MORE ON MORPHISMS

02GX

Contents

This is a chapter of the Stacks Project, version 2c3bdd57, compiled on Jun 18, 2024.

1. Introduction

02GY In this chapter we continue our study of properties of morphisms of schemes. A fundamental reference is [DG67].

2. Thickenings

04EW The following terminology may not be completely standard, but it is convenient.

04EX **Definition 2.1.** Thickenings.

(1) We say a scheme X' is a *thickening* of a scheme X if X is a closed subscheme of X' and the underlying topological spaces are equal.
(2) We say a scheme X' is a *first order thickening* of a scheme X if X is a closed subscheme of X' and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X has square zero.
(3) Given two thickenings $X \subset X'$ and $Y \subset Y'$ a *morphism of thickenings* is a morphism $f' : X' \to Y'$ such that $f'(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subscheme Y. In this situation we set $f = f'|_X : X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.
(4) Let S be a scheme. We similarly define *thickenings over* S, and *morphisms of thickenings over* S. This means that the schemes X, X', Y, Y' above are schemes over S, and that the morphisms $X \to X'$, $Y \to Y'$ and $f' : X' \to Y'$ are morphisms over S.

Finite order thickenings. Let $i_X : X \to X'$ be a thickening. Any local section of the kernel $\mathcal{I} = \text{Ker}(i_X^\sharp)$ is locally nilpotent. Let us say that $X \subset X'$ is a *finite order thickening* if the ideal sheaf $\mathcal{I}$ is "globally" nilpotent, i.e., if there exists an $n \geq 0$ such that $\mathcal{I}^{n+1} = 0$. Technically the class of finite order thickenings $X \subset X'$ is much easier to handle than the general case. Namely, in this case we have a filtration

$$0 = \mathcal{I}^{n+1} \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

and we see that X' is filtered by closed subspaces

$$X = X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over S. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.

First order thickening are described as follows (see Modules, Lemma 28.11).

05YV **Lemma 2.2.** *Let X be a scheme over a base S. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on X where $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_S$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I}$ is its kernel. If

(1) *$\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and*
(2) *$\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module*

then $X' = (X, \mathcal{A})$ is a scheme and $X \to X'$ is a first order thickening over S. Moreover, any first order thickening over S is of this form.

Proof. It is clear that X' is a locally ringed space. Let $U = \mathrm{Spec}(B)$ be an affine open of X. Set $A = \Gamma(U, \mathcal{A})$. Note that since $H^1(U, \mathcal{I}) = 0$ (see Cohomology of Schemes, Lemma 2.2) the map $A \to B$ is surjective. By assumption the kernel $I = \mathcal{I}(U)$ is an ideal of square zero in the ring A. By Schemes, Lemma 6.4 there is a canonical morphism of locally ringed spaces

$$(U, \mathcal{A}|_U) \longrightarrow \mathrm{Spec}(A)$$

coming from the map $B \to \Gamma(U, \mathcal{A})$. Since this morphism fits into the commutative diagram

$$\begin{tikzcd}
(U, \mathcal{O}_X|_U) \arrow[r] \arrow[d] & \mathrm{Spec}(B) \arrow[d] \\
(U, \mathcal{A}|_U) \arrow[r] & \mathrm{Spec}(A)
\end{tikzcd}$$

we see that it is a homeomorphism on underlying topological spaces. Thus to see that it is an isomorphism, it suffices to check it induces an isomorphism on the local rings. For $u \in U$ corresponding to the prime $\mathfrak{p} \subset A$ we obtain a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & I_{\mathfrak{p}} \arrow[r] \arrow[d] & A_{\mathfrak{p}} \arrow[r] \arrow[d] & B_{\mathfrak{p}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_u \arrow[r] & \mathcal{A}_u \arrow[r] & \mathcal{O}_{X,u} \arrow[r] & 0.
\end{tikzcd}$$

The left and right vertical arrows are isomorphisms because $\mathcal{I}$ and $\mathcal{O}_X$ are quasi-coherent sheaves. Hence also the middle map is an isomorphism. Hence every point of $X' = (X, \mathcal{A})$ has an affine neighbourhood and X' is a scheme as desired. $\square$

06AD **Lemma 2.3.** *Any thickening of an affine scheme is affine.*

The case of a finite order thickening is [GD60, Proposition 5.1.9].

Proof. This is a special case of Limits, Proposition 11.2. $\square$

Proof for a finite order thickening. Suppose that $X \subset X'$ is a finite order thickening with X affine. Then we may use Serre's criterion to prove X' is affine. More precisely, we will use Cohomology of Schemes, Lemma 3.1. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{X'}$-module. It suffices to show that $H^1(X', \mathcal{F}) = 0$. Denote $i : X \to X'$ the given closed immersion and denote $\mathcal{I} = \mathrm{Ker}(i^\sharp : \mathcal{O}_{X'} \to i_*\mathcal{O}_X)$. By our discussion of finite order thickenings (following Definition 2.1) there exists an $n \geq 0$ and a filtration

$$0 = \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \ldots \subset \mathcal{F}_0 = \mathcal{F}$$

by quasi-coherent submodules such that $\mathcal{F}_a/\mathcal{F}_{a+1}$ is annihilated by $\mathcal{I}$. Namely, we can take $\mathcal{F}_a = \mathcal{I}^a\mathcal{F}$. Then $\mathcal{F}_a/\mathcal{F}_{a+1} = i_*\mathcal{G}_a$ for some quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}_a$, see Morphisms, Lemma 4.1. We obtain

$$H^1(X', \mathcal{F}_a/\mathcal{F}_{a+1}) = H^1(X', i_*\mathcal{G}_a) = H^1(X, \mathcal{G}_a) = 0$$

The second equality comes from Cohomology of Schemes, Lemma 2.4 and the last equality from Cohomology of Schemes, Lemma 2.2. Thus $\mathcal{F}$ has a finite filtration whose successive quotients have vanishing first cohomology and it follows by a simple induction argument that $H^1(X', \mathcal{F}) = 0$. $\square$

09ZU **Lemma 2.4.** *Let $S \subset S'$ be a thickening of schemes. Let $X' \to S'$ be a morphism and set $X = S \times_{S'} X'$. Then $(X \subset X') \to (S \subset S')$ is a morphism of thickenings. If $S \subset S'$ is a first (resp. finite order) thickening, then $X \subset X'$ is a first (resp. finite order) thickening.*

Proof. Omitted. □

0BPE **Lemma 2.5.** *If $S \subset S'$ and $S' \subset S''$ are thickenings, then so is $S \subset S''$.*

Proof. Omitted. □

0BPF **Lemma 2.6.** *The property of being a thickening is fpqc local. Similarly for first order thickenings.*

Proof. The statement means the following: Let $X \to X'$ be a morphism of schemes and let $\{g_i : X'_i \to X'\}$ be an fpqc covering such that the base change $X_i \to X'_i$ is a thickening for all i. Then $X \to X'$ is a thickening. Since the morphisms g_i are jointly surjective we conclude that $X \to X'$ is surjective. By Descent, Lemma 23.19 we conclude that $X \to X'$ is a closed immersion. Thus $X \to X'$ is a thickening. We omit the proof in the case of first order thickenings. □

3. Morphisms of thickenings

0CF2 If $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings of schemes, then often properties of the morphism f are inherited by f'. There are several variants.

09ZV **Lemma 3.1.** *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings. Then*

1. *f is an affine morphism if and only if f' is an affine morphism,*
2. *f is a surjective morphism if and only if f' is a surjective morphism,*
3. *f is quasi-compact if and only if f' quasi-compact,*
4. *f is universally closed if and only if f' is universally closed,*
5. *f is integral if and only if f' is integral,*
6. *f is (quasi-)separated if and only if f' is (quasi-)separated,*
7. *f is universally injective if and only if f' is universally injective,*
8. *f is universally open if and only if f' is universally open,*
9. *f is quasi-affine if and only if f' is quasi-affine, and*
10. *add more here.*

Proof. Observe that $S \to S'$ and $X \to X'$ are universal homeomorphisms (see for example Morphisms, Lemma 45.6). This immediately implies parts (2), (3), (4), (7), and (8). Part (1) follows from Lemma 2.3 which tells us that there is a 1-to-1 correspondence between affine opens of S and S' and between affine opens of X and X'. Part (9) follows from Limits, Lemma 11.5 and the remark just made about affine opens of S and S'. Part (5) follows from (1) and (4) by Morphisms, Lemma 44.7. Finally, note that

$$S \times_X S = S \times_{X'} S \to S \times_{X'} S' \to S' \times_{X'} S'$$

is a thickening (the two arrows are thickenings by Lemma 2.4). Hence applying (3) and (4) to the morphism $(S \subset S') \to (S \times_X S \to S' \times_{X'} S')$ we obtain (6). □

0D2R **Lemma 3.2.** *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings. Let $\mathcal{L}'$ be an invertible sheaf on X' and denote $\mathcal{L}$ the restriction to X. Then $\mathcal{L}'$ is f'-ample if and only if $\mathcal{L}$ is f-ample.*

Proof. Recall that being relatively ample is a condition for each affine open in the base, see Morphisms, Definition 37.1. By Lemma 2.3 there is a 1-to-1 correspondence between affine opens of S and S'. Thus we may assume S and S' are affine and we reduce to proving that $\mathcal{L}'$ is ample if and only if $\mathcal{L}$ is ample. This is Limits, Lemma 11.4. $\square$

09ZW **Lemma 3.3.** *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings such that $X = S \times_{S'} X'$. If $S \subset S'$ is a finite order thickening, then*

1. *f is a closed immersion if and only if f' is a closed immersion,*
2. *f is locally of finite type if and only if f' is locally of finite type,*
3. *f is locally quasi-finite if and only if f' is locally quasi-finite,*
4. *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
5. *$\Omega_{X/S} = 0$ if and only if $\Omega_{X'/S'} = 0$,*
6. *f is unramified if and only if f' is unramified,*
7. *f is proper if and only if f' is proper,*
8. *f is finite if and only if f' is finite,*
9. *f is a monomorphism if and only if f' is a monomorphism,*
10. *f is an immersion if and only if f' is an immersion, and*
11. *add more here.*

Proof. The properties $\mathcal{P}$ listed in the lemma are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 18.2 and 23.5 and Morphisms, Lemmas 15.4, 20.13, 29.2, 32.10, 35.5, 41.5, and 44.6.

The interesting direction in each case is therefore to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $S \subset S'$ is a first order thickening, see discussion immediately following Definition 2.1.

Most of the proofs will use a reduction to the affine case. Let $U' \subset S'$ be an affine open and let $V' \subset X'$ be an affine open lying over U'. Let $U' = \text{Spec}(A')$ and denote $I \subset A'$ be the ideal defining the closed subscheme $U' \cap S$. Say $V' = \text{Spec}(B')$. Then $V' \cap X = \text{Spec}(B'/IB')$. Setting $A = A'/I$ and $B = B'/IB'$ we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & IB' \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & IA' \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows and $I^2 = 0$.

The translation of (1) into algebra: If $A \to B$ is surjective, then $A' \to B'$ is surjective. This follows from Nakayama's lemma (Algebra, Lemma 20.1).

The translation of (2) into algebra: If $A \to B$ is a finite type ring map, then $A' \to B'$ is a finite type ring map. This follows from Nakayama's lemma (Algebra, Lemma 20.1) applied to a map $A'[x_1, \ldots, x_n] \to B'$ such that $A[x_1, \ldots, x_n] \to B$ is surjective.

Proof of (3). Follows from (2) and that quasi-finiteness of a morphism which is locally of finite type can be checked on fibres, see Morphisms, Lemma 20.6.

Proof of (4). Follows from (2) and that the additional property of "being of relative dimension d" can be checked on fibres (by definition, see Morphisms, Definition 29.1.

The translation of (5) into algebra: If $\Omega_{B/A} = 0$, then $\Omega_{B'/A'} = 0$. By Algebra, Lemma 131.12 we have $0 = \Omega_{B/A} = \Omega_{B'/A'}/I\Omega_{B'/A'}$. Hence $\Omega_{B'/A'} = 0$ by Nakayama's lemma (Algebra, Lemma 20.1).

The translation of (6) into algebra: If $A \to B$ is unramified map, then $A' \to B'$ is unramified. Since $A \to B$ is of finite type we see that $A' \to B'$ is of finite type by (2) above. Since $A \to B$ is unramified we have $\Omega_{B/A} = 0$. By part (5) we have $\Omega_{B'/A'} = 0$. Thus $A' \to B'$ is unramified.

Proof of (7). Follows by combining (2) with results of Lemma 3.1 and the fact that proper equals quasi-compact + separated + locally of finite type + universally closed.

Proof of (8). Follows by combining (2) with results of Lemma 3.1 and using the fact that finite equals integral + locally of finite type (Morphisms, Lemma 44.4).

