y \in Y$. In this case $\{X_{i, y} \rightarrow X_y\}$ is a syntomic covering of the fibre. Hence the locality of $\mathcal{P}_2$ for the syntomic topology on the source follows from Lemma 24.6. $\square$

⁸If S is quasi-separated, then g will be quasi-finite.

26. More on dualizing complexes

0E4M Some lemmas which don't fit anywhere else very well.

0E4N **Lemma 26.1.** *Let $f : X \rightarrow Y$ be a morphism of locally Noetherian schemes. Assume*

- (1) *f is syntomic and surjective, or*
- (2) *f is a surjective flat local complete intersection morphism, or*
- (3) *f is a surjective Gorenstein morphism of finite type.*

*Then $K \in D_{QCoh}(\mathcal{O}_Y)$ is a dualizing complex on Y if and only if Lf^*K is a dualizing complex on X .*

Proof. Taking affine opens and using Derived Categories of Schemes, Lemma 3.5 this translates into Dualizing Complexes, Lemma 26.2. $\square$

27. Duality for proper schemes over fields

0FVU In this section we work out the consequences of the very general material above on dualizing complexes and duality for proper schemes over fields.

0FVV **Lemma 27.1.** *Let X be a proper scheme over a field k . There exists a dualizing complex $\omega_X^\bullet$ with the following properties*

- (1) *$H^i(\omega_X^\bullet)$ is nonzero only for $i \in [-\dim(X), 0]$,*
- (2) *$\omega_X = H^{-\dim(X)}(\omega_X^\bullet)$ is a coherent (S_2) -module whose support is the irreducible components of dimension $\dim(X)$,*
- (3) *the dimension of the support of $H^i(\omega_X^\bullet)$ is at most $-i$,*
- (4) *for $x \in X$ closed the module $H^i(\omega_{X,x}^\bullet) \oplus \dots \oplus H^0(\omega_{X,x}^\bullet)$ is nonzero if and only if $\text{depth}(\mathcal{O}_{X,x}) \leq -i$,*
- (5) *for $K \in D_{QCoh}(\mathcal{O}_X)$ there are functorial isomorphisms⁹*

$$\text{Ext}_X^i(K, \omega_X^\bullet) = \text{Hom}_k(H^{-i}(X, K), k)$$

compatible with shifts and distinguished triangles,

- (6) *there are functorial isomorphisms $\text{Hom}(\mathcal{F}, \omega_X) = \text{Hom}_k(H^{\dim(X)}(X, \mathcal{F}), k)$ for $\mathcal{F}$ quasi-coherent on X , and*
- (7) *if $X \rightarrow \text{Spec}(k)$ is smooth of relative dimension d , then $\omega_X^\bullet \cong \wedge^d \Omega_{X/k}[d]$ and $\omega_X \cong \wedge^d \Omega_{X/k}$.*

Proof. Denote $f : X \rightarrow \text{Spec}(k)$ the structure morphism. Let a be the right adjoint of pushforward of this morphism, see Lemma 3.1. Consider the relative dualizing complex

$$\omega_X^\bullet = a(\mathcal{O}_{\text{Spec}(k)})$$

Compare with Remark 12.5. Since f is proper we have $f^!(\mathcal{O}_{\text{Spec}(k)}) = a(\mathcal{O}_{\text{Spec}(k)})$ by definition, see Section 16. Applying Lemma 17.7 we find that $\omega_X^\bullet$ is a dualizing complex. Moreover, we see that $\omega_X^\bullet$ and ω_X are as in Example 22.1 and as in Example 22.2.

Parts (1), (2), and (3) follow from Lemma 22.4.

⁹This property characterizes $\omega_X^\bullet$ in $D_{QCoh}(\mathcal{O}_X)$ up to unique isomorphism by the Yoneda lemma. Since $\omega_X^\bullet$ is in $D_{Coh}^b(\mathcal{O}_X)$ in fact it suffices to consider $K \in D_{Coh}^b(\mathcal{O}_X)$.

For a closed point $x \in X$ we see that $\omega_{X,x}^\bullet$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$, see Lemma 21.1. Part (4) then follows from Dualizing Complexes, Lemma 20.1.

Part (5) holds by construction as a is the right adjoint to $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D(\mathcal{O}_{\text{Spec}(k)}) = D(k)$ which we can identify with $K \mapsto R\Gamma(X, K)$. We also use that the derived category $D(k)$ of k -modules is the same as the category of graded k -vector spaces.

Part (6) follows from Lemma 22.5 for coherent $\mathcal{F}$ and in general by unwinding (5) for $K = \mathcal{F}[0]$ and $i = -\dim(X)$.

Part (7) follows from Lemma 15.7. $\square$

0FVW **Remark 27.2.** Let k , X , and $\omega_X^\bullet$ be as in Lemma 27.1. The identity on the complex $\omega_X^\bullet$ corresponds, via the functorial isomorphism in part (5), to a map

$$t : H^0(X, \omega_X^\bullet) \longrightarrow k$$

For an arbitrary K in $D_{QCoh}(\mathcal{O}_X)$ the identification $\text{Hom}(K, \omega_X^\bullet)$ with $H^0(X, K)^\vee$ in part (5) corresponds to the pairing

$$\text{Hom}_X(K, \omega_X^\bullet) \times H^0(X, K) \longrightarrow k, \quad (\alpha, \beta) \longmapsto t(\alpha(\beta))$$

This follows from the functoriality of the isomorphisms in (5). Similarly for any $i \in \mathbf{Z}$ we get the pairing

$$\text{Ext}_X^i(K, \omega_X^\bullet) \times H^{-i}(X, K) \longrightarrow k, \quad (\alpha, \beta) \longmapsto t(\alpha(\beta))$$

Here we think of α as a morphism $K[-i] \rightarrow \omega_X^\bullet$ and β as an element of $H^0(X, K[-i])$ in order to define $\alpha(\beta)$. Observe that if K is general, then we only know that this pairing is nondegenerate on one side: the pairing induces an isomorphism of $\text{Hom}_X(K, \omega_X^\bullet)$, resp. $\text{Ext}_X^i(K, \omega_X^\bullet)$ with the k -linear dual of $H^0(X, K)$, resp. $H^{-i}(X, K)$ but in general not vice versa. If K is in $D_{Coh}^b(\mathcal{O}_X)$, then $\text{Hom}_X(K, \omega_X^\bullet)$, $\text{Ext}_X^i(K, \omega_X^\bullet)$, $H^0(X, K)$, and $H^{-i}(X, K)$ are finite dimensional k -vector spaces (by Derived Categories of Schemes, Lemmas 11.5 and 11.4) and the pairings are perfect in the usual sense.

0FVX **Remark 27.3.** We continue the discussion in Remark 27.2 and we use the same notation k , X , $\omega_X^\bullet$, and t . If $\mathcal{F}$ is a coherent $\mathcal{O}_X$ -module we obtain perfect pairings

$$\langle -, - \rangle : \text{Ext}_X^i(\mathcal{F}, \omega_X^\bullet) \times H^{-i}(X, \mathcal{F}) \longrightarrow k, \quad (\alpha, \beta) \longmapsto t(\alpha(\beta))$$

of finite dimensional k -vector spaces. These pairings satisfy the following (obvious) functoriality: if $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a homomorphism of coherent $\mathcal{O}_X$ -modules, then we have

$$\langle \alpha \circ \varphi, \beta \rangle = \langle \alpha, \varphi(\beta) \rangle$$

for $\alpha \in \text{Ext}_X^i(\mathcal{G}, \omega_X^\bullet)$ and $\beta \in H^{-i}(X, \mathcal{F})$. In other words, the k -linear map $\text{Ext}_X^i(\mathcal{G}, \omega_X^\bullet) \rightarrow \text{Ext}_X^i(\mathcal{F}, \omega_X^\bullet)$ induced by φ is, via the pairings, the k -linear dual of the k -linear map $H^{-i}(X, \mathcal{F}) \rightarrow H^{-i}(X, \mathcal{G})$ induced by φ . Formulated in this manner, this still works if φ is a homomorphism of quasi-coherent $\mathcal{O}_X$ -modules.

0FVY **Lemma 27.4.** Let k , X , and $\omega_X^\bullet$ be as in Lemma 27.1. Let $t : H^0(X, \omega_X^\bullet) \rightarrow k$ be as in Remark 27.2. Let $E \in D(\mathcal{O}_X)$ be perfect. Then the pairings

$$H^i(X, \omega_X^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} E^\vee) \times H^{-i}(X, E) \longrightarrow k, \quad (\xi, \eta) \longmapsto t((1_{\omega_X^\bullet} \otimes \epsilon)(\xi \cup \eta))$$

are perfect for all i . Here $\cup$ denotes the cupproduct of Cohomology, Section 31 and $\epsilon : E^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} E \rightarrow \mathcal{O}_X$ is as in Cohomology, Example 50.7.

Proof. By replacing E with $E[-i]$ this reduces to the case $i = 0$. By Cohomology, Lemma 51.2 we see that the pairing is the same as the one discussed in Remark 27.2 whence the result by the discussion in that remark. $\square$

0FVZ **Lemma 27.5.** *Let X be a proper scheme over a field k which is Cohen-Macaulay and equidimensional of dimension d . The module ω_X of Lemma 27.1 has the following properties*

- (1) ω_X is a dualizing module on X (Section 22),
- (2) ω_X is a coherent Cohen-Macaulay module whose support is X ,
- (3) there are functorial isomorphisms $\text{Ext}_X^i(K, \omega_X[d]) = \text{Hom}_k(H^{-i}(X, K), k)$ compatible with shifts and distinguished triangles for $K \in D_{Q\text{Coh}}(X)$,
- (4) there are functorial isomorphisms $\text{Ext}^{d-i}(\mathcal{F}, \omega_X) = \text{Hom}_k(H^i(X, \mathcal{F}), k)$ for $\mathcal{F}$ quasi-coherent on X .

Proof. It is clear from Lemma 27.1 that ω_X is a dualizing module (as it is the left most nonvanishing cohomology sheaf of a dualizing complex). We have $\omega_X^\bullet = \omega_X[d]$ and ω_X is Cohen-Macaulay as X is Cohen-Macaulay, see Lemma 23.1. The other statements follow from this combined with the corresponding statements of Lemma 27.1. $\square$

0FW0 **Remark 27.6.** Let X be a proper Cohen-Macaulay scheme over a field k which is equidimensional of dimension d . Let $\omega_X^\bullet$ and ω_X be as in Lemma 27.1. By Lemma 27.5 we have $\omega_X^\bullet = \omega_X[d]$. Let $t : H^d(X, \omega_X) \rightarrow k$ be the map of Remark 27.2. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$ -module with dual $\mathcal{E}^\vee$. Then we have perfect pairings

$$H^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{E}^\vee) \times H^{d-i}(X, \mathcal{E}) \longrightarrow k, \quad (\xi, \eta) \longmapsto t(1 \otimes \epsilon)(\xi \cup \eta)$$

where $\cup$ is the cup-product and $\epsilon : \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \mathcal{O}_X$ is the evaluation map. This is a special case of Lemma 27.4.

Here is a sanity check for the dualizing complex.

0FW1 **Lemma 27.7.** *Let X be a proper scheme over a field k . Let $\omega_X^\bullet$ and ω_X be as in Lemma 27.1.*

- (1) *If $X \rightarrow \text{Spec}(k)$ factors as $X \rightarrow \text{Spec}(k') \rightarrow \text{Spec}(k)$ for some field k' , then $\omega_X^\bullet$ and ω_X are as in Lemma 27.1 for the morphism $X \rightarrow \text{Spec}(k')$.*
- (2) *If K/k is a field extension, then the pullback of $\omega_X^\bullet$ and ω_X to the base change X_K are as in Lemma 27.1 for the morphism $X_K \rightarrow \text{Spec}(K)$.*

Proof. Denote $f : X \rightarrow \text{Spec}(k)$ the structure morphism and denote $f' : X \rightarrow \text{Spec}(k')$ the given factorization. In the proof of Lemma 27.1 we took $\omega_X^\bullet = a(\mathcal{O}_{\text{Spec}(k)})$ where a is the right adjoint of Lemma 3.1 for f . Thus we have to show $a(\mathcal{O}_{\text{Spec}(k)}) \cong a'(\mathcal{O}_{\text{Spec}(k)})$ where a' is the right adjoint of Lemma 3.1 for f' . Since $k' \subset H^0(X, \mathcal{O}_X)$ we see that k'/k is a finite extension (Cohomology of Schemes, Lemma 19.2). By uniqueness of adjoints we have $a = a' \circ b$ where b is the right adjoint of Lemma 3.1 for $g : \text{Spec}(k') \rightarrow \text{Spec}(k)$. Another way to say this: we have $f^! = (f')^! \circ g^!$. Thus it suffices to show that $\text{Hom}_k(k', k) \cong k'$ as k' -modules, see Example 3.2. This holds because these are k' -vector spaces of the same dimension (namely dimension 1).

Proof of (2). This holds because we have base change for a by Lemma 6.2. See discussion in Remark 12.5. $\square$

28. Relative dualizing complexes

0E2S For a proper, flat morphism of finite presentation we have a rigid relative dualizing complex, see Remark 12.5 and Lemma 12.8. For a separated and finite type morphism $f : X \rightarrow Y$ of Noetherian schemes, we can consider $f^! \mathcal{O}_Y$. In this section we define relative dualizing complexes for morphisms which are flat and locally of finite presentation (but not necessarily quasi-separated or quasi-compact) between schemes (not necessarily locally Noetherian). We show such complexes exist, are unique up to unique isomorphism, and agree with the cases mentioned above. Before reading this section, please read Dualizing Complexes, Section 27.

0E2T **Definition 28.1.** Let $X \rightarrow S$ be a morphism of schemes which is flat and locally of finite presentation. Let $W \subset X \times_S X$ be any open such that the diagonal $\Delta_{X/S} : X \rightarrow X \times_S X$ factors through a closed immersion $\Delta : X \rightarrow W$. A *relative dualizing complex* is a pair (K, ξ) consisting of an object $K \in D(\mathcal{O}_X)$ and a map

$$\xi : \Delta_* \mathcal{O}_X \longrightarrow L\mathrm{pr}_1^* K|_W$$

in $D(\mathcal{O}_W)$ such that

- (1) K is S -perfect (Derived Categories of Schemes, Definition 35.1), and
- (2) ξ defines an isomorphism of $\Delta_* \mathcal{O}_X$ with $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_* \mathcal{O}_X, L\mathrm{pr}_1^* K|_W)$.

By Lemma 9.3 condition (2) is equivalent to the existence of an isomorphism

$$\mathcal{O}_X \longrightarrow R\mathcal{H}om(\mathcal{O}_X, L\mathrm{pr}_1^* K|_W)$$

in $D(\mathcal{O}_X)$ whose pushforward via Δ is equal to ξ . Since $R\mathcal{H}om(\mathcal{O}_X, L\mathrm{pr}_1^* K|_W)$ is independent of the choice of the open W , so is the category of pairs (K, ξ) . If $X \rightarrow S$ is separated, then we can choose $W = X \times_S X$. We will reduce many of the arguments to the case of rings using the following lemma.

0E2U **Lemma 28.2.** *Let $X \rightarrow S$ be a morphism of schemes which is flat and locally of finite presentation. Let (K, ξ) be a relative dualizing complex. Then for any commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & S \end{array}$$

whose horizontal arrows are open immersions, the restriction of K to $\mathrm{Spec}(A)$ corresponds via Derived Categories of Schemes, Lemma 3.5 to a relative dualizing complex for $R \rightarrow A$ in the sense of Dualizing Complexes, Definition 27.1.

Proof. Since formation of $R\mathcal{H}om$ commutes with restrictions to opens we may as well assume $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$. Observe that relatively perfect objects of $D(\mathcal{O}_X)$ are pseudo-coherent and hence are in $D_{Q\mathrm{Coh}}(\mathcal{O}_X)$ (Derived Categories of Schemes, Lemma 10.1). Thus the statement makes sense. Observe that taking Δ_* , $L\mathrm{pr}_1^*$, and $R\mathcal{H}om$ is compatible with what happens on the algebraic side by Derived Categories of Schemes, Lemmas 3.7, 3.8, 10.8. For the last one we observe that $L\mathrm{pr}_1^* K$ is S -perfect (hence bounded below) and that $\Delta_* \mathcal{O}_X$ is a pseudo-coherent object of $D(\mathcal{O}_W)$; translated into algebra this means that A is pseudo-coherent as

an $A \otimes_R A$ -module which follows from More on Algebra, Lemma 82.8 applied to $R \rightarrow A \otimes_R A \rightarrow A$. Thus we recover exactly the conditions in Dualizing Complexes, Definition 27.1. $\square$

0E2V **Lemma 28.3.** *Let $X \rightarrow S$ be a morphism of schemes which is flat and locally of finite presentation. Let (K, ξ) be a relative dualizing complex. Then $\mathcal{O}_X \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(K, K)$ is an isomorphism.*

Proof. Looking affine locally this reduces using Lemma 28.2 to the algebraic case which is Dualizing Complexes, Lemma 27.5. $\square$

0E2W **Lemma 28.4.** *Let $X \rightarrow S$ be a morphism of schemes which is flat and locally of finite presentation. If (K, ξ) and (L, η) are two relative dualizing complexes on X/S , then there is a unique isomorphism $K \rightarrow L$ sending ξ to η .*

Proof. Let $U \subset X$ be an affine open mapping into an affine open of S . Then there is an isomorphism $K|_U \rightarrow L|_U$ by Lemma 28.2 and Dualizing Complexes, Lemma 27.2. The reader can reuse the argument of that lemma in the schemes case to obtain a proof in this case. We will instead use a glueing argument.

Suppose we have an isomorphism $\alpha : K \rightarrow L$. Then $\alpha(\xi) = u\eta$ for some invertible section $u \in H^0(W, \Delta_* \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$. (Because both η and $\alpha(\xi)$ are generators of an invertible $\Delta_* \mathcal{O}_X$ -module by assumption.) Hence after replacing α by $u^{-1}\alpha$ we see that $\alpha(\xi) = \eta$. Since the automorphism group of K is $H^0(X, \mathcal{O}_X^*)$ by Lemma 28.3 there is at most one such α .

Let $\mathcal{B}$ be the collection of affine opens of X which map into an affine open of S . For each $U \in \mathcal{B}$ we have a unique isomorphism $\alpha_U : K|_U \rightarrow L|_U$ mapping ξ to η by the discussion in the previous two paragraphs. Observe that $\text{Ext}^i(K|_U, K|_U) = 0$ for $i < 0$ and any open U of X by Lemma 28.3. By Cohomology, Lemma 45.2 applied to $\text{id} : X \rightarrow X$ we get a unique morphism $\alpha : K \rightarrow L$ agreeing with α_U for all $U \in \mathcal{B}$. Then α sends ξ to η as this is true locally. $\square$

0E2X **Lemma 28.5.** *Let $X \rightarrow S$ be a morphism of schemes which is flat and locally of finite presentation. There exists a relative dualizing complex (K, ξ) .*

Proof. Let $\mathcal{B}$ be the collection of affine opens of X which map into an affine open of S . For each U we have a relative dualizing complex (K_U, ξ_U) for U over S . Namely, choose an affine open $V \subset S$ such that $U \rightarrow X \rightarrow S$ factors through V . Write $U = \text{Spec}(A)$ and $V = \text{Spec}(R)$. By Dualizing Complexes, Lemma 27.4 there exists a relative dualizing complex $K_A \in D(A)$ for $R \rightarrow A$. Arguing backwards through the proof of Lemma 28.2 this determines an V -perfect object $K_U \in D(\mathcal{O}_U)$ and a map

$$\xi : \Delta_* \mathcal{O}_U \rightarrow L\text{pr}_1^* K_U$$

in $D(\mathcal{O}_{U \times_V U})$. Since being V -perfect is the same as being S -perfect and since $U \times_V U = U \times_S U$ we find that (K_U, ξ_U) is as desired.

If $U' \subset U \subset X$ with $U', U \in \mathcal{B}$, then we have a unique isomorphism $\rho_{U'}^U : K_U|_{U'} \rightarrow K_{U'}$ in $D(\mathcal{O}_{U'})$ sending $\xi_U|_{U' \times_S U'}$ to $\xi_{U'}$ by Lemma 28.4 (note that trivially the restriction of a relative dualizing complex to an open is a relative dualizing complex). The uniqueness guarantees that $\rho_{U''}^U = \rho_{U''}^{U'} \circ \rho_{U'}^U$ for $U'' \subset U' \subset U$ in $\mathcal{B}$. Observe that $\text{Ext}^i(K_U, K_U) = 0$ for $i < 0$ for $U \in \mathcal{B}$ by Lemma 28.3 applied to U/S and K_U . Thus the BBD glueing lemma (Cohomology, Theorem 45.8) tells

us there is a unique solution, namely, an object $K \in D(\mathcal{O}_X)$ and isomorphisms $\rho_U : K|_U \rightarrow K_U$ such that we have $\rho_{U'}^U \circ \rho_U|_{U'} = \rho_{U'}$ for all $U' \subset U$, $U, U' \in \mathcal{B}$.

To finish the proof we have to construct the map

$$\xi : \Delta_* \mathcal{O}_X \longrightarrow L\mathrm{pr}_1^* K|_W$$

in $D(\mathcal{O}_W)$ inducing an isomorphism from $\Delta_* \mathcal{O}_X$ to $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_* \mathcal{O}_X, L\mathrm{pr}_1^* K|_W)$. Since we may change W , we choose $W = \bigcup_{U \in \mathcal{B}} U \times_S U$. We can use ρ_U to get isomorphisms

$$R\mathcal{H}om_{\mathcal{O}_W}(\Delta_* \mathcal{O}_X, L\mathrm{pr}_1^* K|_W)|_{U \times_S U} \xrightarrow{\rho_U} R\mathcal{H}om_{\mathcal{O}_{U \times_S U}}(\Delta_* \mathcal{O}_U, L\mathrm{pr}_1^* K_U)$$

As W is covered by the opens $U \times_S U$ we conclude that the cohomology sheaves of $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_* \mathcal{O}_X, L\mathrm{pr}_1^* K|_W)$ are zero except in degree 0. Moreover, we obtain isomorphisms

$$H^0(U \times_S U, R\mathcal{H}om_{\mathcal{O}_W}(\Delta_* \mathcal{O}_X, L\mathrm{pr}_1^* K|_W)) \xrightarrow{\rho_U} H^0((R\mathcal{H}om_{\mathcal{O}_{U \times_S U}}(\Delta_* \mathcal{O}_U, L\mathrm{pr}_1^* K_U))$$

Let τ_U in the LHS be an element mapping to ξ_U under this map. The compatibilities between $\rho_{U'}^U$, ξ_U , $\xi_{U'}$, ρ_U , and $\rho_{U'}$ for $U' \subset U \subset X$ open $U', U \in \mathcal{B}$ imply that $\tau_U|_{U' \times_S U'} = \tau_{U'}$. Thus we get a global section τ of the 0th cohomology sheaf $H^0(R\mathcal{H}om_{\mathcal{O}_W}(\Delta_* \mathcal{O}_X, L\mathrm{pr}_1^* K|_W))$. Since the other cohomology sheaves of $R\mathcal{H}om_{\mathcal{O}_W}(\Delta_* \mathcal{O}_X, L\mathrm{pr}_1^* K|_W)$ are zero, this global section τ determines a morphism ξ as desired. Since the restriction of ξ to $U \times_S U$ gives ξ_U , we see that it satisfies the final condition of Definition 28.1. $\square$

0E2Y **Lemma 28.6.** *Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

of schemes. Assume $X \rightarrow S$ is flat and locally of finite presentation. Let (K, ξ) be a relative dualizing complex for f . Set $K' = L(g')^* K$. Let ξ' be the derived base change of ξ (see proof). Then (K', ξ') is a relative dualizing complex for f' .

Proof. Consider the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ \Delta_{X'/S'} \downarrow & & \downarrow \Delta_{X/S} \\ X' \times_{S'} X' & \xrightarrow{g' \times g'} & X \times_S X \end{array}$$

Choose $W \subset X \times_S X$ open such that $\Delta_{X/S}$ factors through a closed immersion $\Delta : X \rightarrow W$. Choose $W' \subset X' \times_{S'} X'$ open such that $\Delta_{X'/S'}$ factors through a closed immersion $\Delta' : X' \rightarrow W'$ and such that $(g' \times g')(W') \subset W$. Let us still denote $g' \times g' : W' \rightarrow W$ the induced morphism. We have

$$L(g' \times g')^* \Delta_* \mathcal{O}_X = \Delta'_* \mathcal{O}_{X'} \quad \text{and} \quad L(g' \times g')^* L\mathrm{pr}_1^* K|_W = L\mathrm{pr}_1^* K'|_{W'}$$

The first equality holds because X and $X' \times_{S'} X'$ are tor independent over $X \times_S X$ (see for example More on Morphisms, Lemma 69.1). The second holds by transitivity of derived pullback (Cohomology, Lemma 27.2). Thus $\xi' = L(g' \times g')^* \xi$ can be viewed as a map

$$\xi' : \Delta'_* \mathcal{O}_{X'} \longrightarrow L\mathrm{pr}_1^* K'|_{W'}$$

Having said this the proof of the lemma is straightforward. First, K' is S' -perfect by Derived Categories of Schemes, Lemma 35.6. To check that ξ' induces an isomorphism of $\Delta'_* \mathcal{O}_{X'}$ to $R\mathcal{H}om_{\mathcal{O}_{W'}}(\Delta'_* \mathcal{O}_{X'}, Lpr_1^* K'|_{W'})$ we may work affine locally. By Lemma 28.2 we reduce to the corresponding statement in algebra which is proven in Dualizing Complexes, Lemma 27.4. $\square$

0E2Z **Lemma 28.7.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \rightarrow S$ be a proper, flat morphism of finite presentation. The relative dualizing complex $\omega_{X/S}^\bullet$ of Remark 12.5 together with (12.8.1) is a relative dualizing complex in the sense of Definition 28.1.*

Proof. In Lemma 12.7 we proved that $\omega_{X/S}^\bullet$ is S -perfect. Let c be the right adjoint of Lemma 3.1 for the diagonal $\Delta : X \rightarrow X \times_S X$. Then we can apply Δ_* to (12.8.1) to get an isomorphism

$$\Delta_* \mathcal{O}_X \rightarrow \Delta_*(c(Lpr_1^* \omega_{X/S}^\bullet)) = R\mathcal{H}om_{\mathcal{O}_{X \times_S X}}(\Delta_* \mathcal{O}_X, Lpr_1^* \omega_{X/S}^\bullet)$$

The equality holds by Lemmas 9.7 and 9.3. This finishes the proof. $\square$

0E4P **Remark 28.8.** Let $X \rightarrow S$ be a morphism of schemes which is flat, proper, and of finite presentation. By Lemma 28.5 there exists a relative dualizing complex $(\omega_{X/S}^\bullet, \xi)$ in the sense of Definition 28.1. Consider any morphism $g : S' \rightarrow S$ where S' is quasi-compact and quasi-separated (for example an affine open of S). By Lemma 28.6 we see that $(L(g')^* \omega_{X/S}^\bullet, L(g')^* \xi)$ is a relative dualizing complex for the base change $f' : X' \rightarrow S'$ in the sense of Definition 28.1. Let $\omega_{X'/S'}^\bullet$ be the relative dualizing complex for $X' \rightarrow S'$ in the sense of Remark 12.5. Combining Lemmas 28.7 and 28.4 we see that there is a unique isomorphism

$$\omega_{X'/S'}^\bullet \longrightarrow L(g')^* \omega_{X/S}^\bullet$$

compatible with (12.8.1) and $L(g')^* \xi$. These isomorphisms are compatible with morphisms between quasi-compact and quasi-separated schemes over S and the base change isomorphisms of Lemma 12.4 (if we ever need this compatibility we will carefully state and prove it here).