Proof of (9). As f is a monomorphism we have $X = X \times_S X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_S X \subset X' \times_{S'} X')$. We conclude $X' \to X' \times_{S'} X'$ is a closed immersion by (1). In fact, it is a first order thickening as the ideal defining the closed immersion $X' \to X' \times_{S'} X'$ is contained in the pullback of the ideal $\mathcal{I} \subset \mathcal{O}_{S'}$ cutting out S in S'. Indeed, $X = X \times_S X = (X' \times_{S'} X') \times_{S'} S$ is contained in X'. Hence by Morphisms, Lemma 32.7 it suffices to show that $\Omega_{X'/S'} = 0$ which follows from (5) and the corresponding statement for X/S.

Proof of (10). If $f : X \to S$ is an immersion, then it factors as $X \to U \to S$ where $U \to S$ is an open immersion and $X \to U$ is a closed immersion. Let $U' \subset S'$ be the open subscheme whose underlying topological space is the same as U. Then $X' \to S'$ factors through U' and we conclude that $X' \to U'$ is a closed immersion by part (1). This finishes the proof. $\square$

The following lemma is a variant on the preceding one. Rather than assume that the thickenings involved are finite order (which allows us to transfer the property of being locally of finite type from f to f'), we instead take as given that each of f and f' is locally of finite type.

0BPG **Lemma 3.4.** *Let $(f, f') : (X \subset X') \to (Y \to Y')$ be a morphism of thickenings. Assume f and f' are locally of finite type and $X = Y \times_{Y'} X'$. Then*

1. *f is locally quasi-finite if and only if f' is locally quasi-finite,*
2. *f is finite if and only if f' is finite,*
3. *f is a closed immersion if and only if f' is a closed immersion,*
4. *$\Omega_{X/Y} = 0$ if and only if $\Omega_{X'/Y'} = 0$,*
5. *f is unramified if and only if f' is unramified,*
6. *f is a monomorphism if and only if f' is a monomorphism,*
7. *f is an immersion if and only if f' is an immersion,*
8. *f is proper if and only if f' is proper, and*
9. *add more here.*

Proof. The properties $\mathcal{P}$ listed in the lemma are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 18.2 and 23.5

and Morphisms, Lemmas 20.13, 29.2, 32.10, 35.5, 41.5, and 44.6. Hence in each case we need only to prove that if f has the desired property, so does f'.

A morphism is locally quasi-finite if and only if it is locally of finite type and the scheme theoretic fibres are discrete spaces, see Morphisms, Lemma 20.8. Since the underlying topological space is unchanged by passing to a thickening, we see that f' is locally quasi-finite if (and only if) f is. This proves (1).

Case (2) follows from case (5) of Lemma 3.1 and the fact that the finite morphisms are precisely the integral morphisms that are locally of finite type (Morphisms, Lemma 44.4).

Case (3). This follows immediately from Morphisms, Lemma 45.7.

Case (4) follows from the following algebra statement: Let A be a ring and let $I \subset A$ be a locally nilpotent ideal. Let B be a finite type A-algebra. If $\Omega_{(B/IB)/(A/I)} = 0$, then $\Omega_{B/A} = 0$. Namely, the assumption means that $I\Omega_{B/A} = 0$, see Algebra, Lemma 131.12. On the other hand $\Omega_{B/A}$ is a finite B-module, see Algebra, Lemma 131.16. Hence the vanishing of $\Omega_{B/A}$ follows from Nakayama's lemma (Algebra, Lemma 20.1) and the fact that IB is contained in the Jacobson radical of B.

Case (5) follows immediately from (4) and Morphisms, Lemma 35.2.

Proof of (6). As f is a monomorphism we have $X = X \times_Y X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_Y X \subset X' \times_{Y'} X')$. We conclude $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$ is a closed immersion by (3). In fact $\Delta_{X'/Y'}$ is a bijection on underlying sets, hence $\Delta_{X'/Y'}$ is a thickening. On the other hand $\Delta_{X'/Y'}$ is locally of finite presentation by Morphisms, Lemma 21.12. In other words, $\Delta_{X'/Y'}(X')$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{X' \times_{Y'} X'}$ of finite type. Since $\Omega_{X'/Y'} = 0$ by (5) we see that the conormal sheaf of $X' \to X' \times_{Y'} X'$ is zero by Morphisms, Lemma 32.7. In other words, $\mathcal{J}/\mathcal{J}^2 = 0$. This implies $\Delta_{X'/Y'}$ is an isomorphism, for example by Algebra, Lemma 21.5.

Proof of (7). If $f : X \to Y$ is an immersion, then it factors as $X \to V \to Y$ where $V \to Y$ is an open immersion and $X \to V$ is a closed immersion. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is the same as V. Then $X' \to V'$ factors through V' and we conclude that $X' \to V'$ is a closed immersion by part (3).

Case (8) follows from Lemma 3.1 and the definition of proper morphisms as being the quasi-compact, universally closed, and separated morphisms that are locally of finite type. □

4. Picard groups of thickenings

0C6Q Some material on Picard groups of thickenings.

0C6R **Lemma 4.1.** *Let $X \subset X'$ be a first order thickening with ideal sheaf $\mathcal{I}$. Then there is a canonical exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathcal{I}) \arrow[r] & H^0(X', \mathcal{O}_{X'}^*) \arrow[r] & H^0(X, \mathcal{O}_X^*) \arrow[dll, rounded corners] \\ & H^1(X, \mathcal{I}) \arrow[r] & \mathrm{Pic}(X') \arrow[r] & \mathrm{Pic}(X) \arrow[dll, rounded corners] \\ & H^2(X, \mathcal{I}) \arrow[r] & \ldots \arrow[r] & \ldots \end{tikzcd}$$

of abelian groups.

Proof. This is the long exact cohomology sequence associated to the short exact sequence of sheaves of abelian groups

$$0 \to \mathcal{I} \to \mathcal{O}_{X'}^* \to \mathcal{O}_X^* \to 0$$

where the first map sends a local section f of $\mathcal{I}$ to the invertible section $1 + f$ of $\mathcal{O}_{X'}$. We also use the identification of the Picard group of a ringed space with the first cohomology group of the sheaf of invertible functions, see Cohomology, Lemma 6.1. □

0C6S **Lemma 4.2.** *Let $X \subset X'$ be a thickening. Let n be an integer invertible in $\mathcal{O}_X$. Then the map $\mathrm{Pic}(X')[n] \to \mathrm{Pic}(X)[n]$ is bijective.*

Proof for a finite order thickening. By the general principle explained following Definition 2.1 this reduces to the case of a first order thickening. Then may use Lemma 4.1 to see that it suffices to show that $H^1(X, \mathcal{I})[n]$, $H^1(X, \mathcal{I})/n$, and $H^2(X, \mathcal{I})[n]$ are zero. This follows as multiplication by n on $\mathcal{I}$ is an isomorphism as it is an $\mathcal{O}_X$-module. □

Proof in general. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the quasi-coherent ideal sheaf cutting out X. Then we have a short exact sequence of abelian groups

$$0 \to (1 + \mathcal{I})^* \to \mathcal{O}_{X'}^* \to \mathcal{O}_X^* \to 0$$

We obtain a long exact cohomology sequence as in the statement of Lemma 4.1 with $H^i(X, \mathcal{I})$ replaced by $H^i(X, (1 + \mathcal{I})^*)$. Thus it suffices to show that raising to the nth power is an isomorphism $(1 + \mathcal{I})^* \to (1 + \mathcal{I})^*$. Taking sections over affine opens this follows from Algebra, Lemma 32.8. □

5. First order infinitesimal neighbourhood

05YW A natural construction of first order thickenings is the following. Suppose that $i : Z \to X$ be an immersion of schemes. Choose an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme $Z \subset U$. Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals defining Z in U. Then we can consider the closed subscheme $Z' \subset U$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}^2$.

04EY **Definition 5.1.** Let $i : Z \to X$ be an immersion of schemes. The *first order infinitesimal neighbourhood* of Z in X is the first order thickening $Z \subset Z'$ over X described above.

This thickening has the following universal property (which will assuage any fears that the construction above depends on the choice of the open U).

04EZ **Lemma 5.2.** *Let $i : Z \to X$ be an immersion of schemes. The first order infinitesimal neighbourhood Z' of Z in X has the following universal property: Given any commutative diagram*

$$\begin{tikzcd} Z \arrow[d, "i"'] & T \arrow[l, "a"'] \arrow[d] \\ X & T' \arrow[l, "b"'] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z')$ of thickenings over X.

Proof. Let $U \subset X$ be the open used in the construction of Z', i.e., an open such that Z is identified with a closed subscheme of U cut out by the quasi-coherent sheaf of ideals $\mathcal{I}$. Since $|T| = |T'|$ we see that $b(T') \subset U$. Hence we can think of b as a morphism into U. Let $\mathcal{J} \subset \mathcal{O}_{T'}$ be the ideal cutting out T. Since $b(T) \subset Z$ by the diagram above we see that $b^\sharp(b^{-1}\mathcal{I}) \subset \mathcal{J}$. As T' is a first order thickening of T we see that $\mathcal{J}^2 = 0$ hence $b^\sharp(b^{-1}(\mathcal{I}^2)) = 0$. By Schemes, Lemma 4.6 this implies that b factors through Z'. Denote $a' : T' \to Z'$ this factorization and everything is clear. $\square$

04F0 **Lemma 5.3.** *Let $i : Z \to X$ be an immersion of schemes. Let $Z \subset Z'$ be the first order infinitesimal neighbourhood of Z in X. Then the diagram*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

induces a map of conormal sheaves $\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'}$ by Morphisms, Lemma 31.3. This map is an isomorphism.

Proof. This is clear from the construction of Z' above. $\square$

6. Formally unramified morphisms

02H7 Recall that a ring map $R \to A$ is called *formally unramified* (see Algebra, Definition 148.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal of square zero, at most one dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

02H8 **Definition 6.1.** Let $f : X \to S$ be a morphism of schemes. We say f is *formally unramified* if given any solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists at most one dotted arrow making the diagram commute.

We first prove some formal lemmas, i.e., lemmas which can be proved by drawing the corresponding diagrams.

04F1 **Lemma 6.2.** *If $f : X \to S$ is a formally unramified morphism, then given any solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of schemes over S there exists at most one dotted arrow making the diagram commute. In other words, in Definition 6.1 the condition that T be affine may be dropped.

Proof. This is true because a morphism is determined by its restrictions to affine opens. □

02HA **Lemma 6.3.** *A composition of formally unramified morphisms is formally unramified.*

Proof. This is formal. □

02HB **Lemma 6.4.** *A base change of a formally unramified morphism is formally unramified.*

Proof. This is formal. □

02HC **Lemma 6.5.** *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open such that $f(U) \subset V$. If f is formally unramified, so is $f|_U : U \to V$.*

Proof. Consider a solid diagram

$$\begin{tikzcd} U \arrow[d, "f|_U"'] & T \arrow[l, "a"] \arrow[d, "i"] \\ V & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

as in Definition 6.1. If f is formally ramified, then there exists at most one S-morphism $a' : T' \to X$ such that $a'|_T = a$. Hence clearly there exists at most one such morphism into U. □

02HD **Lemma 6.6.** *Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally unramified if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally unramified ring map.*

Proof. This is immediate from the definitions (Definition 6.1 and Algebra, Definition 148.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 6.5. □

Here is a characterization in terms of the sheaf of differentials.

02H9 **Lemma 6.7.** *Let $f : X \to S$ be a morphism of schemes. Then f is formally unramified if and only if $\Omega_{X/S} = 0$.*

Proof. We recall some of the arguments of the proof of Morphisms, Lemma 32.5. Let $W \subset X \times_S X$ be an open such that $\Delta : X \to X \times_S X$ induces a closed immersion into W. Let $\mathcal{J} \subset \mathcal{O}_W$ be the ideal sheaf of this closed immersion. Let $X' \subset W$ be the closed subscheme defined by the quasi-coherent sheaf of ideals $\mathcal{J}^2$. Consider the two morphisms $p_1, p_2 : X' \to X$ induced by the two projections $X \times_S X \to X$. Note that p_1 and p_2 agree when composed with $\Delta : X \to X'$ and that $X \to X'$ is a closed immersion defined by a an ideal whose square is zero. Moreover there is a short exact sequence

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

and $\Omega_{X/S} = \mathcal{J}/\mathcal{J}^2$. Moreover, $\mathcal{J}/\mathcal{J}^2$ is generated by the local sections $p_1^\sharp(f) - p_2^\sharp(f)$ for f a local section of $\mathcal{O}_X$.

Suppose that $f : X \to S$ is formally unramified. By assumption this means that $p_1 = p_2$ when restricted to any affine open $T' \subset X'$. Hence $p_1 = p_2$. By what was said above we conclude that $\Omega_{X/S} = \mathcal{J}/\mathcal{J}^2 = 0$.