0E9W **Lemma 28.9.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . If f is flat, then $f^! \mathcal{O}_Y$ is (the first component of) a relative dualizing complex for X over Y in the sense of Definition 28.1.*

Proof. By Lemma 17.10 we have that $f^! \mathcal{O}_Y$ is Y -perfect. As f is separated the diagonal $\Delta : X \rightarrow X \times_Y X$ is a closed immersion and $\Delta_* \Delta^!(-) = R\mathcal{H}om_{\mathcal{O}_{X \times_Y X}}(\mathcal{O}_X, -)$, see Lemmas 9.7 and 9.3. Hence to finish the proof it suffices to show $\Delta^!(Lpr_1^* f^! \mathcal{O}_Y) \cong \mathcal{O}_X$ where $pr_1 : X \times_Y X \rightarrow X$ is the first projection. We have

$$\mathcal{O}_X = \Delta^! pr_1^! \mathcal{O}_X = \Delta^! pr_1^! Lpr_2^* \mathcal{O}_Y = \Delta^!(Lpr_1^* f^! \mathcal{O}_Y)$$

where $pr_2 : X \times_Y X \rightarrow X$ is the second projection and where we have used the base change isomorphism $pr_1^! \circ Lpr_2^* = Lpr_1^* \circ f^!$ of Lemma 18.1. $\square$

0E30 **Lemma 28.10.** *Let $f : Y \rightarrow X$ and $X \rightarrow S$ be morphisms of schemes which are flat and of finite presentation. Let (K, ξ) and (M, η) be a relative dualizing complex for $X \rightarrow S$ and $Y \rightarrow X$. Set $E = M \otimes_{\mathcal{O}_Y}^L Lf^* K$. Then (E, ζ) is a relative dualizing complex for $Y \rightarrow S$ for a suitable ζ .*

Proof. Using Lemma 28.2 and the algebraic version of this lemma (Dualizing Complexes, Lemma 27.6) we see that E is affine locally the first component of a relative dualizing complex. In particular we see that E is S -perfect since this may be checked affine locally, see Derived Categories of Schemes, Lemma 35.3.

Let us first prove the existence of ζ in case the morphisms $X \rightarrow S$ and $Y \rightarrow X$ are separated so that $\Delta_{X/S}$, $\Delta_{Y/X}$, and $\Delta_{Y/S}$ are closed immersions. Consider the following diagram

$$\begin{array}{ccccc}
 & & Y & \xlongequal{\quad} & Y \\
 & \nearrow q & & \nearrow p & \downarrow f \\
 Y & \xrightarrow{\Delta_{Y/X}} & Y \times_X Y & \xrightarrow{\delta} & Y \times_S Y \\
 & \downarrow m & \downarrow f \times f & \nearrow r & \downarrow \\
 & X & \xrightarrow{\Delta_{X/S}} & X \times_S X &
 \end{array}$$

where p, q, r are the first projections. By Lemma 9.4 we have

$$R\mathcal{H}om_{\mathcal{O}_{Y \times_S Y}}(\Delta_{Y/S,*}\mathcal{O}_Y, Lp^*E) = R\delta_* \left(R\mathcal{H}om_{\mathcal{O}_{Y \times_X Y}}(\Delta_{Y/X,*}\mathcal{O}_Y, R\mathcal{H}om(\mathcal{O}_{Y \times_X Y}, Lp^*E)) \right)$$

By Lemma 10.3 we have

$$R\mathcal{H}om(\mathcal{O}_{Y \times_X Y}, Lp^*E) = R\mathcal{H}om(\mathcal{O}_{Y \times_X Y}, L(f \times f)^* Lr^*K) \otimes_{\mathcal{O}_{Y \times_S Y}}^L Lq^*M$$

By Lemma 10.2 we have

$$R\mathcal{H}om(\mathcal{O}_{Y \times_X Y}, L(f \times f)^* Lr^*K) = Lm^* R\mathcal{H}om(\mathcal{O}_X, Lr^*K)$$

The last expression is isomorphic (via ξ) to $Lm^*\mathcal{O}_X = \mathcal{O}_{Y \times_X Y}$. Hence the expression preceding is isomorphic to Lq^*M . Hence

$$R\mathcal{H}om_{\mathcal{O}_{Y \times_S Y}}(\Delta_{Y/S,*}\mathcal{O}_Y, Lp^*E) = R\delta_* \left(R\mathcal{H}om_{\mathcal{O}_{Y \times_X Y}}(\Delta_{Y/X,*}\mathcal{O}_Y, Lq^*M) \right)$$

The material inside the parentheses is isomorphic to $\Delta_{Y/X,*} * \mathcal{O}_X$ via η . This finishes the proof in the separated case.

In the general case we choose an open $W \subset X \times_S X$ such that $\Delta_{X/S}$ factors through a closed immersion $\Delta : X \rightarrow W$ and we choose an open $V \subset Y \times_X Y$ such that $\Delta_{Y/X}$ factors through a closed immersion $\Delta' : Y \rightarrow V$. Finally, choose an open $W' \subset Y \times_S Y$ whose intersection with $Y \times_X Y$ gives V and which maps into W . Then we consider the diagram

$$\begin{array}{ccccc}
 & & Y & \xlongequal{\quad} & Y \\
 & \nearrow q & & \nearrow p & \downarrow f \\
 Y & \xrightarrow{\Delta'} & V & \xrightarrow{\delta} & W' \\
 & \downarrow m & \downarrow f \times f & \nearrow r & \downarrow \\
 & X & \xrightarrow{\Delta} & W &
 \end{array}$$

and we use exactly the same argument as before. $\square$

29. The fundamental class of an lci morphism

0E9X In this section we will use the computations made in Section 15. Thus our result will suffer from the same kind of non-uniqueness as we have in that section.

0E9Y **Lemma 29.1.** *Let X be a locally ringed space. Let*

$$\mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0$$

be a short exact sequence of $\mathcal{O}_X$ -modules. Assume $\mathcal{E}_1$ and $\mathcal{E}_0$ are locally free of ranks r_1, r_0 . Then there is a canonical map

$$\wedge^{r_0-r_1} \mathcal{F} \longrightarrow \wedge^{r_1}(\mathcal{E}_1^\vee) \otimes \wedge^{r_0} \mathcal{E}_0$$

which is an isomorphism on the stalk at $x \in X$ if and only if $\mathcal{F}$ is locally free of rank $r_0 - r_1$ in an open neighbourhood of x .

Proof. If $r_1 > r_0$ then $\wedge^{r_0-r_1} \mathcal{F} = 0$ by convention and the unique map cannot be an isomorphism. Thus we may assume $r = r_0 - r_1 \geq 0$. Define the map by the formula

$$s_1 \wedge \dots \wedge s_r \mapsto t_1^\vee \wedge \dots \wedge t_{r_1}^\vee \otimes \alpha(t_1) \wedge \dots \wedge \alpha(t_{r_1}) \wedge \tilde{s}_1 \wedge \dots \wedge \tilde{s}_r$$

where $t_1, \dots, t_{r_1}$ is a local basis for $\mathcal{E}_1$, correspondingly $t_1^\vee, \dots, t_{r_1}^\vee$ is the dual basis for $\mathcal{E}_1^\vee$, and s_i is a local lift of s_i to a section of $\mathcal{E}_0$. We omit the proof that this is well defined.

If $\mathcal{F}$ is locally free of rank r , then it is straightforward to verify that the map is an isomorphism. Conversely, assume the map is an isomorphism on stalks at x . Then $\wedge^r \mathcal{F}_x$ is invertible. This implies that $\mathcal{F}_x$ is generated by at most r elements. This can only happen if α has rank r modulo $\mathfrak{m}_x$, i.e., α has maximal rank modulo $\mathfrak{m}_x$. This implies that α has maximal rank in a neighbourhood of x and hence $\mathcal{F}$ is locally free of rank r in a neighbourhood as desired. $\square$

0E9Z **Lemma 29.2.** *Let Y be a Noetherian scheme. Let $f : X \rightarrow Y$ be a local complete intersection morphism which factors as an immersion $X \rightarrow P$ followed by a proper smooth morphism $P \rightarrow Y$. Let r be the locally constant function on X such that $\omega_{X/Y} = H^{-r}(f^! \mathcal{O}_Y)$ is the unique nonzero cohomology sheaf of $f^! \mathcal{O}_Y$, see Lemma 17.11. Then there is a map*

$$\wedge^r \Omega_{X/Y} \longrightarrow \omega_{X/Y}$$

which is an isomorphism on the stalk at a point x if and only if f is smooth at x .

Proof. The assumption implies that X is compactifiable over Y hence $f^!$ is defined, see Section 16. Let $j : W \rightarrow P$ be an open subscheme such that $X \rightarrow P$ factors through a closed immersion $i : X \rightarrow W$. Moreover, we have $f^! = i^! \circ j^! \circ g^!$ where $g : P \rightarrow Y$ is the given morphism. We have $g^! \mathcal{O}_Y = \wedge^d \Omega_{P/Y}[d]$ by Lemma 15.7 where d is the locally constant function giving the relative dimension of P over Y . We have $j^! = j^*$. We have $i^! \mathcal{O}_W = \wedge^c \mathcal{N}[-c]$ where c is the codimension of X in W (a locally constant function on X) and where $\mathcal{N}$ is the normal sheaf of the Koszul-regular immersion i , see Lemma 15.6. Combining the above we find

$$f^! \mathcal{O}_Y = (\wedge^c \mathcal{N} \otimes_{\mathcal{O}_X} \wedge^d \Omega_{P/Y|X}) [d - c]$$

where we have also used Lemma 17.9. Thus $r = d|_X - c$ as locally constant functions on X . The conormal sheaf of $X \rightarrow P$ is the module $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I} \subset \mathcal{O}_W$ is the ideal sheaf of i , see Morphisms, Section 31. Consider the canonical exact sequence

$$\mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{P/Y}|_X \rightarrow \Omega_{X/Y} \rightarrow 0$$

of Morphisms, Lemma 32.15. We obtain our map by an application of Lemma 29.1.

If f is smooth at x , then the map is an isomorphism by an application of Lemma 29.1 and the fact that $\Omega_{X/Y}$ is locally free at x of rank r . Conversely, assume that our map is an isomorphism on stalks at x . Then the lemma shows that $\Omega_{X/Y}$ is free of rank r after replacing X by an open neighbourhood of x . On the other hand, we may also assume that $X = \text{Spec}(A)$ and $Y = \text{Spec}(R)$ where $A = R[x_1, \dots, x_n]/(f_1, \dots, f_m)$ and where $f_1, \dots, f_m$ is a Koszul regular sequence (this follows from the definition of local complete intersection morphisms). Clearly this implies $r = n - m$. We conclude that the rank of the matrix of partials $\partial f_j / \partial x_i$ in the residue field at x is m . Thus after reordering the variables we may assume the determinant of $(\partial f_j / \partial x_i)_{1 \leq i, j \leq m}$ is invertible in an open neighbourhood of x . It follows that $R \rightarrow A$ is smooth at this point, see for example Algebra, Example 137.8. $\square$

0EA0 **Lemma 29.3.** *Let $f : X \rightarrow Y$ be a morphism of schemes. Let $r \geq 0$. Assume*

- (1) *Y is Cohen-Macaulay (Properties, Definition 8.1),*
- (2) *f factors as $X \rightarrow P \rightarrow Y$ where the first morphism is an immersion and the second is smooth and proper,*
- (3) *if $x \in X$ and $\dim(\mathcal{O}_{X,x}) \leq 1$, then f is Koszul at x (More on Morphisms, Definition 62.2), and*
- (4) *if ξ is a generic point of an irreducible component of X , then we have $\text{trdeg}_{\kappa(f(\xi))} \kappa(\xi) = r$.*

Then with $\omega_{X/Y} = H^{-r}(f^! \mathcal{O}_Y)$ there is a map

$$\wedge^r \Omega_{X/Y} \longrightarrow \omega_{X/Y}$$

which is an isomorphism on the locus where f is smooth.

Proof. Let $U \subset X$ be the open subscheme over which f is a local complete intersection morphism. Since f has relative dimension r at all generic points by assumption (4) we see that the locally constant function of Lemma 29.2 is constant with value r and we obtain a map

$$\wedge^r \Omega_{X/Y}|_U = \wedge^r \Omega_{U/Y} \longrightarrow \omega_{U/Y} = \omega_{X/Y}|_U$$

which is an isomorphism in the smooth points of f (this locus is contained in U because a smooth morphism is a local complete intersection morphism). By Lemma 21.5 and the assumption that Y is Cohen-Macaulay the module $\omega_{X/Y}$ is (S_2) . Since U contains all the points of codimension 1 by condition (3) and using Divisors, Lemma 5.11 we see that $j_* \omega_{U/Y} = \omega_{X/Y}$. Hence the map over U extends to X and the proof is complete. $\square$

30. Extension by zero for coherent modules

0G2G The material in this section and the next few can be found in the appendix by Deligne of [Har66].

In this section $j : U \rightarrow X$ will be an open immersion of Noetherian schemes. We are going to consider inverse systems (K_n) in $D_{Coh}^b(\mathcal{O}_X)$ constructed as follows. Let $\mathcal{F}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$ -modules. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals with $V(\mathcal{I}) = X \setminus U$. Then we can set

$$K_n = \mathcal{I}^n \mathcal{F}^\bullet$$

More precisely, K_n is the object of $D_{Coh}^b(\mathcal{O}_X)$ represented by the complex whose term in degree q is the coherent submodule $\mathcal{I}^n \mathcal{F}^q$ of $\mathcal{F}^q$. Observe that the maps $\dots \rightarrow K_3 \rightarrow K_2 \rightarrow K_1$ induce isomorphisms on restriction to U . Let us call such a system a *Deligne system*.

0G2H **Lemma 30.1.** *Let $j : U \rightarrow X$ be an open immersion of Noetherian schemes. Let (K_n) be a Deligne system and denote $K \in D_{Coh}^b(\mathcal{O}_U)$ the value of the constant system $(K_n|_U)$. Let L be an object of $D_{Coh}^b(\mathcal{O}_X)$. Then $\operatorname{colim} \operatorname{Hom}_X(K_n, L) = \operatorname{Hom}_U(K, L|_U)$.*

Proof. Let $L \rightarrow M \rightarrow N \rightarrow L[1]$ be a distinguished triangle in $D_{Coh}^b(\mathcal{O}_X)$. Then we obtain a commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & \operatorname{colim} \operatorname{Hom}_X(K_n, L) & \longrightarrow & \operatorname{colim} \operatorname{Hom}_X(K_n, M) & \longrightarrow & \operatorname{colim} \operatorname{Hom}_X(K_n, N) \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & \operatorname{Hom}_U(K, L|_U) & \longrightarrow & \operatorname{Hom}_U(K, M|_U) & \longrightarrow & \operatorname{Hom}_U(K, N|_U) \longrightarrow \dots \end{array}$$

whose rows are exact by Derived Categories, Lemma 4.2 and Algebra, Lemma 8.8. Hence if the statement of the lemma holds for $N[-1]$, L , N , and $L[1]$ then it holds for M by the 5-lemma. Thus, using the distinguished triangles for the canonical truncations of L (see Derived Categories, Remark 12.4) we reduce to the case that L has only one nonzero cohomology sheaf.

Choose a bounded complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_X$ -modules and a quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ cutting out $X \setminus U$ such that K_n is represented by $\mathcal{I}^n \mathcal{F}^\bullet$. Using “stupid” truncations we obtain compatible termwise split short exact sequences of complexes

$$0 \rightarrow \sigma_{\geq a+1} \mathcal{I}^n \mathcal{F}^\bullet \rightarrow \mathcal{I}^n \mathcal{F}^\bullet \rightarrow \sigma_{\leq a} \mathcal{I}^n \mathcal{F}^\bullet \rightarrow 0$$

which in turn correspond to compatible systems of distinguished triangles in $D_{Coh}^b(\mathcal{O}_X)$. Arguing as above we reduce to the case where $\mathcal{F}^\bullet$ has only one nonzero term. This reduces us to the case discussed in the next paragraph.

Given a coherent $\mathcal{O}_X$ -module $\mathcal{F}$ and a coherent $\mathcal{O}_X$ -module $\mathcal{G}$ we have to show that the canonical map

$$\operatorname{colim} \operatorname{Ext}_X^i(\mathcal{I}^n \mathcal{F}, \mathcal{G}) \longrightarrow \operatorname{Ext}_U^i(\mathcal{F}|_U, \mathcal{G}|_U)$$

is an isomorphism for all $i \geq 0$. For $i = 0$ this is Cohomology of Schemes, Lemma 10.5. Assume $i > 0$.

Injectivity. Let $\xi \in \operatorname{Ext}_X^i(\mathcal{I}^n \mathcal{F}, \mathcal{G})$ be an element whose restriction to U is zero. We have to show there exists an $m \geq n$ such that the restriction of ξ to $\mathcal{I}^m \mathcal{F} = \mathcal{I}^{m-n} \mathcal{I}^n \mathcal{F}$ is zero. After replacing $\mathcal{F}$ by $\mathcal{I}^n \mathcal{F}$ we may assume $n = 0$, i.e., we have $\xi \in \operatorname{Ext}_X^i(\mathcal{F}, \mathcal{G})$ whose restriction to U is zero. By Derived Categories of Schemes, Proposition 11.2 we have $D_{Coh}^b(\mathcal{O}_X) = D^b(\operatorname{Coh}(\mathcal{O}_X))$. Hence we can compute the Ext group in the abelian category of coherent $\mathcal{O}_X$ -modules. This implies there

exists an surjection $\alpha : \mathcal{F}'' \rightarrow \mathcal{F}$ such that $\xi \circ \alpha = 0$ (this is where we use that $i > 0$). Set $\mathcal{F}' = \text{Ker}(\alpha)$ so that we have a short exact sequence

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F}'' \rightarrow \mathcal{F} \rightarrow 0$$

It follows that ξ is the image of an element $\xi' \in \text{Ext}_X^{i-1}(\mathcal{F}', \mathcal{G})$ whose restriction to U is in the image of $\text{Ext}_U^{i-1}(\mathcal{F}''|_U, \mathcal{G}|_U) \rightarrow \text{Ext}_U^{i-1}(\mathcal{F}'|_U, \mathcal{G}|_U)$. By Artin-Rees the inverse systems $(\mathcal{I}^n \mathcal{F}')$ and $(\mathcal{I}^n \mathcal{F}'' \cap \mathcal{F}')$ are pro-isomorphic, see Cohomology of Schemes, Lemma 10.3. Since we have the compatible system of short exact sequences

$$0 \rightarrow \mathcal{F}' \cap \mathcal{I}^n \mathcal{F}'' \rightarrow \mathcal{I}^n \mathcal{F}'' \rightarrow \mathcal{I}^n \mathcal{F} \rightarrow 0$$

we obtain a commutative diagram

$$\begin{array}{ccccc} \text{colim Ext}_X^{i-1}(\mathcal{I}^n \mathcal{F}'', \mathcal{G}) & \longrightarrow & \text{colim Ext}_X^{i-1}(\mathcal{F}' \cap \mathcal{I}^n \mathcal{F}'', \mathcal{G}) & \longrightarrow & \text{colim Ext}_X^i(\mathcal{I}^n \mathcal{F}, \mathcal{G}) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Ext}_U^{i-1}(\mathcal{F}''|_U, \mathcal{G}|_U) & \longrightarrow & \text{Ext}_U^{i-1}(\mathcal{F}'|_U, \mathcal{G}|_U) & \longrightarrow & \text{Ext}_U^i(\mathcal{F}|_U, \mathcal{G}|_U) \end{array}$$

with exact rows. By induction on i and the comment on inverse systems above we find that the left two vertical arrows are isomorphisms. Now ξ gives an element in the top right group which is the image of ξ' in the middle top group, which in turn maps to an element of the bottom middle group coming from some element in the left bottom group. We conclude that ξ maps to zero in $\text{Ext}_X^i(\mathcal{I}^n \mathcal{F}, \mathcal{G})$ for some n as desired.

Surjectivity. Let $\xi \in \text{Ext}_U^i(\mathcal{F}|_U, \mathcal{G}|_U)$. Arguing as above using that $i > 0$ we can find an surjection $\mathcal{H} \rightarrow \mathcal{F}|_U$ of coherent $\mathcal{O}_U$ -modules such that ξ maps to zero in $\text{Ext}_U^i(\mathcal{H}, \mathcal{G}|_U)$. Then we can find a map $\varphi : \mathcal{F}'' \rightarrow \mathcal{F}$ of coherent $\mathcal{O}_X$ -modules whose restriction to U is $\mathcal{H} \rightarrow \mathcal{F}|_U$, see Properties, Lemma 22.4. Observe that the lemma doesn't guarantee φ is surjective but this won't matter (it is possible to pick a surjective φ with a little bit of additional work). Denote $\mathcal{F}' = \text{Ker}(\varphi)$. The short exact sequence

$$0 \rightarrow \mathcal{F}'|_U \rightarrow \mathcal{F}''|_U \rightarrow \mathcal{F}|_U \rightarrow 0$$

shows that ξ is the image of ξ' in $\text{Ext}_U^{i-1}(\mathcal{F}'|_U, \mathcal{G}|_U)$. By induction on i we can find an n such that ξ' is the image of some ξ'_n in $\text{Ext}_X^{i-1}(\mathcal{I}^n \mathcal{F}', \mathcal{G})$. By Artin-Rees we can find an $m \geq n$ such that $\mathcal{F}' \cap \mathcal{I}^m \mathcal{F}'' \subset \mathcal{I}^n \mathcal{F}'$. Using the short exact sequence

$$0 \rightarrow \mathcal{F}' \cap \mathcal{I}^m \mathcal{F}'' \rightarrow \mathcal{I}^m \mathcal{F}'' \rightarrow \mathcal{I}^m \text{Im}(\varphi) \rightarrow 0$$

the image of ξ'_n in $\text{Ext}_X^{i-1}(\mathcal{F}' \cap \mathcal{I}^m \mathcal{F}'', \mathcal{G})$ maps by the boundary map to an element ξ_m of $\text{Ext}_X^i(\mathcal{I}^m \text{Im}(\varphi), \mathcal{G})$ which maps to ξ . Since $\text{Im}(\varphi)$ and $\mathcal{F}$ agree over U we see that $\mathcal{F}/\mathcal{I}^m \text{Im}(\varphi)$ is supported on $X \setminus U$. Hence there exists an $l \geq m$ such that $\mathcal{I}^l \mathcal{F} \subset \mathcal{I}^m \text{Im}(\varphi)$, see Cohomology of Schemes, Lemma 10.2. Taking the image of ξ_m in $\text{Ext}_X^i(\mathcal{I}^l \mathcal{F}, \mathcal{G})$ we win. $\square$

0G4K **Lemma 30.2.** *The result of Lemma 30.1 holds even for $L \in D_{\text{Coh}}^+(\mathcal{O}_X)$.*

Proof. Namely, if (K_n) is a Deligne system then there exists a $b \in \mathbf{Z}$ such that $H^i(K_n) = 0$ for $i > b$. Then $\text{Hom}(K_n, L) = \text{Hom}(K_n, \tau_{\leq b} L)$ and $\text{Hom}(K, L) = \text{Hom}(K, \tau_{\leq b} L)$. Hence using the result of the lemma for $\tau_{\leq b} L$ we win. $\square$

0G4L **Lemma 30.3.** *Let $j : U \rightarrow X$ be an open immersion of Noetherian schemes.*

- (1) Let (K_n) and (L_n) be Deligne systems. Let K and L be the values of the constant systems $(K_n|_U)$ and $(L_n|_U)$. Given a morphism $\alpha : K \rightarrow L$ of $D(\mathcal{O}_U)$ there is a unique morphism of pro-systems $(K_n) \rightarrow (L_n)$ of $D_{\text{Coh}}^b(\mathcal{O}_X)$ whose restriction to U is α .
- (2) Given $K \in D_{\text{Coh}}^b(\mathcal{O}_U)$ there exists a Deligne system (K_n) such that $(K_n|_U)$ is constant with value K .
- (3) The pro-object (K_n) of $D_{\text{Coh}}^b(\mathcal{O}_X)$ of (2) is unique up to unique isomorphism (as a pro-object).

Proof. Part (1) is an immediate consequence of Lemma 30.1 and the fact that morphisms between pro-systems are the same as morphisms between the functors they corepresent, see Categories, Remark 22.7.

Let K be as in (2). We can choose $K' \in D_{\text{Coh}}^b(\mathcal{O}_X)$ whose restriction to U is isomorphic to K , see Derived Categories of Schemes, Lemma 13.2. By Derived Categories of Schemes, Proposition 11.2 we can represent K' by a bounded complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_X$ -modules. Choose a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ whose vanishing locus is $X \setminus U$ (for example choose $\mathcal{I}$ to correspond to the reduced induced subscheme structure on $X \setminus U$). Then we can set K_n equal to the object represented by the complex $\mathcal{I}^n \mathcal{F}^\bullet$ as in the introduction to this section.

Part (3) is immediate from parts (1) and (2). $\square$

0G4M **Lemma 30.4.** *Let $j : U \rightarrow X$ be an open immersion of Noetherian schemes. Let*

$$K \rightarrow L \rightarrow M \rightarrow K[1]$$

be a distinguished triangle of $D_{\text{Coh}}^b(\mathcal{O}_U)$. Then there exists an inverse system of distinguished triangles

$$K_n \rightarrow L_n \rightarrow M_n \rightarrow K_n[1]$$

in $D_{\text{Coh}}^b(\mathcal{O}_X)$ such that (K_n) , (L_n) , (M_n) are Deligne systems and such that the restriction of these distinguished triangles to U is isomorphic to the distinguished triangle we started out with.

Proof. Let (K_n) be as in Lemma 30.3 part (2). Choose an object L' of $D_{\text{Coh}}^b(\mathcal{O}_X)$ whose restriction to U is L (we can do this as the lemma shows). By Lemma 30.1 we can find an n and a morphism $K_n \rightarrow L'$ on X whose restriction to U is the given arrow $K \rightarrow L$. We conclude there is a morphism $K' \rightarrow L'$ of $D_{\text{Coh}}^b(\mathcal{O}_X)$ whose restriction to U is the given arrow $K \rightarrow L$.

By Derived Categories of Schemes, Proposition 11.2 we can find a morphism $\alpha^\bullet : \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ of bounded complexes of coherent $\mathcal{O}_X$ -modules representing $K' \rightarrow L'$. Choose a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ whose vanishing locus is $X \setminus U$. Then we let $K_n = \mathcal{I}^n \mathcal{F}^\bullet$ and $L_n = \mathcal{I}^n \mathcal{G}^\bullet$. Observe that $\alpha^\bullet$ induces a morphism of complexes $\alpha_n^\bullet : \mathcal{I}^n \mathcal{F}^\bullet \rightarrow \mathcal{I}^n \mathcal{G}^\bullet$. From the construction of cones in Derived Categories, Section 9 it is clear that

$$C(\alpha_n)^\bullet = \mathcal{I}^n C(\alpha^\bullet)$$

and hence we can set $M_n = C(\alpha_n)^\bullet$. Namely, we have a compatible system of distinguished triangles (see discussion in Derived Categories, Section 12)

$$K_n \rightarrow L_n \rightarrow M_n \rightarrow K_n[1]$$

whose restriction to U is isomorphic to the distinguished triangle we started out with by axiom TR3 and Derived Categories, Lemma 4.3. $\square$

0G4N **Remark 30.5.** Let $j : U \rightarrow X$ be an open immersion of Noetherian schemes. Sending $K \in D_{Coh}^b(\mathcal{O}_U)$ to a Deligne system whose restriction to U is K determines a functor

$$Rj_! : D_{Coh}^b(\mathcal{O}_U) \longrightarrow \text{Pro-}D_{Coh}^b(\mathcal{O}_X)$$

which is “exact” by Lemma 30.4 and which is “left adjoint” to the functor $j^* : D_{Coh}^b(\mathcal{O}_X) \rightarrow D_{Coh}^b(\mathcal{O}_U)$ by Lemma 30.1.