Conversely, suppose that $\Omega_{X/S} = 0$. Then $X' = X$. Take any pair of morphisms $f'_1, f'_2 : T' \to X$ fitting as dotted arrows in the diagram of Definition 6.1. This gives a morphism $(f'_1, f'_2) : T' \to X \times_S X$. Since $f'_1|_T = f'_2|_T$ and $|T| = |T'|$ we see that the image of T' under (f'_1, f'_2) is contained in the open W chosen above. Since $(f'_1, f'_2)(T) \subset \Delta(X)$ and since T is defined by an ideal of square zero in T' we see that (f'_1, f'_2) factors through X'. As $X' = X$ we conclude $f'_1 = f'_2$ as desired. □

02HE **Lemma 6.8.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *The morphism f is unramified (resp. G-unramified), and*
2. *the morphism f is locally of finite type (resp. locally of finite presentation) and formally unramified.*

Proof. Use Lemma 6.7 and Morphisms, Lemma 35.2. □

7. Universal first order thickenings

04F2 Let $h : Z \to X$ be a morphism of schemes. A *universal first order thickening* of Z over X is a first order thickening $Z \subset Z'$ over X such that given any first order thickening $T \subset T'$ over X and a solid commutative diagram

$$\begin{tikzcd} & Z \arrow[dl] & & T \arrow[ll, "a"'] \arrow[dr] & \\ Z' \arrow[drr] & & & & T' \arrow[llll, dotted, "a'"'] \arrow[dll, "b"] \\ & & X & & \end{tikzcd}$$

there exists a unique dotted arrow making the diagram commute. Note that in this situation $(a, a') : (T \subset T') \to (Z \subset Z')$ is a morphism of thickenings over X. Thus if a universal first order thickening exists, then it is unique up to unique isomorphism. In general a universal first order thickening does not exist, but if h is formally unramified then it does.

04F3 **Lemma 7.1.** *Let $h : Z \to X$ be a formally unramified morphism of schemes. There exists a universal first order thickening $Z \subset Z'$ of Z over X.*

Proof. During this proof we will say $Z \subset Z'$ is a universal first order thickening of Z over X if it satisfies the condition of the lemma. We will construct the universal first order thickening $Z \subset Z'$ over X by glueing, starting with the affine case which is Algebra, Lemma 149.1. We begin with some general remarks.

If a universal first order thickening of Z over X exists, then it is unique up to unique isomorphism. Moreover, suppose that $V \subset Z$ and $U \subset X$ are open subschemes such that $h(V) \subset U$. Let $Z \subset Z'$ be a universal first order thickening of Z over X. Let $V' \subset Z'$ be the open subscheme such that $V = Z \cap V'$. Then we claim that $V \subset V'$ is the universal first order thickening of V over U. Namely, suppose given any diagram

$$\begin{tikzcd} V \arrow[d, "h"'] & T \arrow[l, "a"'] \arrow[d] \\ U & T' \arrow[l, "b"] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening over U. By the universal property of Z' we obtain $(a, a') : (T \subset T') \to (Z \subset Z')$. But since we have equality $|T| = |T'|$ of underlying topological spaces we see that $a'(T') \subset V'$. Hence we may think of (a, a') as a morphism of thickenings $(a, a') : (T \subset T') \to (V \subset V')$ over U. Uniqueness is clear also. In a completely similar manner one proves that if $h(Z) \subset U$ and $Z \subset Z'$ is a universal first order thickening over U, then $Z \subset Z'$ is a universal first order thickening over X.

Before we glue affine pieces let us show that the lemma holds if Z and X are affine. Say $X = \mathrm{Spec}(R)$ and $Z = \mathrm{Spec}(S)$. By Algebra, Lemma 149.1 there exists a first order thickening $Z \subset Z'$ over X which has the universal property of the lemma for diagrams

$$\begin{tikzcd} Z \arrow[d, "h"'] & T \arrow[l, "a"'] \arrow[d] \\ X & T' \arrow[l, "b"] \end{tikzcd}$$

where T, T' are affine. Given a general diagram we can choose an affine open covering $T' = \bigcup T'_i$ and we obtain morphisms $a'_i : T'_i \to Z'$ over X such that $a'_i|_{T_i} = a|_{T_i}$. By uniqueness we see that a'_i and a'_j agree on any affine open of $T'_i \cap T'_j$. Hence the morphisms a'_i glue to a global morphism $a' : T' \to Z'$ over X as desired. Thus the lemma holds if X and Z are affine.

Choose an affine open covering $Z = \bigcup Z_i$ such that each Z_i maps into an affine open U_i of X. By Lemma 6.5 the morphisms $Z_i \to U_i$ are formally unramified. Hence by the affine case we obtain universal first order thickenings $Z_i \subset Z'_i$ over U_i. By the general remarks above $Z_i \subset Z'_i$ is also a universal first order thickening of Z_i over X. Let $Z'_{i,j} \subset Z'_i$ be the open subscheme such that $Z_i \cap Z_j = Z'_{i,j} \cap Z_i$.

By the general remarks we see that both $Z'_{i,j}$ and $Z'_{j,i}$ are universal first order thickenings of $Z_i \cap Z_j$ over X. Thus, by the first of our general remarks, we see that there is a canonical isomorphism $\varphi_{ij} : Z'_{i,j} \to Z'_{j,i}$ inducing the identity on $Z_i \cap Z_j$. We claim that these morphisms satisfy the cocycle condition of Schemes, Section 14. (Verification omitted. Hint: Use that $Z'_{i,j} \cap Z'_{i,k}$ is the universal first order thickening of $Z_i \cap Z_j \cap Z_k$ which determines it up to unique isomorphism by what was said above.) Hence we can use the results of Schemes, Section 14 to get a first order thickening $Z \subset Z'$ over X which the property that the open subscheme $Z'_i \subset Z'$ with $Z_i = Z'_i \cap Z$ is a universal first order thickening of Z_i over X.

It turns out that this implies formally that Z' is a universal first order thickening of Z over X. Namely, we have the universal property for any diagram

$$\begin{tikzcd} Z \arrow[d, "h"'] & T \arrow[l, "a"'] \arrow[d] \\ X & T' \arrow[l, "b"] \end{tikzcd}$$

where $a(T)$ is contained in some Z_i. Given a general diagram we can choose an open covering $T' = \bigcup T'_i$ such that $a(T_i) \subset Z_i$. We obtain morphisms $a'_i : T'_i \to Z'$ over X such that $a'_i|_{T_i} = a|_{T_i}$. We see that a'_i and a'_j necessarily agree on $T'_i \cap T'_j$ since both $a'_i|_{T'_i \cap T'_j}$ and $a'_j|_{T'_i \cap T'_j}$ are solutions of the problem of mapping into the universal first order thickening $Z'_i \cap Z'_j$ of $Z_i \cap Z_j$ over X. Hence the morphisms a'_i glue to a global morphism $a' : T' \to Z'$ over X as desired. This finishes the proof. □

04F4 **Definition 7.2.** Let $h : Z \to X$ be a formally unramified morphism of schemes.

(1) The *universal first order thickening* of Z over X is the thickening $Z \subset Z'$ constructed in Lemma 7.1.
(2) The *conormal sheaf of Z over X* is the conormal sheaf of Z in its universal first order thickening Z' over X.

We often denote the conormal sheaf $\mathcal{C}_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$

on Z. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

04F5 **Lemma 7.3.** *Let $i : Z \to X$ be an immersion of schemes. Then*

(1) *i is formally unramified,*
(2) *the universal first order thickening of Z over X is the first order infinitesimal neighbourhood of Z in X of Definition 5.1, and*
(3) *the conormal sheaf of i in the sense of Morphisms, Definition 31.1 agrees with the conormal sheaf of i in the sense of Definition 7.2.*

Proof. By Morphisms, Lemmas 35.7 and 35.8 an immersion is unramified, hence formally unramified by Lemma 6.8. The other assertions follow by combining Lemmas 5.2 and 5.3 and the definitions. □

04F6 **Lemma 7.4.** *Let $Z \to X$ be a formally unramified morphism of schemes. Then the universal first order thickening Z' is formally unramified over X.*

Proof. There are two proofs. The first is to show that $\Omega_{Z'/X} = 0$ by working affine locally and applying Algebra, Lemma 149.5. Then Lemma 6.7 implies what we want. The second is a direct argument as follows.

Let $T \subset T'$ be a first order thickening. Let

$$\begin{tikzcd} Z' \arrow[d] & T \arrow[l, "c"'] \arrow[d] \\ X & T' \arrow[l] \arrow[ul, "{a,b}"'] \end{tikzcd}$$

be a commutative diagram. Consider two morphisms $a, b : T' \to Z'$ fitting into the diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since Z' is a first order thickening of Z, we see that T' is a first order thickening of T'_a. Moreover, since $c = a|_T$ we see that $T_0 = T \cap T'_a$ (scheme theoretically). As T' is a first order thickening of T it follows that T'_a is a first order thickening of T_0. Now $a|_{T'_a}$ and $b|_{T'_a}$ are morphisms of T'_a into Z' over X which agree on T_0 as morphisms into Z. Hence by the universal property of Z' we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus a and b are morphism from the first order thickening T' of T'_a whose restrictions to T'_a agree as morphisms into Z. Thus using the universal property of Z' once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired. □

04F7 **Lemma 7.5.** *Consider a commutative diagram of schemes*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

with h and h' formally unramified. Let $Z \subset Z'$ be the universal first order thickening of Z over X. Let $W \subset W'$ be the universal first order thickening of W over Y. There exists a canonical morphism $(f, f') : (Z, Z') \to (W, W')$ of thickenings over Y which fits into the following commutative diagram

$$\begin{tikzcd} & & Z' \arrow[dl] \arrow[d, "f'"] \\ Z \arrow[urr] \arrow[r] \arrow[d, "f"'] & X \arrow[d] & W' \arrow[dl] \\ W \arrow[urr] \arrow[r] & Y & \end{tikzcd}$$

In particular the morphism (f, f') of thickenings induces a morphism of conormal sheaves $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$.

Proof. The first assertion is clear from the universal property of W'. The induced map on conormal sheaves is the map of Morphisms, Lemma 31.3 applied to $(Z \subset Z') \to (W \subset W')$. □

04F8 **Lemma 7.6.** *Let*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

be a fibre product diagram in the category of schemes with h' formally unramified. Then h is formally unramified and if $W \subset W'$ is the universal first order thickening of W over Y, then $Z = X \times_Y W \subset X \times_Y W'$ is the universal first order thickening of Z over X. In particular the canonical map $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ of Lemma 7.5 is surjective.

Proof. The morphism h is formally unramified by Lemma 6.4. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Morphisms, Lemma 31.4 for why this implies that the map of conormal sheaves is surjective. □

04F9 **Lemma 7.7.** *Let*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

be a fibre product diagram in the category of schemes with h' formally unramified and g flat. In this case the corresponding map $Z' \to W'$ of universal first order thickenings is flat, and $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ is an isomorphism.

Proof. Flatness is preserved under base change, see Morphisms, Lemma 25.8. Hence the first statement follows from the description of W' in Lemma 7.6. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Morphisms, Lemma 31.4 for why this implies that the map of conormal sheaves is an isomorphism. □

04FA **Lemma 7.8.** *Taking the universal first order thickenings commutes with taking opens. More precisely, let $h : Z \to X$ be a formally unramified morphism of schemes. Let $V \subset Z$, $U \subset X$ be opens such that $h(V) \subset U$. Let Z' be the universal first order thickening of Z over X. Then $h|_V : V \to U$ is formally unramified and the universal first order thickening of V over U is the open subscheme $V' \subset Z'$ such that $V = Z \cap V'$. In particular, $\mathcal{C}_{Z/X}|_V = \mathcal{C}_{V/U}$.*

Proof. The first statement is Lemma 6.5. The compatibility of universal thickenings can be deduced from the proof of Lemma 7.1, or from Algebra, Lemma 149.4 or deduced from Lemma 7.7. □

04FB **Lemma 7.9.** *Let $h : Z \to X$ be a formally unramified morphism of schemes over S. Let $Z \subset Z'$ be the universal first order thickening of Z over X with structure morphism $h' : Z' \to X$. The canonical map*

$$c_{h'} : (h')^*\Omega_{X/S} \longrightarrow \Omega_{Z'/S}$$

induces an isomorphism $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$.

Proof. The map $c_{h'}$ is the map defined in Morphisms, Lemma 32.8. If $i : Z \to Z'$ is the given closed immersion, then $i^*c_{h'}$ is a map $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$. Checking that it is an isomorphism reduces to the affine case by localization, see Lemma 7.8 and Morphisms, Lemma 32.3. In this case the result is Algebra, Lemma 149.5. $\square$

04FC **Lemma 7.10.** *Let $h : Z \to X$ be a formally unramified morphism of schemes over S. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to h^*\Omega_{X/S} \to \Omega_{Z/S} \to 0.$$

The first arrow is induced by $\text{d}_{Z'/S}$ where Z' is the universal first order neighbourhood of Z over X.

Proof. We know that there is a canonical exact sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

see Morphisms, Lemma 32.15. Hence the result follows on applying Lemma 7.9. $\square$

067V **Lemma 7.11.** *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[rd, "j"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

be a commutative diagram of schemes where i and j are formally unramified. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 7.5 and the second from Lemma 7.10.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 7.10 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"] \arrow[rd, "j'"'] & Z' \arrow[r] \arrow[d] & X \arrow[d] \\ & Z'' \arrow[r] & Y \end{tikzcd}$$

Apply Morphisms, Lemma 32.18 to the left triangle to get an exact sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

As Z'' is formally unramified over Y (see Lemma 7.4) we have $\Omega_{Z'/Z''} = \Omega_{Z/Y}$ (by combining Lemma 6.7 and Morphisms, Lemma 32.9). Then we have $(i')^*\Omega_{Z'/Y} = i^*\Omega_{X/Y}$ by Lemma 7.9. $\square$

06AE **Lemma 7.12.** *Let $Z \to Y \to X$ be formally unramified morphisms of schemes.*

1. *If $Z \subset Z'$ is the universal first order thickening of Z over X and $Y \subset Y'$ is the universal first order thickening of Y over X, then there is a morphism $Z' \to Y'$ and $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y.*
2. *There is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 7.5 and $i : Z \to Y$ is the first morphism.

Proof. The map $h : Z' \to Y'$ in (1) comes from Lemma 7.5. The assertion that $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y is clear from the universal properties of Z' and Y'. By Morphisms, Lemma 31.5 we have an exact sequence

$$(i')^*\mathcal{C}_{Y \times_{Y'} Z'/Z'} \to \mathcal{C}_{Z/Z'} \to \mathcal{C}_{Z/Y \times_{Y'} Z'} \to 0$$

where $i' : Z \to Y \times_{Y'} Z'$ is the given morphism. By Morphisms, Lemma 31.4 there exists a surjection $h^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{Y \times_{Y'} Z'/Z'}$. Combined with the equalities $\mathcal{C}_{Y/Y'} = \mathcal{C}_{Y/X}$, $\mathcal{C}_{Z/Z'} = \mathcal{C}_{Z/X}$, and $\mathcal{C}_{Z/Y \times_{Y'} Z'} = \mathcal{C}_{Z/Y}$ this proves the lemma. □

8. Formally étale morphisms

02HF Recall that a ring map $R \to A$ is called *formally étale* (see Algebra, Definition 150.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rd, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal of square zero, there exists exactly one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

02HG **Definition 8.1.** Let $f : X \to S$ be a morphism of schemes. We say f is *formally étale* if given any solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists exactly one dotted arrow making the diagram commute.

It is clear that a formally étale morphism is formally unramified. Hence if $f : X \to S$ is formally étale, then $\Omega_{X/S}$ is zero, see Lemma 6.7.

04FD **Lemma 8.2.** *If $f : X \to S$ is a formally étale morphism, then given any solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of schemes over S there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 8.1 the condition that T be affine may be dropped.

Proof. Let $T' = \bigcup T'_i$ be an affine open covering, and let $T_i = T \cap T'_i$. Then we get morphisms $a'_i : T'_i \to X$ fitting into the diagram. By uniqueness we see that a'_i and a'_j agree on any affine open subscheme of $T'_i \cap T'_j$. Hence a'_i and a'_j agree on $T'_i \cap T'_j$. Thus we see that the morphisms a'_i glue to a global morphism $a' : T' \to X$. The uniqueness of a' we have seen in Lemma 6.2. □

02HI **Lemma 8.3.** *A composition of formally étale morphisms is formally étale.*

Proof. This is formal. ☐

02HJ **Lemma 8.4.** *A base change of a formally étale morphism is formally étale.*

Proof. This is formal. ☐

02HK **Lemma 8.5.** *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open subschemes such that $f(U) \subset V$. If f is formally étale, so is $f|_U : U \to V$.*

Proof. Consider a solid diagram

$$\begin{tikzcd} U \arrow[d, "f|_U"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ V & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

as in Definition 8.1. If f is formally ramified, then there exists exactly one S-morphism $a' : T' \to X$ such that $a'|_T = a$. Since $|T'| = |T|$ we conclude that $a'(T') \subset U$ which gives our unique morphism from T' into U. ☐

04FE **Lemma 8.6.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is formally étale,*
(2) *f is formally unramified and the universal first order thickening of X over S is equal to X,*
(3) *f is formally unramified and $\mathcal{C}_{X/S} = 0$, and*
(4) *$\Omega_{X/S} = 0$ and $\mathcal{C}_{X/S} = 0$.*

Proof. Actually, the last assertion only make sense because $\Omega_{X/S} = 0$ implies that $\mathcal{C}_{X/S}$ is defined via Lemma 6.7 and Definition 7.2. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that f is formally unramified. Hence we may assume f is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening X' of X over S and the fact that $X = X' \Leftrightarrow \mathcal{C}_{X/S} = 0$ since after all by definition $\mathcal{C}_{X/S} = \mathcal{C}_{X/X'}$ is the ideal sheaf of X in X'. ☐

04FF **Lemma 8.7.** *An unramified flat morphism is formally étale.*

Proof. Say $X \to S$ is unramified and flat. Then $\Delta : X \to X \times_S X$ is an open immersion, see Morphisms, Lemma 35.13. We have to show that $\mathcal{C}_{X/S}$ is zero. Consider the two projections $p, q : X \times_S X \to X$. As f is formally unramified (see Lemma 6.8), q is formally unramified (see Lemma 6.4). As f is flat, p is flat, see Morphisms, Lemma 25.8. Hence $p^*\mathcal{C}_{X/S} = \mathcal{C}_q$ by Lemma 7.7 where $\mathcal{C}_q$ denotes the conormal sheaf of the formally unramified morphism $q : X \times_S X \to X$. But $\Delta(X) \subset X \times_S X$ is an open subscheme which maps isomorphically to X via q. Hence by Lemma 7.8 we see that $\mathcal{C}_q|_{\Delta(X)} = \mathcal{C}_{X/X} = 0$. In other words, the pullback of $\mathcal{C}_{X/S}$ to X via the identity morphism is zero, i.e., $\mathcal{C}_{X/S} = 0$. ☐

02HL **Lemma 8.8.** *Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally étale if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally étale ring map.*

Proof. This is immediate from the definitions (Definition 8.1 and Algebra, Definition 150.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 6.5. □

02HM **Lemma 8.9.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *The morphism f is étale, and*
2. *the morphism f is locally of finite presentation and formally étale.*

Proof. Assume f is étale. An étale morphism is locally of finite presentation, flat and unramified, see Morphisms, Section 36. Hence f is locally of finite presentation and formally étale, see Lemma 8.7.

Conversely, suppose that f is locally of finite presentation and formally étale. Being étale is local in the Zariski topology on X and S, see Morphisms, Lemma 36.2. By Lemma 8.5 we can cover X by affine opens U which map into affine opens V such that $U \to V$ is formally étale (and of finite presentation, see Morphisms, Lemma 21.2). By Lemma 8.8 we see that the ring maps $\mathcal{O}(V) \to \mathcal{O}(U)$ are formally étale (and of finite presentation). We win by Algebra, Lemma 150.2. (We will give another proof of this implication when we discuss formally smooth morphisms.) □

9. Infinitesimal deformations of maps

04BU In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening X' of X can be seen as a sheaf on X.

04FG **Lemma 9.1.** *Let S be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings over S. Let $(a, a'), (b, b') : (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over S. Assume that*

1. *$a = b$, and*
2. *the two maps $a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ (Morphisms, Lemma 31.3) are equal.*

Then the map $(a')^\sharp - (b')^\sharp$ factors as

$$\mathcal{O}_{Y'} \to \mathcal{O}_Y \xrightarrow{D} a_*\mathcal{C}_{X/X'} \to a_*\mathcal{O}_{X'}$$

where D is an $\mathcal{O}_S$-derivation.

Proof. Instead of working on Y we work on X. The advantage is that the pullback functor a^{-1} is exact. Using (1) and (2) we obtain a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}_{X/X'} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & a^{-1}\mathcal{C}_{Y/Y'} \arrow[r] \arrow[u] & a^{-1}\mathcal{O}_{Y'} \arrow[r] \arrow[u, shift left, "(a')^\sharp"'] \arrow[u, shift right, "(b')^\sharp"] & a^{-1}\mathcal{O}_Y \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Now it is a general fact that in such a situation the difference of the $\mathcal{O}_S$-algebra maps $(a')^\sharp$ and $(b')^\sharp$ is an $\mathcal{O}_S$-derivation from $a^{-1}\mathcal{O}_Y$ to $\mathcal{C}_{X/X'}$. By adjointness of the functors a^{-1} and a_* this is the same thing as an $\mathcal{O}_S$-derivation from $\mathcal{O}_Y$ into $a_*\mathcal{C}_{X/X'}$. Some details omitted. □

Note that in the situation of the lemma above we may write D as

04BV (9.1.1) $$D = \mathrm{d}_{Y/S} \circ \theta$$

where θ is an $\mathcal{O}_Y$-linear map $\theta : \Omega_{Y/S} \to a_*\mathcal{C}_{X/X'}$. Of course, then by adjunction again we may view θ as an $\mathcal{O}_X$-linear map $\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$.

02H5 **Lemma 9.2.** *Let S be a scheme. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over S. Let*

$$\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$$

be an $\mathcal{O}_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 9.1 hold and the derivation D and θ are related by Equation (9.1.1).

Proof. We simply set $b = a$ and we define $(b')^\sharp$ to be the map

$$(a')^\sharp + D : a^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

where D is as in Equation (9.1.1). We omit the verification that $(b')^\sharp$ is a map of sheaves of $\mathcal{O}_S$-algebras and that (1) and (2) of Lemma 9.1 hold. Equation (9.1.1) holds by construction. $\square$

0CK1 **Remark 9.3.** Assumptions and notation as in Lemma 9.2. The action of a local section θ on a' is sometimes indicated by $\theta \cdot a'$. Note that this means nothing else than the fact that $(a')^\sharp$ and $(\theta \cdot a')^\sharp$ differ by a derivation D which is related to θ by Equation (9.1.1).

04FH **Lemma 9.4.** *Let S be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be first order thickenings over S. Assume given a morphism $a : X \to Y$ and a map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ of $\mathcal{O}_X$-modules. For an open subscheme $U' \subset X'$ consider morphisms $a' : U' \to Y'$ such that*

1. *a' is a morphism over S,*
2. *$a'|_U = a|_U$, and*
3. *the induced map $a^*\mathcal{C}_{Y/Y'}|_U \to \mathcal{C}_{X/X'}|_U$ is the restriction of A to U.*

Here $U = X \cap U'$. Then the rule

04FI (9.4.1) $$U' \mapsto \{a' : U' \to Y' \text{ such that (1), (2), (3) hold.}\}$$

defines a sheaf of sets on X'.

Proof. Denote $\mathcal{F}$ the rule of the lemma. The restriction mapping $\mathcal{F}(U') \to \mathcal{F}(V')$ for $V' \subset U' \subset X'$ of $\mathcal{F}$ is really the restriction map $a' \mapsto a'|_{V'}$. With this definition in place it is clear that $\mathcal{F}$ is a sheaf since morphisms are defined locally. $\square$

In the following lemma we identify sheaves on X and any thickening of X.

04FJ **Lemma 9.5.** *Same notation and assumptions as in Lemma 9.4. There is an action of the sheaf*

$$\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})$$

on the sheaf (9.4.1). Moreover, the action is simply transitive for any open $U' \subset X'$ over which the sheaf (9.4.1) has a section.

Proof. This is a combination of Lemmas 9.1, 9.2, and 9.4. $\square$

04FK **Remark 9.6.** A special case of Lemmas 9.1, 9.2, 9.4, and 9.5 is where $Y = Y'$. In this case the map A is always zero. The sheaf of Lemma 9.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y \text{ over } S \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})$.

0CK2 **Remark 9.7.** Another special case of Lemmas 9.1, 9.2, 9.4, and 9.5 is where S itself is a thickening $Z \subset Z' = S$ and $Y = Z \times_{Z'} Y'$. Picture

$$\begin{tikzcd} (X \subset X') \arrow[rr, dotted, "{(a,?)}"'] \arrow[dr, "{(g,g')}"'] & & (Y \subset Y') \arrow[dl, "{(h,h')}"] \\ & (Z \subset Z') & \end{tikzcd}$$

In this case the map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is determined by a: the map $h^*\mathcal{C}_{Z/Z'} \to \mathcal{C}_{Y/Y'}$ is surjective (because we assumed $Y = Z \times_{Z'} Y'$), hence the pullback $g^*\mathcal{C}_{Z/Z'} = a^*h^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'}$ is surjective, and the composition $g^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ has to be the canonical map induced by g'. Thus the sheaf of Lemma 9.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y' \text{ over } Z' \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'})$.

04FL **Lemma 9.8.** *Let S be a scheme. Let $X \subset X'$ be a first order thickening over S. Let Y be a scheme over S. Let $a', b' : X' \to Y$ be two morphisms over S with $a = a'|_X = b'|_X$. This gives rise to a commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "a"'] & X' \arrow[d, "{(b',a')}"] \\ Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y \end{tikzcd}$$

Since the horizontal arrows are immersions with conormal sheaves $\mathcal{C}_{X/X'}$ and $\Omega_{Y/S}$, by Morphisms, Lemma 31.3, we obtain a map $\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$. Then this θ and the derivation D of Lemma 9.1 are related by Equation (9.1.1).