0G4P **Remark 30.6.** Let (A_n) and (B_n) be inverse systems of a category $\mathcal{C}$. Let us say a linear-pro-morphism from (A_n) to (B_n) is given by a compatible family of morphisms $\varphi_n : A_{cn+d} \rightarrow B_n$ for all $n \geq 1$ for some fixed integers $c, d \geq 1$. We’ll say $(\varphi_n : A_{cn+d} \rightarrow B_n)$ and $(\psi_n : A_{c'n+d'} \rightarrow B_n)$ determine the same morphism if there exist $c'' \geq \max(c, c')$ and $d'' \geq \max(d, d')$ such that the two induced morphisms $A_{c''n+d''} \rightarrow B_n$ are the same for all n . It seems likely that Deligne systems (K_n) with given value on U are well defined up to linear-pro-isomorphisms. If we ever need this we will carefully formulate and prove this here.

0G4Q **Lemma 30.7.** Let $j : U \rightarrow X$ be an open immersion of Noetherian schemes. Let

$$K_n \rightarrow L_n \rightarrow M_n \rightarrow K_n[1]$$

be an inverse system of distinguished triangles in $D_{Coh}^b(\mathcal{O}_X)$. If (K_n) and (M_n) are pro-isomorphic to Deligne systems, then so is (L_n) .

Proof. Observe that the systems $(K_n|_U)$ and $(M_n|_U)$ are essentially constant as they are pro-isomorphic to constant systems. Denote K and M their values. By Derived Categories, Lemma 42.2 we see that the inverse system $L_n|_U$ is essentially constant as well. Denote L its value. Let $N \in D_{Coh}^b(\mathcal{O}_X)$. Consider the commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & \text{colim Hom}_X(M_n, N) & \longrightarrow & \text{colim Hom}_X(L_n, N) & \longrightarrow & \text{colim Hom}_X(K_n, N) \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & \text{Hom}_U(M, N|_U) & \longrightarrow & \text{Hom}_U(L, N|_U) & \longrightarrow & \text{Hom}_U(K, N|_U) \longrightarrow \dots \end{array}$$

By Lemma 30.1 and the fact that isomorphic ind-systems have the same colimit, we see that the vertical arrows two to the right and two to the left of the middle one are isomorphisms. By the 5-lemma we conclude that the middle vertical arrow is an isomorphism. Now, if (L'_n) is a Deligne system whose restriction to U has constant value L (which exists by Lemma 30.3), then we have $\text{colim Hom}_X(L'_n, N) = \text{Hom}_U(L, N|_U)$ as well. Hence the pro-systems (L_n) and (L'_n) are pro-isomorphic by Categories, Remark 22.7. $\square$

0G4R **Lemma 30.8.** Let X be a Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}^\bullet$ be a complex of coherent $\mathcal{O}_X$ -modules. Let $p \in \mathbf{Z}$. Set $\mathcal{H} = H^p(\mathcal{F}^\bullet)$ and $\mathcal{H}_n = H^p(\mathcal{I}^n \mathcal{F}^\bullet)$. Then there are canonical $\mathcal{O}_X$ -module maps

$$\dots \rightarrow \mathcal{H}_3 \rightarrow \mathcal{H}_2 \rightarrow \mathcal{H}_1 \rightarrow \mathcal{H}$$

There exists a $c > 0$ such that for $n \geq c$ the image of $\mathcal{H}_n \rightarrow \mathcal{H}$ is contained in $\mathcal{I}^{n-c} \mathcal{H}$ and there is a canonical $\mathcal{O}_X$ -module map $\mathcal{I}^n \mathcal{H} \rightarrow \mathcal{H}_{n-c}$ such that the compositions

$$\mathcal{I}^n \mathcal{H} \rightarrow \mathcal{H}_{n-c} \rightarrow \mathcal{I}^{n-2c} \mathcal{H} \quad \text{and} \quad \mathcal{H}_n \rightarrow \mathcal{I}^{n-c} \mathcal{H} \rightarrow \mathcal{H}_{n-2c}$$

are the canonical ones. In particular, the inverse systems $(\mathcal{H}_n)$ and $(\mathcal{I}^n \mathcal{H})$ are isomorphic as pro-objects of $\text{Mod}(\mathcal{O}_X)$.

Proof. If X is affine, translated into algebra this is More on Algebra, Lemma 101.1. In the general case, argue exactly as in the proof of that lemma replacing the reference to Artin-Rees in algebra with a reference to Cohomology of Schemes, Lemma 10.3. Details omitted. $\square$

0G4S **Lemma 30.9.** *Let $j : U \rightarrow X$ be an open immersion of Noetherian schemes. Let $a \leq b$ be integers. Let (K_n) be an inverse system of $D_{\text{Coh}}^b(\mathcal{O}_X)$ such that $H^i(K_n) = 0$ for $i \notin [a, b]$. The following are equivalent*

- (1) (K_n) is pro-isomorphic to a Deligne system,
- (2) for every $p \in \mathbf{Z}$ there exists a coherent $\mathcal{O}_X$ -module $\mathcal{F}$ such that the pro-systems $(H^p(K_n))$ and $(\mathcal{I}^n \mathcal{F})$ are pro-isomorphic.

Proof. Assume (1). To prove (2) holds we may assume (K_n) is a Deligne system. By definition we may choose a bounded complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_X$ -modules and a quasi-coherent sheaf of ideals cutting out $X \setminus U$ such that K_n is represented by $\mathcal{I}^n \mathcal{F}^\bullet$. Thus the result follows from Lemma 30.8.

Assume (2). We will prove that (K_n) is as in (1) by induction on $b - a$. If $a = b$ then (1) holds essentially by assumption. If $a < b$ then we consider the compatible system of distinguished triangles

$$\tau_{\leq a} K_n \rightarrow K_n \rightarrow \tau_{\geq a+1} K_n \rightarrow (\tau_{\leq a} K_n)[1]$$

See Derived Categories, Remark 12.4. By induction on $b - a$ we know that $\tau_{\leq a} K_n$ and $\tau_{\geq a+1} K_n$ are pro-isomorphic to Deligne systems. We conclude by Lemma 30.7. $\square$

0G4T **Lemma 30.10.** *Let $j : U \rightarrow X$ be an open immersion of Noetherian schemes. Let (K_n) be an inverse system in $D_{\text{Coh}}^b(\mathcal{O}_X)$. Let $X = W_1 \cup \dots \cup W_r$ be an open covering. The following are equivalent*

- (1) (K_n) is pro-isomorphic to a Deligne system,
- (2) for each i the restriction $(K_n|_{W_i})$ is pro-isomorphic to a Deligne system with respect to the open immersion $U \cap W_i \rightarrow W_i$.

Proof. By induction on r . If $r = 1$ then the result is clear. Assume $r > 1$. Set $V = W_1 \cup \dots \cup W_{r-1}$. By induction we see that $(K_n|_V)$ is a Deligne system. This reduces us to the discussion in the next paragraph.

Assume $X = V \cup W$ is an open covering and $(K_n|_W)$ and $(K_n|_V)$ are pro-isomorphic to Deligne systems. We have to show that (K_n) is pro-isomorphic to a Deligne system. Observe that $(K_n|_{V \cap W})$ is pro-isomorphic to a Deligne system (it follows immediately from the construction of Deligne systems that restrictions to opens preserves them). In particular the pro-systems $(K_n|_{U \cap V})$, $(K_n|_{U \cap W})$, and $(K_n|_{U \cap V \cap W})$ are essentially constant. It follows from the distinguished triangles in Cohomology, Lemma 33.2 and Derived Categories, Lemma 42.2 that $(K_n|_U)$ is essentially constant. Denote $K \in D_{\text{Coh}}^b(\mathcal{O}_U)$ the value of this system. Let L be an

object of $D_{Coh}^b(\mathcal{O}_X)$. Consider the diagram

$$\begin{array}{ccc}
\mathrm{colim} \, \mathrm{Ext}^{-1}(K_n|_V, L|_V) \oplus \mathrm{colim} \, \mathrm{Ext}^{-1}(K_n|_W, L|_W) & \longrightarrow & \mathrm{Ext}^{-1}(K|_{U \cap V}, L|_{U \cap V}) \oplus \mathrm{Ext}^{-1}(K|_{U \cap W}, L|_{U \cap W}) \\
\downarrow & & \downarrow \\
\mathrm{colim} \, \mathrm{Ext}^{-1}(K_n|_{V \cap W}, L|_{V \cap W}) & \longrightarrow & \mathrm{Ext}^{-1}(K|_{U \cap V \cap W}, L|_{U \cap V \cap W}) \\
\downarrow & & \downarrow \\
\mathrm{colim} \, \mathrm{Hom}(K_n, L) & \longrightarrow & \mathrm{Hom}(K|_U, L|_U) \\
\downarrow & & \downarrow \\
\mathrm{colim} \, \mathrm{Hom}(K_n|_V, L|_V) \oplus \mathrm{colim} \, \mathrm{Hom}(K_n|_W, L|_W) & \longrightarrow & \mathrm{Hom}(K|_{U \cap V}, L|_{U \cap V}) \oplus \mathrm{Hom}(K|_{U \cap W}, L|_{U \cap W}) \\
\downarrow & & \downarrow \\
\mathrm{colim} \, \mathrm{Hom}(K_n|_{V \cap W}, L|_{V \cap W}) & \longrightarrow & \mathrm{Hom}(K|_{U \cap V \cap W}, L|_{U \cap V \cap W})
\end{array}$$

The vertical sequences are exact by Cohomology, Lemma 33.3 and the fact that filtered colimits are exact. All horizontal arrows except for the middle one are isomorphisms by Lemma 30.1 and the fact that pro-isomorphic systems have the same colimits. Hence the middle one is an isomorphism too by the 5-lemma. It follows that (K_n) is pro-isomorphic to a Deligne system for K . Namely, if (K'_n) is a Deligne system whose restriction to U has constant value K (which exists by Lemma 30.3), then we have $\mathrm{colim} \, \mathrm{Hom}_X(K'_n, L) = \mathrm{Hom}_U(K, L|_U)$ as well. Hence the pro-systems (K_n) and (K'_n) are pro-isomorphic by Categories, Remark 22.7. $\square$

0G4U **Lemma 30.11.** *Let $j : U \rightarrow X$ be an open immersion of Noetherian schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals with $V(\mathcal{I}) = X \setminus U$. Let K be in $D_{Coh}^b(\mathcal{O}_X)$. Then*

$$K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{I}^n$$

is pro-isomorphic to a Deligne system with constant value $K|_U$ over U .

Proof. By Lemma 30.10 the question is local on X . Thus we may assume X is the spectrum of a Noetherian ring. In this case the statement follows from the algebra version which is More on Algebra, Lemma 101.6. $\square$

31. Preliminaries to compactly supported cohomology

0G4V In Situation 16.1 let $f : X \rightarrow Y$ be a morphism in the category FTS_S . Using the constructions in the previous section, we will construct a functor

$$Rf_! : D_{Coh}^b(\mathcal{O}_X) \longrightarrow \mathrm{Pro}\text{-}D_{Coh}^b(\mathcal{O}_Y)$$

which reduces to the functor of Remark 30.5 if f is an open immersion and in general is constructed using a compactification of f . Before we do this, we need the following lemmas to prove our construction is well defined.

0G4W **Lemma 31.1.** *Let $f : X \rightarrow Y$ be a proper morphism of Noetherian schemes. Let $V \subset Y$ be an open subscheme and set $U = f^{-1}(V)$. Picture*

$$\begin{array}{ccc} U & \xrightarrow{\quad j \quad} & X \\ g \downarrow & & \downarrow f \\ V & \xrightarrow{\quad j' \quad} & Y \end{array}$$

Then we have a canonical isomorphism $Rj'_! \circ Rg_ \rightarrow Rf_* \circ Rj_!$ of functors $D_{Coh}^b(\mathcal{O}_U) \rightarrow \text{Pro-}D_{Coh}^b(\mathcal{O}_Y)$ where $Rj_!$ and $Rj'_!$ are as in Remark 30.5.*

First proof. Let K be an object of $D_{Coh}^b(\mathcal{O}_U)$. Let (K_n) be a Deligne system for $U \rightarrow X$ whose restriction to U is constant with value K . Of course this means that (K_n) represents $Rj_!K$ in $\text{Pro-}D_{Coh}^b(\mathcal{O}_X)$. Observe that both $Rj'_!Rg_*K$ and $Rf_*Rj_!K$ restrict to the constant pro-object with value Rg_*K on V . This is immediate for the first one and for the second one it follows from the fact that $(Rf_*K_n)|_V = Rg_*(K_n|_U) = Rg_*K$. By the uniqueness of Deligne systems in Lemma 30.3 it suffices to show that (Rf_*K_n) is pro-isomorphic to a Deligne system. The lemma referenced will also show that the isomorphism we obtain is functorial.