Proof. Omitted. Hint: The equality may be checked on affine opens where it comes from the following computation. If f is a local section of $\mathcal{O}_Y$, then $1 \otimes f - f \otimes 1$ is a local section of $\mathcal{C}_{Y/(Y \times_S Y)}$ corresponding to $\mathrm{d}_{Y/S}(f)$. It is mapped to the local section $(a')^\sharp(f) - (b')^\sharp(f) = D(f)$ of $\mathcal{C}_{X/X'}$. In other words, $\theta(\mathrm{d}_{Y/S}(f)) = D(f)$. $\square$

For later purposes we need a result that roughly states that the construction of Lemma 9.2 is compatible with étale localization.

04BX **Lemma 9.9.** *Let*

$$\begin{tikzcd} X_1 \arrow[d] & X_2 \arrow[l, "f"'] \arrow[d] \\ S_1 & S_2 \arrow[l] \end{tikzcd}$$

be a commutative diagram of schemes with $X_2 \to X_1$ and $S_2 \to S_1$ étale. Then the map $c_f : f^*\Omega_{X_1/S_1} \to \Omega_{X_2/S_2}$ of Morphisms, Lemma 32.8 is an isomorphism.

Proof. We recall that an étale morphism $U \to V$ is a smooth morphism with $\Omega_{U/V} = 0$. Using this we see that Morphisms, Lemma 32.9 implies $\Omega_{X_2/S_2} = \Omega_{X_2/S_1}$ and Morphisms, Lemma 34.16 implies that the map $f^*\Omega_{X_1/S_1} \to \Omega_{X_2/S_1}$ (for the morphism f seen as a morphism over S_1) is an isomorphism. Hence the lemma follows. $\square$

04BY **Lemma 9.10.** *Consider a commutative diagram of first order thickenings*

$$\begin{tikzcd} (T_2 \subset T_2') \arrow[r, "(a_2, a_2')"'] \arrow[d, "(h,h')"'] & (X_2 \subset X_2') \arrow[d, "(f,f')"] \\ (T_1 \subset T_1') \arrow[r, "(a_1, a_1')"] & (X_1 \subset X_1') \end{tikzcd} \quad \textit{and a commutative diagram of schemes} \quad \begin{tikzcd} X_2' \arrow[r] \arrow[d] & S_2 \arrow[d] \\ X_1' \arrow[r] & S_1 \end{tikzcd}$$

with $X_2 \to X_1$ and $S_2 \to S_1$ étale. For any $\mathcal{O}_{T_1}$-linear map $\theta_1 : a_1^*\Omega_{X_1/S_1} \to \mathcal{C}_{T_1/T_1'}$ let θ_2 be the composition

$$a_2^*\Omega_{X_2/S_2} \Longrightarrow h^*a_1^*\Omega_{X_1/S_1} \xrightarrow{h^*\theta_1} h^*\mathcal{C}_{T_1/T_1'} \longrightarrow \mathcal{C}_{T_2/T_2'}$$

(equality sign is explained in the proof). Then the diagram

$$\begin{tikzcd} T_2' \arrow[r, "\theta_2 \cdot a_2'"'] \arrow[d] & X_2' \arrow[d] \\ T_1' \arrow[r, "\theta_1 \cdot a_1'"] & X_1' \end{tikzcd}$$

commutes where the actions $\theta_2 \cdot a_2'$ and $\theta_1 \cdot a_1'$ are as in Remark 9.3.

Proof. The equality sign comes from the identification $f^*\Omega_{X_1/S_1} = \Omega_{X_2/S_2}$ of Lemma 9.9. Namely, using this we have $a_2^*\Omega_{X_2/S_2} = a_2^*f^*\Omega_{X_1/S_1} = h^*a_1^*\Omega_{X_1/S_1}$ because $f \circ a_2 = a_1 \circ h$. Having said this, the commutativity of the diagram may be checked on affine opens. Hence we may assume the schemes in the initial big diagram are affine. Thus we obtain commutative diagrams

$$\begin{tikzcd} (B_2', I_2) & (A_2', J_2) \arrow[l, "a_2'"'] \\ (B_1', I_1) \arrow[u, "h'"] & (A_1', J_1) \arrow[l, "a_1'"] \arrow[u, "f'"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_2' & R_2 \arrow[l] \\ A_1' \arrow[u] & R_1 \arrow[l] \arrow[u] \end{tikzcd}$$

The notation signifies that I_1, I_2, J_1, J_2 are ideals of square zero and maps of pairs are ring maps sending ideals into ideals. Set $A_1 = A_1'/J_1$, $A_2 = A_2'/J_2$, $B_1 = B_1'/I_1$, and $B_2 = B_2'/I_2$. We are given that

$$A_2 \otimes_{A_1} \Omega_{A_1/R_1} \longrightarrow \Omega_{A_2/R_2}$$

is an isomorphism. Then $\theta_1 : B_1 \otimes_{A_1} \Omega_{A_1/R_1} \to I_1$ is B_1-linear. This gives an R_1-derivation $D_1 = \theta_1 \circ \text{d}_{A_1/R_1} : A_1 \to I_1$. In a similar way we see that $\theta_2 : B_2 \otimes_{A_2} \Omega_{A_2/R_2} \to I_2$ gives rise to a R_2-derivation $D_2 = \theta_2 \circ \text{d}_{A_2/R_2} : A_2 \to I_2$. The construction of θ_2 implies the following compatibility between θ_1 and θ_2: for every $x \in A_1$ we have

$$h'(D_1(x)) = D_2(f'(x))$$

as elements of I_2. We may view D_1 as a map $A_1' \to B_1'$ using $A_1' \to A_1 \xrightarrow{D_1} I_1 \to B_1$ similarly we may view D_2 as a map $A_2' \to B_2'$. Then the displayed equality holds for $x \in A_1'$. By the construction of the action in Lemma 9.2 and Remark 9.3 we know that $\theta_1 \cdot a_1'$ corresponds to the ring map $a_1' + D_1 : A_1' \to B_1'$ and $\theta_2 \cdot a_2'$ corresponds to the ring map $a_2' + D_2 : A_2' \to B_2'$. By the displayed equality we obtain that $h' \circ (a_1' + D_1) = (a_2' + D_2) \circ f'$ as desired. □

04BZ **Remark 9.11.** Lemma 9.10 can be improved in the following way. Suppose that we have commutative diagrams as in Lemma 9.10 but we do not assume that $X_2 \to X_1$ and $S_2 \to S_1$ are étale. Next, suppose we have $\theta_1 : a_1^*\Omega_{X_1/S_1} \to \mathcal{C}_{T_1/T_1'}$ and $\theta_2 : a_2^*\Omega_{X_2/S_2} \to \mathcal{C}_{T_2/T_2'}$ such that

$$\begin{tikzcd} f_*\mathcal{O}_{X_2} \arrow[r, "f_*D_2"'] & f_*a_{2,*}\mathcal{C}_{T_2/T_2'} \\ \mathcal{O}_{X_1} \arrow[u, "f^\sharp"] \arrow[r, "D_1"] & a_{1,*}\mathcal{C}_{T_1/T_1'} \arrow[u, "\text{induced by } (h')^\sharp"'] \end{tikzcd}$$

is commutative where D_i corresponds to θ_i as in Equation (9.1.1). Then we have the conclusion of Lemma 9.10. The importance of the condition that both $X_2 \to X_1$ and $S_2 \to S_1$ are étale is that it allows us to construct a θ_2 from θ_1.

10. Infinitesimal deformations of schemes

063X The following simple lemma is often a convenient tool to check whether an infinitesimal deformation of a map is flat.

063Y **Lemma 10.1.** *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of first order thickenings. Assume that f is flat. Then the following are equivalent*

(1) *f' is flat and $X = S \times_{S'} X'$, and*
(2) *the canonical map $f^*\mathcal{C}_{S/S'} \to \mathcal{C}_{X/X'}$ is an isomorphism.*

Proof. As the problem is local on X' we may assume that X, X', S, S' are affine schemes. Say $S' = \mathrm{Spec}(A')$, $X' = \mathrm{Spec}(B')$, $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ with $A = A'/I$ and $B = B'/J$ for some square zero ideals. Then we obtain the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. The canonical map of the lemma is the map

$$I \otimes_A B = I \otimes_{A'} B' \longrightarrow J.$$

The assumption that f is flat signifies that $A \to B$ is flat.

Assume (1). Then $A' \to B'$ is flat and $J = IB'$. Flatness implies $\mathrm{Tor}_1^{A'}(B', A) = 0$ (see Algebra, Lemma 75.8). This means $I \otimes_{A'} B' \to B'$ is injective (see Algebra, Remark 75.9). Hence we see that $I \otimes_A B \to J$ is an isomorphism.

Assume (2). Then it follows that $J = IB'$, so that $X = S \times_{S'} X'$. Moreover, we get $\mathrm{Tor}_1^{A'}(B', A'/I) = 0$ by reversing the implications in the previous paragraph. Hence B' is flat over A' by Algebra, Lemma 99.8. □

The following lemma is the "nilpotent" version of the "critère de platitude par fibres", see Section 16.

06AF **Lemma 10.2.** *Consider a commutative diagram*

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\ & (S \subset S') & \end{tikzcd}$$

of thickenings. Assume

1. X' *is flat over* S',
2. f *is flat,*
3. $S \subset S'$ *is a finite order thickening, and*
4. $X = S \times_{S'} X'$ *and* $Y = S \times_{S'} Y'$.

Then f' *is flat and* Y' *is flat over* S' *at all points in the image of* f'.

Proof. Immediate consequence of Algebra, Lemma 101.8. $\square$

Many properties of morphisms of schemes are preserved under flat deformations.

06AG **Lemma 10.3.** *Consider a commutative diagram*

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\ & (S \subset S') & \end{tikzcd}$$

of thickenings. Assume $S \subset S'$ *is a finite order thickening,* X' *flat over* S', $X = S \times_{S'} X'$, *and* $Y = S \times_{S'} Y'$. *Then*

06AH (1) f *is flat if and only if* f' *is flat,*
06AI (2) f *is an isomorphism if and only if* f' *is an isomorphism,*
06AJ (3) f *is an open immersion if and only if* f' *is an open immersion,*
06AK (4) f *is quasi-compact if and only if* f' *is quasi-compact,*
06AL (5) f *is universally closed if and only if* f' *is universally closed,*
06AM (6) f *is (quasi-)separated if and only if* f' *is (quasi-)separated,*
06AN (7) f *is a monomorphism if and only if* f' *is a monomorphism,*
06AP (8) f *is surjective if and only if* f' *is surjective,*
06AQ (9) f *is universally injective if and only if* f' *is universally injective,*
06AR (10) f *is affine if and only if* f' *is affine,*
06AS (11) f *is locally of finite type if and only if* f' *is locally of finite type,*
06AT (12) f *is locally quasi-finite if and only if* f' *is locally quasi-finite,*
06AU (13) f *is locally of finite presentation if and only if* f' *is locally of finite presentation,*
06AV (14) f *is locally of finite type of relative dimension* d *if and only if* f' *is locally of finite type of relative dimension* d,
06AW (15) f *is universally open if and only if* f' *is universally open,*
06AX (16) f *is syntomic if and only if* f' *is syntomic,*
06AY (17) f *is smooth if and only if* f' *is smooth,*
06AZ (18) f *is unramified if and only if* f' *is unramified,*
06B0 (19) f *is étale if and only if* f' *is étale,*
06B1 (20) f *is proper if and only if* f' *is proper,*

06B2 (21) *f is integral if and only if f' is integral,*
06B3 (22) *f is finite if and only if f' is finite,*
06B4 (23) *f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and*
(24) *add more here.*

Proof. The assumptions on X and Y mean that f is the base change of f' by $X \to X'$. The properties $\mathcal{P}$ listed in (1) – (23) above are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 18.2, 19.3, 21.12, and 23.5 and Morphisms, Lemmas 9.4, 10.4, 11.8, 15.4, 20.13, 21.4, 29.2, 30.4, 34.5, 35.5, 36.4, 41.5, 44.6, and 48.4.

The interesting direction in each case is therefore to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $S \subset S'$ is a first order thickening, see discussion immediately following Definition 2.1. We make a couple of general remarks which we will use without further mention in the arguments below. (I) Let $W' \subset S'$ be an affine open and let $U' \subset X'$ and $V' \subset Y'$ be affine opens lying over W' with $f'(U') \subset V'$. Let $W' = \text{Spec}(R')$ and denote $I \subset R'$ be the ideal defining the closed subscheme $W' \cap S$. Say $U' = \text{Spec}(B')$ and $V' = \text{Spec}(A')$. Then we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & IB' \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & IA' \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. Moreover $IB' \cong I \otimes_R B$, see proof of Lemma 10.1. (II) The morphisms $X \to X'$ and $Y \to Y'$ are universal homeomorphisms. Hence the topology of the maps f and f' (after any base change) is identical. (III) If f is flat, then f' is flat and $Y' \to S'$ is flat at every point in the image of f', see Lemma 10.2.

Ad (1). This is general remark (III).

Ad (2). Assume f is an isomorphism. By (III) we see that $Y' \to S'$ is flat. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $V \cong f^{-1}(V) = U = Y \times_{Y'} U'$ is affine. By Lemma 2.3 we see that U' is affine. Thus we have a diagram as in the general remark (I) and moreover $IA \cong I \otimes_R A$ because $R' \to A'$ is flat. Then $IB' \cong I \otimes_R B \cong I \otimes_R A \cong IA'$ and $A \cong B$. By the exactness of the rows in the diagram above we see that $A' \cong B'$, i.e., $U' \cong V'$. Thus f' is an isomorphism.