Proof that (Rf_*K_n) is pro-isomorphic to a Deligne system. First, we observe that the question is independent of the choice of the Deligne system (K_n) corresponding to K (by the aforementioned uniqueness). By Lemmas 30.4 and 30.7 if we have a distinguished triangle

$$K \rightarrow L \rightarrow M \rightarrow K[1]$$

in $D_{Coh}^b(\mathcal{O}_U)$ and the result holds for K and M , then the result holds for L . Using the distinguished triangles of canonical truncations (Derived Categories, Remark 12.4) we reduce to the problem studied in the next paragraph.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ -module. Let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals cutting out $Y \setminus V$. Denote $\mathcal{J}^n \mathcal{F}$ the image of $f^* \mathcal{J}^n \otimes \mathcal{F} \rightarrow \mathcal{F}$. We have to show that $(Rf_*(\mathcal{J}^n \mathcal{F}))$ is a Deligne system. By Lemma 30.10 the question is local on Y . Thus we may assume $Y = \text{Spec}(A)$ is affine and $\mathcal{J}$ corresponds to an ideal $I \subset A$. By Lemma 30.9 it suffices to show that the inverse system of cohomology modules $(H^p(X, I^n \mathcal{F}))$ is pro-isomorphic to the inverse system $(I^n M)$ for some finite A -module M . This is shown in Cohomology of Schemes, Lemma 20.3. $\square$

Second proof. Let K be an object of $D_{Coh}^b(\mathcal{O}_U)$. Let L be an object of $D_{Coh}^b(\mathcal{O}_Y)$. We will construct a bijection

$$\text{Hom}_{\text{Pro-}D_{Coh}^b(\mathcal{O}_Y)}(Rj'_!Rg_*K, L) \longrightarrow \text{Hom}_{\text{Pro-}D_{Coh}^b(\mathcal{O}_Y)}(Rf_*Rj_!K, L)$$

functorial in K and L . Fixing K this will determine an isomorphism of pro-objects $Rf_*Rj_!K \rightarrow Rj'_!Rg_*K$ by Categories, Remark 22.7 and varying K we obtain that this determines an isomorphism of functors. To actually produce the isomorphism

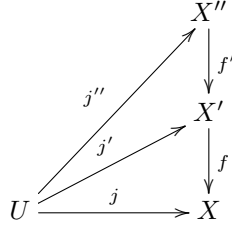
we use the sequence of functorial equalities

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Pro-}D_{\mathrm{Coh}}^b(\mathcal{O}_Y)}(Rj'_!Rg_*K, L) &= \mathrm{Hom}_V(Rg_*K, L|_V) \\
&= \mathrm{Hom}_U(K, g^!(L|_V)) \\
&= \mathrm{Hom}_U(K, f^!L|_U) \\
&= \mathrm{Hom}_{\mathrm{Pro-}D_{\mathrm{Coh}}^b(\mathcal{O}_X)}(Rj_!K, f^!L) \\
&= \mathrm{Hom}_{\mathrm{Pro-}D_{\mathrm{Coh}}^b(\mathcal{O}_Y)}(Rf_*Rj_!K, L)
\end{aligned}$$

The first equality is true by Lemma 30.1. The second equality is true because g is proper (as the base change of f to V) and hence $g^!$ is the right adjoint of pushforward by construction, see Section 16. The third equality holds as $g^!(L|_V) = f^!L|_U$ by Lemma 17.2. Since $f^!L$ is in $D_{\mathrm{Coh}}^+(\mathcal{O}_X)$ by Lemma 17.6 the fourth equality follows from Lemma 30.2. The fifth equality holds again because $f^!$ is the right adjoint to Rf_* as f is proper. $\square$

0G4X **Lemma 31.2.** *Let $j : U \rightarrow X$ be an open immersion of Noetherian schemes. Let $j' : U \rightarrow X'$ be a compactification of U over X (see proof) and denote $f : X' \rightarrow X$ the structure morphism. Then we have a canonical isomorphism $Rj_! \rightarrow Rf_* \circ R(j')_!$ of functors $D_{\mathrm{Coh}}^b(\mathcal{O}_U) \rightarrow \mathrm{Pro-}D_{\mathrm{Coh}}^b(\mathcal{O}_X)$ where $Rj_!$ and $Rj'_!$ are as in Remark 30.5.*

Proof. The fact that X' is a compactification of U over X means precisely that $f : X' \rightarrow X$ is proper, that j' is an open immersion, and $j = f \circ j'$. See More on Flatness, Section 32. If $j'(U) = f^{-1}(j(U))$, then the lemma follows immediately from Lemma 31.1. If $j'(U) \neq f^{-1}(j(U))$, then denote $X'' \subset X'$ the scheme theoretic closure of $j' : U \rightarrow X'$ and denote $j'' : U \rightarrow X''$ the corresponding open immersion. Picture



By More on Flatness, Lemma 32.1 part (c) and the discussion above we have isomorphisms $Rf'_* \circ Rj''_! = Rj'_!$ and $R(f \circ f')_* \circ Rj''_! = Rj_!$. Since $R(f \circ f')_* = Rf_* \circ Rf'_*$ we conclude. $\square$

0G4Y **Remark 31.3.** Let $X \supset U \supset U'$ be open subschemes of a Noetherian scheme X . Denote $j : U \rightarrow X$ and $j' : U' \rightarrow X$ the inclusion morphisms. We claim there is a canonical map

$$Rj'_!(K|_{U'}) \longrightarrow Rj_!K$$

functorial for K in $D_{\mathrm{Coh}}^b(\mathcal{O}_U)$. Namely, by Lemma 30.1 we have for any L in $D_{\mathrm{Coh}}^b(\mathcal{O}_X)$ the map

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Pro-}D_{\mathrm{Coh}}^b(\mathcal{O}_X)}(Rj_!K, L) &= \mathrm{Hom}_U(K, L|_U) \\
&\rightarrow \mathrm{Hom}_{U'}(K|_{U'}, L|_{U'}) \\
&= \mathrm{Hom}_{\mathrm{Pro-}D_{\mathrm{Coh}}^b(\mathcal{O}_X)}(Rj'_!(K|_{U'}), L)
\end{aligned}$$

functorial in L and K' . The functoriality in L shows by Categories, Remark 22.7 that we obtain a canonical map $Rj'_!(K|_{U'}) \rightarrow Rj_!K$ which is functorial in K by the functoriality of the arrow above in K .

Here is an explicit construction of this arrow. Namely, suppose that $\mathcal{F}^\bullet$ is a bounded complex of coherent $\mathcal{O}_X$ -modules whose restriction to U represents K in the derived category. We have seen in the proof of Lemma 30.3 that such a complex always exists. Let $\mathcal{I}$, resp. $\mathcal{I}'$ be a quasi-coherent sheaf of ideals on X with $V(\mathcal{I}) = X \setminus U$, resp. $V(\mathcal{I}') = X \setminus U'$. After replacing $\mathcal{I}$ by $\mathcal{I} + \mathcal{I}'$ we may assume $\mathcal{I}' \subset \mathcal{I}$. By construction $Rj_!K$, resp. $Rj'_!(K|_{U'})$ is represented by the inverse system (K_n) , resp. (K'_n) of $D_{Coh}^b(\mathcal{O}_X)$ with

$$K_n = \mathcal{I}^n \mathcal{F}^\bullet \quad \text{resp.} \quad K'_n = (\mathcal{I}')^n \mathcal{F}^\bullet$$

Clearly the map constructed above is given by the maps $K'_n \rightarrow K_n$ coming from the inclusions $(\mathcal{I}')^n \subset \mathcal{I}^n$.

32. Compactly supported cohomology for coherent modules

0G4Z In Situation 16.1 given a morphism $f : X \rightarrow Y$ in FTS_S , we will define a functor

$$Rf_! : D_{Coh}^b(\mathcal{O}_X) \longrightarrow \text{Pro-}D_{Coh}^b(\mathcal{O}_Y)$$

Namely, we choose a compactification $j : X \rightarrow \overline{X}$ over Y which is possible by More on Flatness, Theorem 33.8 and Lemma 32.2. Denote $\overline{f} : \overline{X} \rightarrow Y$ the structure morphism. Then we set

$$Rf_!K = R\overline{f}_* Rj_!K$$

for $K \in D_{Coh}^b(\mathcal{O}_X)$ where $Rj_!$ is as in Remark 30.5.

0G50 **Lemma 32.1.** *The functor $Rf_!$ is, up to isomorphism, independent of the choice of the compactification.*

In fact, the functor $Rf_!$ will be characterized as a “left adjoint” to $f^!$ which will determine it up to unique isomorphism.

Proof. Consider the category of compactifications of X over Y , which is cofiltered according to More on Flatness, Theorem 33.8 and Lemmas 32.1 and 32.2. To every choice of a compactification

$$j : X \rightarrow \overline{X}, \quad \overline{f} : \overline{X} \rightarrow Y$$

the construction above associates the functor $R\overline{f}_* \circ Rj_!$. Suppose given a morphism $g : \overline{X}_1 \rightarrow \overline{X}_2$ between compactifications $j_i : X \rightarrow \overline{X}_i$ over Y . Then we get an isomorphism

$$R\overline{f}_{2,*} \circ Rj_{2,!} = R\overline{f}_{2,*} \circ Rg_* \circ j_{1,!} = R\overline{f}_{1,*} \circ Rj_{1,!}$$

using Lemma 31.2 in the first equality. In this way we see our functor is independent of the choice of compactification up to isomorphism. $\square$

0G51 **Proposition 32.2.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S . Then the functors $Rf_!$ and $f^!$ are adjoint in the following sense: for all $K \in D_{Coh}^b(\mathcal{O}_X)$ and $L \in D_{Coh}^+(\mathcal{O}_Y)$ we have*

$$\text{Hom}_X(K, f^!L) = \text{Hom}_{\text{Pro-}D_{Coh}^+(\mathcal{O}_Y)}(Rf_!K, L)$$

bifunctorially in K and L .

Proof. Choose a compactification $j : X \rightarrow \overline{X}$ over Y and denote $\overline{f} : \overline{X} \rightarrow Y$ the structure morphism. Then we have

$$\begin{aligned} \mathrm{Hom}_X(K, f^!L) &= \mathrm{Hom}_X(K, j^*\overline{f}^!L) \\ &= \mathrm{Hom}_{\mathrm{Pro-}D_{\mathrm{Coh}}^+(\mathcal{O}_{\overline{X}})}(Rj_!K, \overline{f}^!L) \\ &= \mathrm{Hom}_{\mathrm{Pro-}D_{\mathrm{Coh}}^+(\mathcal{O}_Y)}(Rf_*Rj_!K, L) \\ &= \mathrm{Hom}_{\mathrm{Pro-}D_{\mathrm{Coh}}^+(\mathcal{O}_Y)}(Rf_!K, L) \end{aligned}$$

The first equality follows immediately from the construction of $f^!$ in Section 16. By Lemma 17.6 we have $\overline{f}^!L$ in $D_{\mathrm{Coh}}^+(\mathcal{O}_{\overline{X}})$ hence the second equality follows from Lemma 30.2. Since $\overline{f}$ is proper the functor $\overline{f}^!$ is the right adjoint of pushforward by construction. This is why we have the third equality. The fourth equality holds because $Rf_! = Rf_*Rj_!$. $\square$

0G52 **Lemma 32.3.** *In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S. Let*

$$K \rightarrow L \rightarrow M \rightarrow K[1]$$

be a distinguished triangle of $D_{\mathrm{Coh}}^b(\mathcal{O}_X)$. Then there exists an inverse system of distinguished triangles

$$K_n \rightarrow L_n \rightarrow M_n \rightarrow K_n[1]$$

in $D_{\mathrm{Coh}}^b(\mathcal{O}_Y)$ such that the pro-systems (K_n) , (L_n) , and (M_n) give $Rf_!K$, $Rf_!L$, and $Rf_!M$.

Proof. Choose a compactification $j : X \rightarrow \overline{X}$ over Y and denote $\overline{f} : \overline{X} \rightarrow Y$ the structure morphism. Choose an inverse system of distinguished triangles

$$\overline{K}_n \rightarrow \overline{L}_n \rightarrow \overline{M}_n \rightarrow \overline{K}_n[1]$$

in $D_{\mathrm{Coh}}^b(\mathcal{O}_{\overline{X}})$ as in Lemma 30.4 corresponding to the open immersion j and the given distinguished triangle. Take $K_n = R\overline{f}_*\overline{K}_n$ and similarly for L_n and M_n . This works by the very definition of $Rf_!$. $\square$

0G53 **Remark 32.4.** Let $\mathcal{C}$ be a category. Suppose given an inverse system

$$\dots \xrightarrow{\alpha_4} (M_{3,n}) \xrightarrow{\alpha_3} (M_{2,n}) \xrightarrow{\alpha_2} (M_{1,n})$$

of inverse systems in the category of pro-objects of $\mathcal{C}$. In other words, the arrows α_i are morphisms of pro-objects. By Categories, Example 22.6 we can represent each α_i by a pair (m_i, a_i) where $m_i : \mathbf{N} \rightarrow \mathbf{N}$ is an increasing function and $a_{i,n} : M_{i,m_i(n)} \rightarrow M_{i-1,n}$ is a morphism of $\mathcal{C}$ making the diagrams

$$\begin{array}{ccccc} \dots & \longrightarrow & M_{i,m_i(3)} & \longrightarrow & M_{i,m_i(2)} & \longrightarrow & M_{i,m_i(1)} \\ & & \downarrow a_{i,3} & & \downarrow a_{i,2} & & \downarrow a_{i,1} \\ \dots & \longrightarrow & M_{i-1,3} & \longrightarrow & M_{i-1,2} & \longrightarrow & M_{i-1,1} \end{array}$$

commute. By replacing $m_i(n)$ by $\max(n, m_i(n))$ and adjusting the morphisms $a_i(n)$ accordingly (as in the example referenced) we may assume that $m_i(n) \geq n$. In this situation consider the inverse system

$$\dots \rightarrow M_{4,m_4(m_3(m_2(4)))} \rightarrow M_{3,m_3(m_2(3))} \rightarrow M_{2,m_2(2)} \rightarrow M_{1,1}$$

with general term

$$M_k = M_{k,m_k(m_{k-1}(\dots(m_2(k))\dots))}$$

For any object N of $\mathcal{C}$ we have

$$\operatorname{colim}_i \operatorname{colim}_n \operatorname{Mor}_{\mathcal{C}}(M_{i,n}, N) = \operatorname{colim}_k \operatorname{Mor}_{\mathcal{C}}(M_k, N)$$

We omit the details. In other words, we see that the inverse system (M_k) has the property

$$\operatorname{colim}_i \operatorname{Mor}_{\operatorname{Pro-}\mathcal{C}}((M_{i,n}), N) = \operatorname{Mor}_{\operatorname{Pro-}\mathcal{C}}((M_k), N)$$

This property determines the inverse system (M_k) up to pro-isomorphism by the discussion in Categories, Remark 22.7. In this way we can turn certain inverse systems in $\operatorname{Pro-}\mathcal{C}$ into pro-objects with countable index categories.