Ad (3). Assume f is an open immersion. Then f is an isomorphism of X with an open subscheme $V \subset Y$. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is V. Then f' is a map from X' to V' which is an isomorphism by (2). Hence f' is an open immersion.

Ad (4). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (5). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (6). Note that $X \times_Y X = Y \times_{Y'} (X' \times_{Y'} X')$ so that $X' \times_{Y'} X'$ is a thickening of $X \times_Y X$. Hence the topology of the maps $\Delta_{X/Y}$ and $\Delta_{X'/Y'}$ matches and we win. See also Lemma 3.1 for a more general statement.

Ad (7). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. The base change of $\Delta_{X'/Y'}$ by $S \to S'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (2) we conclude that $\Delta_{X'/Y'}$ is an isomorphism.

Ad (8). This is clear. See also Lemma 3.1 for a more general statement.

Ad (9). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (10). Assume f is affine. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $U = Y \times_{Y'} U'$ is affine. By Lemma 2.3 we see that U' is affine. Hence f' is affine. See also Lemma 3.1 for a more general statement.

Ad (11). Via remark (I) comes down to proving $A' \to B'$ is of finite type if $A \to B$ is of finite type. Suppose that $x_1, \ldots, x_n \in B'$ are elements whose images in B generate B as an A-algebra. Then $A'[x_1, \ldots, x_n] \to B$ is surjective as both $A'[x_1, \ldots, x_n] \to B$ is surjective and $I \otimes_R A[x_1, \ldots, x_n] \to I \otimes_R B$ is surjective. See also Lemma 3.3 for a more general statement.

Ad (12). Follows from (11) and that quasi-finiteness of a morphism of finite type can be checked on fibres, see Morphisms, Lemma 20.6. See also Lemma 3.3 for a more general statement.

Ad (13). Via remark (I) comes down to proving $A' \to B'$ is of finite presentation if $A \to B$ is of finite presentation. We may assume that $B' = A'[x_1, \ldots, x_n]/K'$ for some ideal K' by (11). We get a short exact sequence

$$0 \to K' \to A'[x_1, \ldots, x_n] \to B' \to 0$$

As B' is flat over R' we see that $K' \otimes_{R'} R$ is the kernel of the surjection $A[x_1, \ldots, x_n] \to B$. By assumption on $A \to B$ there exist finitely many $f'_1, \ldots, f'_m \in K'$ whose images in $A[x_1, \ldots, x_n]$ generate this kernel. Since I is nilpotent we see that $f'_1, \ldots, f'_m$ generate K' by Nakayama's lemma, see Algebra, Lemma 20.1.

Ad (14). Follows from (11) and general remark (II). See also Lemma 3.3 for a more general statement.

Ad (15). Immediate from general remark (II). See also Lemma 3.1 for a more general statement.

Ad (16). Assume f is syntomic. By (13) f' is locally of finite presentation, by general remark (III) f' is flat and the fibres of f' are the fibres of f. Hence f' is syntomic by Morphisms, Lemma 30.11.

Ad (17). Assume f is smooth. By (13) f' is locally of finite presentation, by general remark (III) f' is flat, and the fibres of f' are the fibres of f. Hence f' is smooth by Morphisms, Lemma 34.3.

Ad (18). Assume f unramified. By (11) f' is locally of finite type and the fibres of f' are the fibres of f. Hence f' is unramified by Morphisms, Lemma 35.12. See also Lemma 3.3 for a more general statement.

Ad (19). Assume f étale. By (13) f' is locally of finite presentation, by general remark (III) f' is flat, and the fibres of f' are the fibres of f. Hence f' is étale by Morphisms, Lemma 36.8.

Ad (20). This follows from a combination of (6), (11), (4), and (5). See also Lemma 3.3 for a more general statement.

Ad (21). Combine (5) and (10) with Morphisms, Lemma 44.7. See also Lemma 3.1 for a more general statement.

Ad (22). Combine (21), and (11) with Morphisms, Lemma 44.4. See also Lemma 3.3 for a more general statement.

Ad (23). Assume f finite locally free. By (22) we see that f' is finite, by general remark (III) f' is flat, and by (13) f' is locally of finite presentation. Hence f' is finite locally free by Morphisms, Lemma 48.2. □

The following lemma is the "locally nilpotent" version of the "critère de platitude par fibres", see Section 16.

0CF3 **Lemma 10.4.** *Consider a commutative diagram*

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[rd] & & (Y \subset Y') \arrow[ld] \\ & (S \subset S') & \end{tikzcd}$$

of thickenings. Assume

1. $Y' \to S'$ *is locally of finite type,*
2. $X' \to S'$ *is flat and locally of finite presentation,*
3. f *is flat, and*
4. $X = S \times_{S'} X'$ *and* $Y = S \times_{S'} Y'$.

Then f' is flat and for all $y' \in Y'$ in the image of f' the local ring $\mathcal{O}_{Y',y'}$ is flat and essentially of finite presentation over $\mathcal{O}_{S',s'}$.

Proof. Immediate consequence of Algebra, Lemma 128.10. □

Many properties of morphisms of schemes are preserved under flat deformations as in the lemma above.

0CF4 **Lemma 10.5.** *Consider a commutative diagram*

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[rd] & & (Y \subset Y') \arrow[ld] \\ & (S \subset S') & \end{tikzcd}$$

of thickenings. Assume $Y' \to S'$ locally of finite type, $X' \to S'$ flat and locally of finite presentation, $X = S \times_{S'} X'$, and $Y = S \times_{S'} Y'$. Then

0CF5 (1) *f is flat if and only if f' is flat,*
0CF6 (2) *f is an isomorphism if and only if f' is an isomorphism,*
0CF7 (3) *f is an open immersion if and only if f' is an open immersion,*
0CF8 (4) *f is quasi-compact if and only if f' is quasi-compact,*
0CF9 (5) *f is universally closed if and only if f' is universally closed,*

0CFA (6) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
0CFB (7) *f is a monomorphism if and only if f' is a monomorphism,*
0CFC (8) *f is surjective if and only if f' is surjective,*
0CFD (9) *f is universally injective if and only if f' is universally injective,*
0CFE (10) *f is affine if and only if f' is affine,*
0CFF (11) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
0CFG (12) *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
0CFH (13) *f is universally open if and only if f' is universally open,*
0CFI (14) *f is syntomic if and only if f' is syntomic,*
0CFJ (15) *f is smooth if and only if f' is smooth,*
0CFK (16) *f is unramified if and only if f' is unramified,*
0CFL (17) *f is étale if and only if f' is étale,*
0CFM (18) *f is proper if and only if f' is proper,*
0CFN (19) *f is finite if and only if f' is finite,*
0CFP (20) *f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and*
(21) *add more here.*

Proof. The assumptions on X and Y mean that f is the base change of f' by $X \to X'$. The properties $\mathcal{P}$ listed in (1) – (20) above are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 18.2, 19.3, 21.12, and 23.5 and Morphisms, Lemmas 9.4, 10.4, 11.8, 20.13, 29.2, 30.4, 34.5, 35.5, 36.4, 41.5, 44.6, and 48.4.

The interesting direction in each case is therefore to assume that f has the property and deduce that f' has it too. We make a couple of general remarks which we will use without further mention in the arguments below. (I) Let $W' \subset S'$ be an affine open and let $U' \subset X'$ and $V' \subset Y'$ be affine opens lying over W' with $f'(U') \subset V'$. Let $W' = \text{Spec}(R')$ and denote $I \subset R'$ be the ideal defining the closed subscheme $W' \cap S$. Say $U' = \text{Spec}(B')$ and $V' = \text{Spec}(A')$. Then we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & IB' \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & IA' \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. (II) The morphisms $X \to X'$ and $Y \to Y'$ are universal homeomorphisms. Hence the topology of the maps f and f' (after any base change) is identical. (III) If f is flat, then f' is flat and $Y' \to S'$ is flat at every point in the image of f', see Lemma 10.2.

Ad (1). This is general remark (III).

Ad (2). Assume f is an isomorphism. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $V \cong f^{-1}(V) = U = Y \times_{Y'} U'$ is affine. By Lemma 2.3 we see that U' is affine. Thus we have a diagram as in the general remark (I). By Algebra, Lemma 126.11 we see that $A' \to B'$ is an isomorphism, i.e., $U' \cong V'$. Thus f' is an isomorphism.

Ad (3). Assume f is an open immersion. Then f is an isomorphism of X with an open subscheme $V \subset Y$. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is V. Then f' is a map from X' to V' which is an isomorphism by (2). Hence f' is an open immersion.

Ad (4). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (5). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (6). Note that $X \times_Y X = Y \times_{Y'} (X' \times_{Y'} X')$ so that $X' \times_{Y'} X'$ is a thickening of $X \times_Y X$. Hence the topology of the maps $\Delta_{X/Y}$ and $\Delta_{X'/Y'}$ matches and we win. See also Lemma 3.1 for a more general statement.

Ad (7). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. Observe that $X' \times_{Y'} X' \to S'$ is locally of finite type. The base change of $\Delta_{X'/Y'}$ by $S \to S'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (2) we conclude that $\Delta_{X'/Y'}$ is an isomorphism.

Ad (8). This is clear. See also Lemma 3.1 for a more general statement.

Ad (9). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (10). Assume f is affine. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $U = Y \times_{Y'} U'$ is affine. By Lemma 2.3 we see that U' is affine. Hence f' is affine. See also Lemma 3.1 for a more general statement.

Ad (11). Follows from the fact that f' is locally of finite type (by Morphisms, Lemma 15.8) and that quasi-finiteness of a morphism of finite type can be checked on fibres, see Morphisms, Lemma 20.6.

Ad (12). Follows from general remark (II) and the fact that f' is locally of finite type (Morphisms, Lemma 15.8).

Ad (13). Immediate from general remark (II). See also Lemma 3.1 for a more general statement.

Ad (14). Assume f is syntomic. By Morphisms, Lemma 21.11 f' is locally of finite presentation. By general remark (III) f' is flat. The fibres of f' are the fibres of f. Hence f' is syntomic by Morphisms, Lemma 30.11.

Ad (15). Assume f is smooth. By Morphisms, Lemma 21.11 f' is locally of finite presentation. By general remark (III) f' is flat. The fibres of f' are the fibres of f. Hence f' is smooth by Morphisms, Lemma 34.3.

Ad (16). Assume f unramified. By Morphisms, Lemma 15.8 f' is locally of finite type. The fibres of f' are the fibres of f. Hence f' is unramified by Morphisms, Lemma 35.12.

Ad (17). Assume f étale. By Morphisms, Lemma 21.11 f' is locally of finite presentation. By general remark (III) f' is flat. The fibres of f' are the fibres of f. Hence f' is étale by Morphisms, Lemma 36.8.

Ad (18). This follows from a combination of (6), the fact that f is locally of finite type (Morphisms, Lemma 15.8), (4), and (5).

Ad (19). Combine (5), (10), Morphisms, Lemma 44.7, the fact that f is locally of finite type (Morphisms, Lemma 15.8), and Morphisms, Lemma 44.4.

Ad (20). Assume f finite locally free. By (19) we see that f' is finite. By general remark (III) f' is flat. By Morphisms, Lemma 21.11 f' is locally of finite presentation. Hence f' is finite locally free by Morphisms, Lemma 48.2. $\square$

0D4F **Lemma 10.6** (Deformations of projective schemes)**.** *Let $f : X \to S$ be a morphism of schemes which is proper, flat, and of finite presentation. Let $\mathcal{L}$ be f-ample. Assume S is quasi-compact. There exists a $d_0 \geq 0$ such that for every cartesian diagram*

$$\begin{tikzcd} X \arrow[r, "i'"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ S \arrow[r, "i"] & S' \end{tikzcd} \quad \text{and} \quad \begin{matrix} \text{invertible } \mathcal{O}_{X'}\text{-module} \\ \mathcal{L}' \text{ with } \mathcal{L} \cong (i')^*\mathcal{L}' \end{matrix}$$

where $S \subset S'$ is a thickening and f' is proper, flat, of finite presentation we have

1. *$R^p(f')_*(\mathcal{L}')^{\otimes d} = 0$ for all $p > 0$ and $d \geq d_0$,*
2. *$\mathcal{A}'_d = (f')_*(\mathcal{L}')^{\otimes d}$ is finite locally free for $d \geq d_0$,*
3. *$\mathcal{A}' = \mathcal{O}_{S'} \oplus \bigoplus_{d \geq d_0} \mathcal{A}'_d$ is a quasi-coherent $\mathcal{O}_{S'}$-algebra of finite presentation,*
4. *there is a canonical isomorphism $r' : X' \to \underline{Proj}_{S'}(\mathcal{A}')$, and*
5. *there is a canonical isomorphism $\theta' : (r')^*\mathcal{O}_{\underline{Proj}_{S'}(\mathcal{A}')}(1) \to \mathcal{L}'$.*

The construction of $\mathcal{A}'$, r', θ' is functorial in the data $(X', S', i, i', f', \mathcal{L}')$.

Proof. We first describe the maps r' and θ'. Observe that $\mathcal{L}'$ is f'-ample, see Lemma 3.2. There is a canonical map of quasi-coherent graded $\mathcal{O}_{S'}$-algebras $\mathcal{A}' \to \bigoplus_{d \geq 0}(f')_*(\mathcal{L}')^{\otimes d}$ which is an isomorphism in degrees $\geq d_0$. Hence this induces an isomorphism on relative Proj compatible with the Serre twists of the structure sheaf, see Constructions, Lemma 18.4. Hence we get the morphism r' by Morphisms, Lemma 37.4 (which in turn appeals to the construction given in Constructions, Lemma 19.1) and it is an isomorphism by Morphisms, Lemma 43.17. We get the map θ' from Constructions, Lemma 19.1. By Properties, Lemma 28.2 we find that θ' is an isomorphism (this also uses that the morphism r' over affine opens of S' is the same as the morphism from Properties, Lemma 26.9 as is explained in the proof of Morphisms, Lemma 43.17).

Assuming the vanishing and local freeness stated in parts (1) and (2), the functoriality of the construction can be seen as follows. Suppose that $h : T \to S'$ is a morphism of schemes, denote $f_T : X'_T \to T$ the base change of f' and $\mathcal{L}_T$ the pullback of $\mathcal{L}$ to X'_T. By cohomology and base change (as formulated in Derived Categories of Schemes, Lemma 22.5 for example) we have the corresponding vanishing over T and moreover $h^*\mathcal{A}'_d = f_{T,*}\mathcal{L}_T^{\otimes d}$ (and thus the local freeness of pushforwards as well as the finite generation of the corresponding graded $\mathcal{O}_T$-algebra $\mathcal{A}_T$). Hence the morphism $r_T : X_T \to \underline{\text{Proj}}_T(\bigoplus f_{T,*}\mathcal{L}_T^{\otimes d})$ is simply the base change of r' to T and the pullback of θ' is the map θ_T.

Having said all of the above, we see that it suffices to prove (1), (2), and (3). Pick d_0 such that $R^pf_*\mathcal{L}^{\otimes d} = 0$ for all $d \geq d_0$ and $p > 0$, see Cohomology of Schemes, Lemma 16.1. We claim that d_0 works.

By cohomology and base change (Derived Categories of Schemes, Lemma 30.4) we see that $E'_d = Rf'_*(\mathcal{L}')^{\otimes d}$ is a perfect object of $D(\mathcal{O}_{S'})$ and its formation commutes with arbitrary base change. In particular, $E_d = Li^*E'_d = Rf_*\mathcal{L}^{\otimes d}$. By Derived Categories of Schemes, Lemma 32.4 we see that for $d \geq d_0$ the complex E_d is isomorphic to the finite locally free $\mathcal{O}_S$-module $f_*\mathcal{L}^{\otimes d}$ placed in cohomological degree 0. Then by Derived Categories of Schemes, Lemma 31.3 we conclude that E'_d is isomorphic to a finite locally free module placed in cohomological degree 0. Of course this means that $E'_d = \mathcal{A}'_d[0]$, that $R^pf'_*(\mathcal{L}')^{\otimes d} = 0$ for $p > 0$, and that $\mathcal{A}'_d$ is finite locally free. This proves (1) and (2).

The last thing we have to show is finite presentation of $\mathcal{A}'$ as a sheaf of $\mathcal{O}_{S'}$-algebras (this notion was introduced in Properties, Section 22). Let $U' = \mathrm{Spec}(R') \subset S'$ be an affine open. Then $A' = \mathcal{A}'(U')$ is a graded R'-algebra whose graded parts are finite projective R'-modules. We have to show that A' is a finitely presented R'-algebra. We will prove this by reduction to the Noetherian case. Namely, we can find a finite type $\mathbf{Z}$-subalgebra $R'_0 \subset R'$ and a pair[1] $(X'_0, \mathcal{L}'_0)$ over R'_0 whose base change is $(X'_{U'}, \mathcal{L}'|_{X'_{U'}})$, see Limits, Lemmas 10.2, 10.3, 13.1, 8.7, and 4.15. Cohomology of Schemes, Lemma 16.1 implies $A'_0 = \bigoplus_{d \geq 0} H^0(X'_0, (\mathcal{L}'_0)^{\otimes d})$ is a finitely generated graded R'_0-algebra and implies there exists a d'_0 such that $H^p(X'_0, (\mathcal{L}'_0)^{\otimes d}) = 0$, $p > 0$ for $d \geq d'_0$. By the arguments given above applied to $X'_0 \to \mathrm{Spec}(R'_0)$ and $\mathcal{L}'_0$ we see that $(A'_0)_d$ is a finite projective R'_0-module and that

$$A'_d = \mathcal{A}'_d(U') = H^0(X'_{U'}, (\mathcal{L}')^{\otimes d}|_{X'_{U'}}) = H^0(X'_0, (\mathcal{L}'_0)^{\otimes d}) \otimes_{R'_0} R' = (A'_0)_d \otimes_{R'_0} R'$$

for $d \geq d'_0$. Now a small twist in the argument is that we don't know that we can choose d'_0 equal to d_0[2]. To get around this we use the following sequence of arguments to finish the proof:

(a) The algebra $B = R'_0 \oplus \bigoplus_{d \geq \max(d_0, d'_0)} (A'_0)_d$ is an R'_0-algebra of finite type: apply the Artin-Tate lemma to $B \subset A'_0$, see Algebra, Lemma 51.7.
(b) As R'_0 is Noetherian we see that B is an R'_0-algebra of finite presentation.
(c) By right exactness of tensor product we see that $B \otimes_{R'_0} R'$ is an R'-algebra of finite presentation.
(d) By the displayed equalities this exactly says that $C = R' \oplus \bigoplus_{d \geq \max(d_0, d'_0)} A'_d$ is an R'-algebra of finite presentation.
(e) The quotient A'/C is the direct sum of the finite projective R'-modules A'_d, $d_0 \leq d \leq \max(d_0, d'_0)$, hence finitely presented as R'-module.
(f) The quotient A'/C is finitely presented as a C-module by Algebra, Lemma 6.4.
(g) Thus A' is finitely presented as a C-module by Algebra, Lemma 5.3.
(h) By Algebra, Lemma 7.4 this implies A' is finitely presented as a C-algebra.
(i) Finally, by Algebra, Lemma 6.2 applied to $R' \to C \to A'$ this implies A' is finitely presented as an R'-algebra.

This finishes the proof. □

11. Formally smooth morphisms

02GZ Michael Artin's position on differential criteria of smoothness (e.g., Morphisms, Lemma 34.14) is that they are basically useless (in practice). In this section we introduce the notion of a formally smooth morphism $X \to S$. Such a morphism

[1] With the same properties as those enjoyed by $X' \to S'$ and $\mathcal{L}'$, i.e., $X'_0 \to \mathrm{Spec}(R'_0)$ is flat and proper and $\mathcal{L}'_0$ is ample.

[2] Actually, one can reduce to this case by doing more limit arguments.

is characterized by the property that T-valued points of X lift to infinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 11.7. It turns out that this criterion is often easier to use than the differential criteria mentioned above.

Recall that a ring map $R \to A$ is called *formally smooth* (see Algebra, Definition 138.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rd, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

02H0 **Definition 11.1.** Let $f : X \to S$ be a morphism of schemes. We say f is *formally smooth* if given any solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists a dotted arrow making the diagram commute.

In the cases of formally unramified and formally étale morphisms the condition that T' be affine could be dropped, see Lemmas 6.2 and 8.2. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow Zariski locally on T'. In fact, analyzing the proof of Lemma 11.10 shows that this would be equivalent to the definition as it currently stands. In particular, being formally smooth is Zariski local on the source (and in fact it is smooth local on the source, insert future reference here).

02H1 **Lemma 11.2.** *A composition of formally smooth morphisms is formally smooth.*

Proof. Omitted. ☐

02H2 **Lemma 11.3.** *A base change of a formally smooth morphism is formally smooth.*

Proof. Omitted, but see Algebra, Lemma 138.2 for the algebraic version. ☐

02HH **Lemma 11.4.** *Let $f : X \to S$ be a morphism of schemes. Then f is formally étale if and only if f is formally smooth and formally unramified.*

Proof. Omitted. ☐

02H3 **Lemma 11.5.** *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open subschemes such that $f(U) \subset V$. If f is formally smooth, so is $f|_U : U \to V$.*

Proof. Consider a solid diagram

$$\begin{tikzcd} U \arrow[d, "f|_U"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ V & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

as in Definition 11.1. If f is formally smooth, then there exists an S-morphism $a' : T' \to X$ such that $a'|_T = a$. Since the underlying sets of T and T' are the same we see that a' is a morphism into U (see Schemes, Section 3). And it clearly is a V-morphism as well. Hence the dotted arrow above as desired. $\square$

02H4 **Lemma 11.6.** *Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally smooth if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally smooth ring map.*

Proof. This is immediate from the definitions (Definition 11.1 and Algebra, Definition 138.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 6.5. $\square$

The following lemma is the main result of this section. It is a victory of the functorial point of view in that it implies (combined with Limits, Proposition 6.1) that we can recognize whether a morphism $f : X \to S$ is smooth in terms of "simple" properties of the functor $h_X : Sch/S \to Sets$.

02H6 **Lemma 11.7** (Infinitesimal lifting criterion)**.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *The morphism f is smooth, and*
2. *the morphism f is locally of finite presentation and formally smooth.*

Proof. Assume $f : X \to S$ is locally of finite presentation and formally smooth. Consider a pair of affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ such that $f(U) \subset V$. By Lemma 11.5 we see that $U \to V$ is formally smooth. By Lemma 11.6 we see that $R \to A$ is formally smooth. By Morphisms, Lemma 21.2 we see that $R \to A$ is of finite presentation. By Algebra, Proposition 138.13 we see that $R \to A$ is smooth. Hence by the definition of a smooth morphism we see that $X \to S$ is smooth.

Conversely, assume that $f : X \to S$ is smooth. Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

as in Definition 11.1. We will show the dotted arrow exists thereby proving that f is formally smooth.

Let $\mathcal{F}$ be the sheaf of sets on T' of Lemma 9.4 in the special case discussed in Remark 9.6. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 9.5. Our goal is simply to show that $\mathcal{F}(T) \neq \emptyset$. In other words we are trying to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor on T (see Cohomology, Section 4). There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}_t \neq \emptyset$ for all $t \in T$ (see Cohomology, Definition 4.1). (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T, \mathcal{H}) = 0$ (see Cohomology, Lemma 4.3 – we may use either cohomology of $\mathcal{H}$ as an abelian sheaf or as an $\mathcal{O}_T$-module, see Cohomology, Lemma 13.3).

First we prove (I). To see this, for every $t \in T$ we can choose an affine open $U \subset T$ neighbourhood of t such that $a(U)$ is contained in an affine open $\mathrm{Spec}(A) = W \subset X$ which maps to an affine open $\mathrm{Spec}(R) = V \subset S$. By Morphisms, Lemma 34.2 the ring map $R \to A$ is smooth. Hence by Algebra, Proposition 138.13 the ring map $R \to A$ is formally smooth. Lemma 11.6 in turn implies that $W \to V$ is formally smooth. Hence we can lift $a|_U : U \to W$ to a V-morphism $a' : U' \to W \subset X$ showing that $\mathcal{F}(U) \neq \emptyset$.

Finally we prove (II). By Morphisms, Lemma 32.13 we see that $\Omega_{X/S}$ is of finite presentation (it is even finite locally free by Morphisms, Lemma 34.12). Hence $a^*\Omega_{X/S}$ is of finite presentation (see Modules, Lemma 11.4). Hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'})$ is quasi-coherent by the discussion in Schemes, Section 24. Thus by Cohomology of Schemes, Lemma 2.2 we have $H^1(T, \mathcal{H}) = 0$ as desired. □

Locally projective quasi-coherent modules are defined in Properties, Section 21.