0G54 **Remark 32.5.** In Situation 16.1 let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be composable morphisms of FTS_S . Let us define the composition

$$Rg_! \circ Rf_! : D_{Coh}^b(\mathcal{O}_X) \longrightarrow \operatorname{Pro-}D_{Coh}^b(\mathcal{O}_Z)$$

Namely, by the very construction of $Rf_!$ for K in $D_{Coh}^b(\mathcal{O}_X)$ the output $Rf_!K$ is the pro-isomorphism class of an inverse system (M_n) in $D_{Coh}^b(\mathcal{O}_Y)$. Then, since $Rg_!$ is constructed similarly, we see that

$$\dots \rightarrow Rg_!M_3 \rightarrow Rg_!M_2 \rightarrow Rg_!M_1$$

is an inverse system of $\operatorname{Pro-}D_{Coh}^b(\mathcal{O}_Y)$. By the discussion in Remark 32.4 there is a unique pro-isomorphism class, which we will denote $Rg_!Rf_!K$, of inverse systems in $D_{Coh}^b(\mathcal{O}_Z)$ such that

$$\operatorname{Hom}_{\operatorname{Pro-}D_{Coh}^b(\mathcal{O}_Z)}(Rg_!Rf_!K, L) = \operatorname{colim}_n \operatorname{Hom}_{\operatorname{Pro-}D_{Coh}^b(\mathcal{O}_Z)}(Rg_!M_n, L)$$

We omit the discussion necessary to see that this construction is functorial in K as it will immediately follow from the next lemma.

0G55 **Lemma 32.6.** In Situation 16.1 let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be composable morphisms of FTS_S . With notation as in Remark 32.5 we have $Rg_! \circ Rf_! = R(g \circ f)_!$.

Proof. By the discussion in Categories, Remark 22.7 it suffices to show that we obtain the same answer if we compute Hom into L in $D_{Coh}^b(\mathcal{O}_Z)$. To do this we compute, using the notation in Remark 32.5, as follows

$$\begin{aligned} \operatorname{Hom}_Z(Rg_!Rf_!K, L) &= \operatorname{colim}_n \operatorname{Hom}_Z(Rg_!M_n, L) \\ &= \operatorname{colim}_n \operatorname{Hom}_Y(M_n, g^!L) \\ &= \operatorname{Hom}_Y(Rf_!K, g^!L) \\ &= \operatorname{Hom}_X(K, f^!g^!L) \\ &= \operatorname{Hom}_X(K, (g \circ f)^!L) \\ &= \operatorname{Hom}_Z(R(g \circ f)_!K, L) \end{aligned}$$

The first equality is the definition of $Rg_!Rf_!K$. The second equality is Proposition 32.2 for g . The third equality is the fact that $Rf_!K$ is given by (M_n) . The fourth equality is Proposition 32.2 for f . The fifth equality is Lemma 16.3. The sixth is Proposition 32.2 for $g \circ f$. $\square$

0G56 **Remark 32.7.** In Situation 16.1 let $f : X \rightarrow Y$ be a morphism of FTS_S and let $U \subset X$ be an open. Set $g = f|_U : U \rightarrow Y$. Then there is a canonical morphism

$$Rg_!(K|_U) \longrightarrow Rf_!K$$

functorial in K in $D_{Coh}^b(\mathcal{O}_X)$ which can be defined in at least 3 ways.

- (1) Denote $i : U \rightarrow X$ the inclusion morphism. We have $Rg_! = Rf_! \circ Ri_!$ by Lemma 32.6 and we can use $Rf_!$ applied to the map $Ri_!(K|_U) \rightarrow K$ which is a special case of Remark 31.3.
- (2) Choose a compactification $j : X \rightarrow \overline{X}$ of X over Y with structure morphism $\overline{f} : \overline{X} \rightarrow Y$. Set $j' = j \circ i : U \rightarrow \overline{X}$. We can use that $Rf_! = R\overline{f}_* \circ Rj_!$ and $Rg_! = R\overline{f}_* \circ Rj'_!$ and we can use $R\overline{f}_*$ applied to the map $Rj'_!(K|_U) \rightarrow Rj_!K$ of Remark 31.3.
- (3) We can use

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Pro-}D_{\mathrm{Coh}}^b(\mathcal{O}_Y)}(Rf_!K, L) &= \mathrm{Hom}_X(K, f^!L) \\
&\rightarrow \mathrm{Hom}_U(K|_U, f^!L|_U) \\
&= \mathrm{Hom}_U(K|_U, g^!L) \\
&= \mathrm{Hom}_{\mathrm{Pro-}D_{\mathrm{Coh}}^b(\mathcal{O}_Y)}(Rg_!(K|_U), L)
\end{aligned}$$

functorial in L and K . Here we have used Proposition 32.2 twice and the construction of upper shriek functors which shows that $g^! = i^* \circ f^!$. The functoriality in L shows by Categories, Remark 22.7 that we obtain a canonical map $Rg_!(K|_U) \rightarrow Rf_!K$ in $\mathrm{Pro-}D_{\mathrm{Coh}}^b(\mathcal{O}_Y)$ which is functorial in K by the functoriality of the arrow above in K .

Each of these three constructions gives the same arrow; we omit the details.

0G57 **Remark 32.8.** Let us generalize the covariance of compactly supported cohomology given in Remark 32.7 to étale morphisms. Namely, in Situation 16.1 suppose given a commutative diagram

$$\begin{array}{ccc}
U & \xrightarrow{h} & X \\
& \searrow g & \swarrow f \\
& & Y
\end{array}$$

of FTS_S with h étale. Then there is a canonical morphism

$$Rg_!(h^*K) \longrightarrow Rf_!K$$

functorial in K in $D_{\mathrm{Coh}}^b(\mathcal{O}_X)$. We define this transformation using the sequence of maps

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Pro-}D_{\mathrm{Coh}}^b(\mathcal{O}_Y)}(Rf_!K, L) &= \mathrm{Hom}_X(K, f^!L) \\
&\rightarrow \mathrm{Hom}_U(h^*K, h^*(f^!L)) \\
&= \mathrm{Hom}_U(h^*K, h^!f^!L) \\
&= \mathrm{Hom}_U(h^*K, g^!L) \\
&= \mathrm{Hom}_{\mathrm{Pro-}D_{\mathrm{Coh}}^b(\mathcal{O}_Y)}(Rg_!(h^*K), L)
\end{aligned}$$

functorial in L and K . Here we have used Proposition 32.2 twice, we have used the equality $h^* = h^!$ of Lemma 18.2, and we have used the equality $h^! \circ f^! = g^!$ of Lemma 16.3. The functoriality in L shows by Categories, Remark 22.7 that we obtain a canonical map $Rg_!(h^*K) \rightarrow Rf_!K$ in $\mathrm{Pro-}D_{\mathrm{Coh}}^b(\mathcal{O}_Y)$ which is functorial in K by the functoriality of the arrow above in K .

0G58 **Remark 32.9.** In Remarks 32.7 and 32.8 we have seen that the construction of compactly supported cohomology is covariant with respect to open immersions

and étale morphisms. In fact, the correct generality is that given a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ & \searrow g & \swarrow f \\ & Y & \end{array}$$

of FTS_S with h flat and quasi-finite there exists a canonical transformation

$$Rg_! \circ h^* \longrightarrow Rf_!$$

As in Remark 32.8 this map can be constructed using a transformation of functors $h^* \rightarrow h^!$ on $D_{Coh}^+(\mathcal{O}_X)$. Recall that $h^!K = h^*K \otimes \omega_{U/X}$ where $\omega_{U/X} = h^!\mathcal{O}_X$ is the relative dualizing sheaf of the flat quasi-finite morphism h (see Lemmas 17.9 and 21.6). Recall that $\omega_{U/X}$ is the same as the relative dualizing module which will be constructed in Discriminants, Remark 2.11 by Discriminants, Lemma 15.1. Thus we can use the trace element $\tau_{U/X} : \mathcal{O}_U \rightarrow \omega_{U/X}$ which will be constructed in Discriminants, Remark 4.7 to define our transformation. If we ever need this, we will precisely formulate and prove the result here.

33. Duality for compactly supported cohomology

0G59 Let k be a field. Let U be a separated scheme of finite type over k . Let K be an object of $D_{Coh}^b(\mathcal{O}_U)$. Let us define the compactly supported cohomology $H_c^i(U, K)$ of K as follows. Choose an open immersion $j : U \rightarrow X$ into a scheme proper over k and a Deligne system (K_n) for $j : U \rightarrow X$ whose restriction to U is constant with value K . Then we set

$$H_c^i(U, K) = \lim H^i(X, K_n)$$

We view this as a topological k -vector space using the limit topology (see More on Algebra, Section 36). There are several points to make here.

First, this definition is independent of the choice of X and (K_n) . Namely, if $p : U \rightarrow \text{Spec}(k)$ denotes the structure morphism, then we already know that $Rp_!K = (R\Gamma(X, K_n))$ is well defined up to pro-isomorphism in $D(k)$ hence so is the limit defining $H_c^i(U, K)$.

Second, it may seem more natural to use the expression

$$H^i(R\lim R\Gamma(X, K_n)) = R\Gamma(X, R\lim K_n)$$

but this would give the same answer: since the k -vector spaces $H^j(X, K_n)$ are finite dimensional, these inverse systems satisfy Mittag-Leffler and hence $R^1\lim$ terms of Cohomology, Lemma 37.1 vanish.

If $U' \subset U$ is an open subscheme, then there is a canonical map

$$H_c^i(U', K|_{U'}) \longrightarrow H_c^i(U, K)$$

functorial for K in $D_{Coh}^b(\mathcal{O}_U)$. See for example Remark 32.7. In fact, using Remark 32.8 we see that more generally such a map exists for an étale morphism $U' \rightarrow U$ of separated schemes of finite type over k .

If V is a k -vector space then we put a topology on $\text{Hom}_k(V, k)$ as follows: write $V = \bigcup V_i$ as the filtered union of its finite dimensional k -subvector spaces and use the limit topology on $\text{Hom}_k(V, k) = \lim \text{Hom}_k(V_i, k)$. If $\dim_k V < \infty$ then the topology on $\text{Hom}_k(V, k)$ is discrete. More generally, if $V = \text{colim}_n V_n$ is written as a directed

colimit of finite dimensional vector spaces, then $\mathrm{Hom}_k(V, k) = \lim \mathrm{Hom}_k(V_n, k)$ as topological vector spaces.

0G5A **Lemma 33.1.** *Let $p : U \rightarrow \mathrm{Spec}(k)$ be separated of finite type where k is a field. Let $\omega_{U/k}^\bullet = p^! \mathcal{O}_{\mathrm{Spec}(k)}$. There are canonical isomorphisms*

$$\mathrm{Hom}_k(H^i(U, K), k) = H_c^{-i}(U, R\mathcal{H}om_{\mathcal{O}_U}(K, \omega_{U/k}^\bullet))$$

of topological k -vector spaces functorial for K in $D_{\mathrm{Coh}}^b(\mathcal{O}_U)$.

Proof. Choose a compactification $j : U \rightarrow X$ over k . Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent ideal sheaf with $V(\mathcal{I}) = X \setminus U$. By Derived Categories of Schemes, Proposition 11.2 we may choose $M \in D_{\mathrm{Coh}}^b(\mathcal{O}_X)$ with $K = M|_U$. We have

$$H^i(U, K) = \mathrm{Ext}_U^i(\mathcal{O}_U, M|_U) = \mathrm{colim} \mathrm{Ext}_X^i(\mathcal{I}^n, M) = \mathrm{colim} H^i(X, R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M))$$

by Lemma 30.1. Since $\mathcal{I}^n$ is a coherent $\mathcal{O}_X$ -module, we have $\mathcal{I}^n$ in $D_{\mathrm{Coh}}^-(\mathcal{O}_X)$, hence $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M)$ is in $D_{\mathrm{Coh}}^+(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 11.5.

Let $\omega_{X/k}^\bullet = q^! \mathcal{O}_{\mathrm{Spec}(k)}$ where $q : X \rightarrow \mathrm{Spec}(k)$ is the structure morphism, see Section 27. We find that

$$\begin{aligned} \mathrm{Hom}_k(H^i(X, R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M)), k) \\ &= \mathrm{Ext}_X^{-i}(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M), \omega_{X/k}^\bullet) \\ &= H^{-i}(X, R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M), \omega_{X/k}^\bullet)) \end{aligned}$$

by Lemma 27.1. By Lemma 2.4 part (1) the canonical map

$$R\mathcal{H}om_{\mathcal{O}_X}(M, \omega_{X/k}^\bullet) \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{I}^n \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^n, M), \omega_{X/k}^\bullet)$$

is an isomorphism. Observe that $\omega_{U/k}^\bullet = \omega_{X/k}^\bullet|_U$ because $p^!$ is constructed as $q^!$ composed with restriction to U . Hence $R\mathcal{H}om_{\mathcal{O}_X}(M, \omega_{X/k}^\bullet)$ is an object of $D_{\mathrm{Coh}}^b(\mathcal{O}_X)$ which restricts to $R\mathcal{H}om_{\mathcal{O}_U}(K, \omega_{U/k}^\bullet)$ on U . Hence by Lemma 30.11 we conclude that

$$\lim H^{-i}(X, R\mathcal{H}om_{\mathcal{O}_X}(M, \omega_{X/k}^\bullet) \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{I}^n)$$

is an avatar for the right hand side of the equality of the lemma. Combining all the isomorphisms obtained in this manner we get the isomorphism of the lemma. $\square$

0G5B **Lemma 33.2.** *With notation as in Lemma 33.1 suppose $U' \subset U$ is an open subscheme. Then the diagram*

$$\begin{array}{ccc} \mathrm{Hom}_k(H^i(U, K), k) & \xrightarrow{\quad} & H_c^{-i}(U, R\mathcal{H}om_{\mathcal{O}_U}(K, \omega_{U/k}^\bullet)) \\ \uparrow & & \uparrow \\ \mathrm{Hom}_k(H^i(U', K|_{U'}), k) & \xrightarrow{\quad} & H_c^{-i}(U', R\mathcal{H}om_{\mathcal{O}_{U'}}(K, \omega_{U'/k}^\bullet)) \end{array}$$

is commutative. Here the horizontal arrows are the isomorphisms of Lemma 33.1, the vertical arrow on the left is the contragredient to the restriction map $H^i(U, K) \rightarrow H^i(U', K|_{U'})$, and the right vertical arrow is Remark 32.7 (see discussion before the lemma).

Proof. We strongly urge the reader to skip this proof. Choose X and M as in the proof of Lemma 33.1. We are going to drop the subscript $\mathcal{O}_X$ from $R\mathcal{H}om$ and $\otimes^{\mathbf{L}}$. We write

$$H^i(U, K) = \operatorname{colim} H^i(X, R\mathcal{H}om(\mathcal{I}^n, M))$$

and

$$H^i(U', K|_{U'}) = \operatorname{colim} H^i(X, R\mathcal{H}om((\mathcal{I}')^n, M))$$

as in the proof of Lemma 33.1 where we choose $\mathcal{I}' \subset \mathcal{I}$ as in the discussion in Remark 31.3 so that the map $H^i(U, K) \rightarrow H^i(U', K|_{U'})$ is induced by the maps $(\mathcal{I}')^n \rightarrow \mathcal{I}^n$. We similarly write

$$H_c^i(U, R\mathcal{H}om(K, \omega_{U/k}^\bullet)) = \lim H^i(X, R\mathcal{H}om(M, \omega_{X/k}^\bullet) \otimes^{\mathbf{L}} \mathcal{I}^n)$$

and

$$H_c^i(U', R\mathcal{H}om(K|_{U'}, \omega_{U'/k}^\bullet)) = \lim H^i(X, R\mathcal{H}om(M, \omega_{X/k}^\bullet) \otimes^{\mathbf{L}} (\mathcal{I}')^n)$$

so that the arrow $H_c^i(U', R\mathcal{H}om(K|_{U'}, \omega_{U'/k}^\bullet)) \rightarrow H_c^i(U, R\mathcal{H}om(K, \omega_{U/k}^\bullet))$ is similarly deduced from the maps $(\mathcal{I}')^n \rightarrow \mathcal{I}^n$. The diagrams

$$\begin{array}{ccc} R\mathcal{H}om(M, \omega_{X/k}^\bullet) \otimes^{\mathbf{L}} \mathcal{I}^n & \longrightarrow & R\mathcal{H}om(R\mathcal{H}om(\mathcal{I}^n, M), \omega_{X/k}^\bullet) \\ \uparrow & & \uparrow \\ R\mathcal{H}om(M, \omega_{X/k}^\bullet) \otimes^{\mathbf{L}} (\mathcal{I}')^n & \longrightarrow & R\mathcal{H}om(R\mathcal{H}om((\mathcal{I}')^n, M), \omega_{X/k}^\bullet) \end{array}$$

commute because the construction of the horizontal arrows in Cohomology, Lemma 42.9 is functorial in all three entries. Hence we finally come down to the assertion that the diagrams

$$\begin{array}{ccc} \operatorname{Hom}_k(H^i(X, R\mathcal{H}om(\mathcal{I}^n, M)), k) & \longrightarrow & H^{-i}(X, R\mathcal{H}om(R\mathcal{H}om(\mathcal{I}^n, M), \omega_{X/k}^\bullet)) \\ \uparrow & & \uparrow \\ \operatorname{Hom}_k(H^i(X, R\mathcal{H}om((\mathcal{I}')^n, M)), k) & \longrightarrow & H^{-i}(X, R\mathcal{H}om(R\mathcal{H}om((\mathcal{I}')^n, M), \omega_{X/k}^\bullet)) \end{array}$$

commute. This is true because the duality isomorphism

$$\operatorname{Hom}_k(H^i(X, L), k) = \operatorname{Ext}_X^{-i}(L, \omega_{X/k}^\bullet) = H^{-i}(X, R\mathcal{H}om(L, \omega_{X/k}^\bullet))$$

is functorial for L in $D_{Q\text{Coh}}(\mathcal{O}_X)$. $\square$

0G5C **Lemma 33.3.** *Let X be a proper scheme over a field k . Let $K \in D_{\text{Coh}}^b(\mathcal{O}_X)$ with $H^i(K) = 0$ for $i < 0$. Set $\mathcal{F} = H^0(K)$. Let $Z \subset X$ be closed with complement $U = X \setminus Z$. Then*

$$H_c^0(U, K|_U) \subset H^0(X, \mathcal{F})$$

is given by those global sections of $\mathcal{F}$ which vanish in an open neighbourhood of Z .

Proof. Consider the map $H_c^0(U, K|_U) \rightarrow H_X^0(X, K) = H^0(X, K) = H^0(X, \mathcal{F})$ of Remark 32.7. To study this we represent K by a bounded complex $\mathcal{F}^\bullet$ with $\mathcal{F}^i = 0$ for $i < 0$. Then we have by definition

$$H_c^0(U, K|_U) = \lim H^0(X, \mathcal{I}^n \mathcal{F}^\bullet) = \lim \operatorname{Ker}(H^0(X, \mathcal{I}^n \mathcal{F}^0) \rightarrow H^0(X, \mathcal{I}^n \mathcal{F}^1))$$

By Artin-Rees (Cohomology of Schemes, Lemma 10.3) this is the same as $\lim H^0(X, \mathcal{I}^n \mathcal{F})$. Thus the arrow $H_c^0(U, K|_U) \rightarrow H^0(X, \mathcal{F})$ is injective and the image consists of those

global sections of $\mathcal{F}$ which are contained in the subsheaf $\mathcal{I}^n \mathcal{F}$ for any n . The characterization of these as the sections which vanish in a neighbourhood of Z comes from Krull's intersection theorem (Algebra, Lemma 51.4) by looking at stalks of $\mathcal{F}$. See discussion in Algebra, Remark 51.6 for the case of functions. $\square$

34. Lichtenbaum's theorem

0G5D The theorem below was conjectured by Lichtenbaum and proved by Grothendieck (see [Har67]). There is a very nice proof of the theorem by Kleiman in [Kle67]. A generalization of the theorem to the case of cohomology with supports can be found in [Lyu91]. The most interesting part of the argument is contained in the proof of the following lemma.

0G5E **Lemma 34.1.** *Let U be a variety. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$ -module. If $H^d(U, \mathcal{F})$ is nonzero, then $\dim(U) \geq d$ and if equality holds, then U is proper.*

Proof. By the Grothendieck's vanishing result in Cohomology, Proposition 20.7 we conclude that $\dim(U) \geq d$. Assume $\dim(U) = d$. Choose a compactification $U \rightarrow X$ such that U is dense in X . (This is possible by More on Flatness, Theorem 33.8 and Lemma 32.2.) After replacing X by its reduction we find that X is a proper variety of dimension d and we see that U is proper if and only if $U = X$. Set $Z = X \setminus U$. We will show that $H^d(U, \mathcal{F})$ is zero if Z is nonempty.

Choose a coherent $\mathcal{O}_X$ -module $\mathcal{G}$ whose restriction to U is $\mathcal{F}$, see Properties, Lemma 22.5. Let $\omega_X^\bullet$ denote the dualizing complex of X as in Section 27. Set $\omega_U^\bullet = \omega_X^\bullet|_U$. Then $H^d(U, \mathcal{F})$ is dual to

$$H_c^{-d}(U, R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}, \omega_U^\bullet))$$

by Lemma 33.1. By Lemma 27.1 we see that the cohomology sheaves of $\omega_X^\bullet$ vanish in degrees $< -d$ and $H^{-d}(\omega_X^\bullet) = \omega_X$ is a coherent $\mathcal{O}_X$ -module which is (S_2) and whose support is X . In particular, ω_X is torsion free, see Divisors, Lemma 11.10. Thus we see that the cohomology sheaf

$$H^{-d}(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \omega_X^\bullet)) = \mathcal{H}om(\mathcal{G}, \omega_X)$$

is torsion free, see Divisors, Lemma 11.12. Consequently this sheaf has no nonzero sections vanishing on any nonempty open of X (those would be torsion sections). Thus it follows from Lemma 33.3 that $H_c^{-d}(U, R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}, \omega_U^\bullet))$ is zero, and hence $H^d(U, \mathcal{F})$ is zero as desired. $\square$

0G5F **Theorem 34.2.** *Let X be a nonempty separated scheme of finite type over a field k . Let $d = \dim(X)$. The following are equivalent*

- (1) $H^d(X, \mathcal{F}) = 0$ for all coherent $\mathcal{O}_X$ -modules $\mathcal{F}$ on X ,
- (2) $H^d(X, \mathcal{F}) = 0$ for all quasi-coherent $\mathcal{O}_X$ -modules $\mathcal{F}$ on X , and
- (3) no irreducible component $X' \subset X$ of dimension d is proper over k .

Proof. Assume there exists an irreducible component $X' \subset X$ (which we view as an integral closed subscheme) which is proper and has dimension d . Let $\omega_{X'}$ be a dualizing module of X' over k , see Lemma 27.1. Then $H^d(X', \omega_{X'})$ is nonzero as it is dual to $H^0(X', \mathcal{O}_{X'})$ by the lemma. Hence we see that $H^d(X, \omega_{X'}) = H^d(X', \omega_{X'})$ is nonzero and we conclude that (1) does not hold. In this way we see that (1) implies (3).

Let us prove that (3) implies (1). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ -module such that $H^d(X, \mathcal{F})$ is nonzero. Choose a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_m = \mathcal{F}$$

as in Cohomology of Schemes, Lemma 12.3. We obtain exact sequences

$$H^d(X, \mathcal{F}_i) \rightarrow H^d(X, \mathcal{F}_{i+1}) \rightarrow H^d(X, \mathcal{F}_{i+1}/\mathcal{F}_i)$$

Thus for some $i \in \{1, \dots, m\}$ we find that $H^d(X, \mathcal{F}_{i+1}/\mathcal{F}_i)$ is nonzero. By our choice of the filtration this means that there exists an integral closed subscheme $Z \subset X$ and a nonzero coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ such that $H^d(Z, \mathcal{I})$ is nonzero. By Lemma 34.1 we conclude $\dim(Z) = d$ and Z is proper over k contradicting (3). Hence (3) implies (1).

Finally, let us show that (1) and (2) are equivalent for any Noetherian scheme X . Namely, (2) trivially implies (1). On the other hand, assume (1) and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ -module. Then we can write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as the filtered colimit of its coherent submodules, see Properties, Lemma 22.3. Then we have $H^d(X, \mathcal{F}) = \operatorname{colim} H^d(X, \mathcal{F}_i) = 0$ by Cohomology, Lemma 19.1. Thus (2) is true. $\square$

35. Other chapters

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- (1) Introduction
- (2) Conventions
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- (4) Categories
- (5) Topology
- (6) Sheaves on Spaces
- (7) Sites and Sheaves
- (8) Stacks
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- (22) Differential Graded Algebra
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Schemes

- (26) Schemes

(27) Constructions of Schemes

- (28) Properties of Schemes
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| (85) Simplicial Spaces | (117) Auto Generated Index |
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References

- [Con00] Brian Conrad, *Grothendieck duality and base change*, Lecture Notes in Mathematics, vol. 1750, Springer-Verlag, Berlin, 2000.
- [Har66] Robin Hartshorne, *Residues and duality, lecture notes of a seminar on the work of a. grothendieck*, Lecture Notes in Math., vol. 20, Springer-Verlag, 1966.
- [Har67] ———, *Local cohomology*, A seminar given by A. Grothendieck, Harvard University, Fall, vol. 1961, Springer-Verlag, Berlin-New York, 1967.
- [Kie72] Reinhardt Kiehl, *Ein "Descente"-Lemma und Grothendiecks Projektionssatz für nicht-noethersche Schemata*, Math. Ann. **198** (1972), 287–316.
- [Kle67] Steven L. Kleiman, *On the vanishing of $H^n(X, \mathcal{F})$ for an n -dimensional variety*, Proc. Amer. Math. Soc. **18** (1967), 940–944.

- [Lip09] Joseph Lipman, *Notes on derived functors and Grothendieck duality*, Foundations of Grothendieck duality for diagrams of schemes, Lecture Notes in Math., vol. 1960, Springer, Berlin, 2009, pp. 1–259.
- [LN07] Joseph Lipman and Amnon Neeman, *Quasi-perfect scheme-maps and boundedness of the twisted inverse image functor*, Illinois J. Math. **51** (2007), no. 1, 209–236.
- [Lyu91] Gennady Lyubeznik, *A generalization of Lichtenbaum’s theorem on the cohomological dimension of algebraic varieties*, Math. Z. **208** (1991), no. 3, 463–466.
- [Nee96] Amnon Neeman, *The Grothendieck duality theorem via Bousfield’s techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236.
- [Nee11] A. Neeman, *Rigid dualizing complexes*, Bull. Iranian Math. Soc. **37** (2011), no. 2, 273–290.
- [Nee14] Amnon Neeman, *An improvement on the base-change theorem and the functor f -shriek*.
- [vdB97] Michel van den Bergh, *Existence theorems for dualizing complexes over non-commutative graded and filtered rings*, J. Algebra **195** (1997), no. 2, 662–679.
- [Yek10] Amnon Yekutieli, *Rigid dualizing complexes via differential graded algebras (survey)*, Triangulated categories, London Math. Soc. Lecture Note Ser., vol. 375, Cambridge Univ. Press, Cambridge, 2010, pp. 452–463.
- [YZ09] Amnon Yekutieli and James J. Zhang, *Rigid dualizing complexes over commutative rings*, Algebr. Represent. Theory **12** (2009), no. 1, 19–52.