06B5 **Lemma 11.8.** *Let $f : X \to Y$ be a formally smooth morphism of schemes. Then $\Omega_{X/Y}$ is locally projective on X.*

Proof. Choose $U \subset X$ and $V \subset Y$ affine open such that $f(U) \subset V$. By Lemma 11.5 $f|_U : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see Lemma 11.6. Hence by Algebra, Lemma 138.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega_{\Gamma(U,\mathcal{O}_U)/\Gamma(V,\mathcal{O}_V)}$ is projective. Hence $\Omega_{U/V}$ is locally projective, see Properties, Section 21. □

0D0E **Lemma 11.9.** *Let T be an affine scheme. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_T$-modules. Consider $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(\mathcal{F}, \mathcal{G})$. If $\mathcal{F}$ is locally projective, then $H^1(T, \mathcal{H}) = 0$.*

Proof. By the definition of a locally projective sheaf on a scheme (see Properties, Definition 21.1) we see that $\mathcal{F}$ is a direct summand of a free $\mathcal{O}_T$-module. Hence we may assume that $\mathcal{F} = \bigoplus_{i \in I} \mathcal{O}_T$ is a free module. In this case $\mathcal{H} = \prod_{i \in I} \mathcal{G}$ is a product of quasi-coherent modules. By Cohomology, Lemma 11.12 we conclude that $H^1 = 0$ because the cohomology of a quasi-coherent sheaf on an affine scheme is zero, see Cohomology of Schemes, Lemma 2.2. □

0D0F **Lemma 11.10.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

1. *f is formally smooth,*
2. *for every $x \in X$ there exist opens $x \in U \subset X$ and $f(x) \in V \subset Y$ with $f(U) \subset V$ such that $f|_U : U \to V$ is formally smooth,*
3. *for every pair of affine opens $U \subset X$ and $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is formally smooth, and*
4. *there exists an affine open covering $Y = \bigcup V_j$ and for each j an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is a formally smooth ring map for all j and i.*

Proof. The implications (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), and (2) $\Rightarrow$ (4) follow from Lemma 11.5. The implication (3) $\Rightarrow$ (4) is immediate.

Assume (4). The proof that f is formally smooth is the same as the second part of the proof of Lemma 11.7. Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[lu, dashed, "a"'] \end{tikzcd}$$

as in Definition 11.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on T' of Lemma 9.4 as in the special case discussed in Remark 9.6. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on T with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 9.5. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology, Definition 4.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ locally has a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T, \mathcal{H}) = 0$, see Cohomology, Lemma 4.3.

First we prove (I). To see this, for every $t \in T$ we can choose an affine open $W \subset T$ neighbourhood of t such that $a(W)$ is contained in U_{ji} for some i, j. Let $W' \subset T'$ be the corresponding open subscheme. By assumption (4) we can lift $a|_W : W \to U_{ji}$ to a V_j-morphism $a' : W' \to U_{ji}$ showing that $\mathcal{F}(W')$ is nonempty.

Finally we prove (II). By Lemma 11.8 we see that Ω_{U_{ji}/V_j} locally projective. Hence $\Omega_{X/Y}$ is locally projective, see Properties, Lemma 21.2. Hence $a^*\Omega_{X/Y}$ is locally projective, see Properties, Lemma 21.3. Hence

$$H^1(T, \mathcal{H}) = H^1(T, \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'}) = 0$$

by Lemma 11.9 as desired. □

06B6 **Lemma 11.11.** *Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume f is formally smooth. Then*

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

(see Morphisms, Lemma 32.9) is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $B \to C$ formally smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Algebra, Lemma 131.7 is exact. This is Algebra, Lemma 138.9. □

06B7 **Lemma 11.12.** *Let $h : Z \to X$ be a formally unramified morphism of schemes over S. Assume that Z is formally smooth over S. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to h^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 7.10 is short exact.

Proof. Let $Z \to Z'$ be the universal first order thickening of Z over X. From the proof of Lemma 7.10 we see that our sequence is identified with the sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

Since $Z \to S$ is formally smooth we can locally on Z' find a left inverse $Z' \to Z$ over S to the inclusion map $Z \to Z'$. Thus the sequence is locally split, see Morphisms, Lemma 32.16. $\square$

067W **Lemma 11.13.** *Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[dr, "j"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

be a commutative diagram of schemes where i and j are formally unramified and f is formally smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 7.11 is exact and locally split.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 7.10 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"'] \arrow[dr, "j'"'] & Z' \arrow[r, "a"'] \arrow[d, "k"] & X \arrow[d, "f"] \\ & Z'' \arrow[r, "b"] & Y \end{tikzcd}$$

In the proof of Lemma 7.11 we identified the sequence above with the sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

Let $U'' \subset Z''$ be an affine open. Denote $U \subset Z$ and $U' \subset Z'$ the corresponding affine open subschemes. As f is formally smooth there exists a morphism $h : U'' \to X$ which agrees with i on U and such that $f \circ h$ equals $b|_{U''}$. Since Z' is the universal first order thickening we obtain a unique morphism $g : U'' \to Z'$ such that $g = a \circ h$. The universal property of Z'' implies that $k \circ g$ is the inclusion map $U'' \to Z''$. Hence g is a left inverse to k. Picture

$$\begin{tikzcd} U \arrow[r] \arrow[d] & Z' \arrow[d, "k"] \\ U'' \arrow[r] \arrow[ur, "g"] & Z'' \end{tikzcd}$$

Thus g induces a map $\mathcal{C}_{Z/Z'}|_U \to \mathcal{C}_{Z/Z''}|_U$ which is a left inverse to the map $\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'}$ over U. $\square$

12. Smoothness over a Noetherian base

02HW It turns out that if the base is Noetherian then we can get away with less in the formulation of formal smoothness. In some sense the following lemmas are the beginning of deformation theory.

02HX **Lemma 12.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Assume that S is locally Noetherian and f locally of finite type. The following are equivalent:*

(1) *f is smooth at x,*

(2) *for every solid commutative diagram*

$$\begin{tikzcd} X & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ S \arrow[from=u, "f"'] & \mathrm{Spec}(B') \arrow[l, "\beta"] \arrow[ul, dashed] \end{tikzcd}$$

where $B' \to B$ is a surjection of local rings with $\mathrm{Ker}(B' \to B)$ *of square zero, and α mapping the closed point of* $\mathrm{Spec}(B)$ *to x there exists a dotted arrow making the diagram commute,*

(3) *same as in (2) but with $B' \to B$ ranging over small extensions (see Algebra, Definition 141.1), and*

(4) *same as in (2) but with $B' \to B$ ranging over small extensions such that α induces an isomorphism $\kappa(x) \to \kappa(\mathfrak{m})$ where $\mathfrak{m} \subset B$ is the maximal ideal.*

Proof. Choose an affine neighbourhood $V \subset S$ of $f(x)$ and choose an affine neighbourhood $U \subset X$ of x such that $f(U) \subset V$. For any "test" diagram as in (2) the morphism α will map $\mathrm{Spec}(B)$ into U and the morphism β will map $\mathrm{Spec}(B')$ into V (see Schemes, Section 13). Hence the lemma reduces to the morphism $f|_U : U \to V$ of affines. (Indeed, V is Noetherian and $f|_U$ is of finite type, see Properties, Lemma 5.2 and Morphisms, Lemma 15.2.) In this affine case the lemma is identical to Algebra, Lemma 141.2. $\square$

Sometimes it is useful to know that one only needs to check the lifting criterion for small extensions "centered" at points of finite type (see Morphisms, Section 16).

02HY **Lemma 12.2.** *Let $f : X \to S$ be a morphism of schemes. Assume that S is locally Noetherian and f locally of finite type. The following are equivalent:*

(1) *f is smooth,*

(2) *for every solid commutative diagram*

$$\begin{tikzcd} X & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ S \arrow[from=u, "f"'] & \mathrm{Spec}(B') \arrow[l, "\beta"] \arrow[ul, dashed] \end{tikzcd}$$

where $B' \to B$ is a small extension of Artinian local rings and β of finite type (!) there exists a dotted arrow making the diagram commute.

Proof. If f is smooth, then the infinitesimal lifting criterion (Lemma 11.7) says f is formally smooth and (2) holds.

Assume (2). The set of points $x \in X$ where f is not smooth forms a closed subset T of X. By the discussion in Morphisms, Section 16, if $T \neq \emptyset$ there exists a point $x \in T \subset X$ such that the morphism

$$\mathrm{Spec}(\kappa(x)) \to X \to S$$

is of finite type (namely, pick any point x of T which is closed in an affine open of X). By Morphisms, Lemma 16.2 given any local Artinian ring B' with residue field $\kappa(x)$ then any morphism $\beta : \mathrm{Spec}(B') \to S$ is of finite type. Thus we see that all the diagrams used in Lemma 12.1 (4) correspond to diagrams as in the current lemma (2). Whence $X \to S$ is smooth a x a contradiction. $\square$

Here is a useful application.

0A43 **Lemma 12.3.** *Let $f : X \to S$ be a finite type morphism of locally Noetherian schemes. Let $Z \subset S$ be a closed subscheme with nth infinitesimal neighbourhood $Z_n \subset S$. Set $X_n = Z_n \times_S X$.*

1. *If $X_n \to Z_n$ is smooth for all n, then f is smooth at every point of $f^{-1}(Z)$.*
2. *If $X_n \to Z_n$ is étale for all n, then f is étale at every point of $f^{-1}(Z)$.*

Proof. Assume $X_n \to Z_n$ is smooth for all n. Let $x \in X$ be a point lying over a point of Z. Given a small extension $B' \to B$ and morphisms α, β as in Lemma 12.1 part (3) the maximal ideal of B' is nilpotent (as B' is Artinian) and hence the morphism β factors through Z_n and α factors through X_n for a suitable n. Thus the lifting property for $X_n \to Z_n$ kicks in to get the desired dotted arrow in the diagram. This proves (1). Part (2) follows from (1) and the fact that a morphism is étale if and only if it is smooth of relative dimension 0. $\square$

0D4G **Lemma 12.4.** *Let $f : X \to S$ be a morphism of locally Noetherian schemes. Let $Z \subset S$ be a closed subscheme with nth infinitesimal neighbourhood $Z_n \subset S$. Set $X_n = Z_n \times_S X$. If $X_n \to Z_n$ is flat for all n, then f is flat at every point of $f^{-1}(Z)$.*

Proof. This is a translation of Algebra, Lemma 99.11 into the language of schemes. $\square$

13. The naive cotangent complex

0D0G This section is the continuation of Modules, Section 31 which in turn continues the discussion in Algebra, Section 134.

0D0H **Definition 13.1.** Let $f : X \to Y$ be a morphism of schemes. The *naive cotangent complex of f* is the complex defined in Modules, Definition 31.6. Notation: NL_f or $NL_{X/Y}$.

0D0I **Lemma 13.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ be affine opens with $f(U) \subset V$. There is a canonical map*

$$\widetilde{NL_{A/R}} \longrightarrow NL_{X/Y}|_U$$

of complexes which is an isomorphism in $D(\mathcal{O}_U)$.

Proof. From the construction of $NL_{X/Y}$ in Modules, Section 31 we see there is a canonical map of complexes $NL_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)} \to NL_{X/Y}(U)$ of $A = \mathcal{O}_X(U)$-modules, which is compatible with further restrictions. Using the canonical map $R \to f^{-1}\mathcal{O}_Y(U)$ we obtain a canonical map $NL_{A/R} \to NL_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)}$ of complexes of A-modules. Using the universal property of the $\widetilde{\ }$ functor (see Schemes, Lemma 7.1) we obtain a map as in the statement of the lemma. We may check this map is an isomorphism on cohomology sheaves by checking it induces isomorphisms on stalks. This follows from Algebra, Lemma 134.11 and 134.13 and Modules, Lemma 31.4 (and the description of the stalks of $\mathcal{O}_X$ and $f^{-1}\mathcal{O}_Y$ at a point $\mathfrak{p} \in \mathrm{Spec}(A)$ as $A_\mathfrak{p}$ and $R_\mathfrak{q}$ where $\mathfrak{q} = R \cap \mathfrak{p}$; references used are Schemes, Lemma 5.4 and Sheaves, Lemma 21.5). $\square$

0D0J **Lemma 13.3.** *Let $f : X \to Y$ be a morphism of schemes. The cohomology sheaves of the complex $NL_{X/Y}$ are quasi-coherent, zero outside degrees -1, 0 and equal to $\Omega_{X/Y}$ in degree 0.*

Proof. By construction of the naive cotangent complex in Modules, Section 31 we have that $NL_{X/Y}$ is a complex sitting in degrees -1, 0 and that its cohomology in degree 0 is $\Omega_{X/Y}$. The sheaf of differentials is quasi-coherent (by Morphisms, Lemma 32.7). To finish the proof it suffices to show that $H^{-1}(NL_{X/Y})$ is quasi-coherent. This follows by checking over affines using Lemma 13.2. $\square$

0D0K **Lemma 13.4.** *Let $f : X \to Y$ be a morphism of schemes. If f is locally of finite presentation, then $NL_{X/Y}$ is locally on X quasi-isomorphic to a complex*

$$\ldots \to 0 \to \mathcal{F}^{-1} \to \mathcal{F}^0 \to 0 \to \ldots$$

of quasi-coherent $\mathcal{O}_X$-modules with $\mathcal{F}^0$ of finite presentation and $\mathcal{F}^{-1}$ of finite type.

Proof. By Lemma 13.2 it suffices to show that $NL_{A/R}$ has this shape if $R \to A$ is a finitely presented ring map. Write $A = R[x_1, \ldots, x_n]/I$ with I finitely generated. Then I/I^2 is a finite A-module and $NL_{A/R}$ is quasi-isomorphic to

$$\ldots \to 0 \to I/I^2 \to \bigoplus\nolimits_{i=1,\ldots,n} A\text{d}x_i \to 0 \to \ldots$$

by Algebra, Section 134 and in particular Algebra, Lemma 134.2. $\square$

0D0L **Lemma 13.5.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

1. *f is formally smooth,*
2. *$H^{-1}(NL_{X/Y}) = 0$ and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is locally projective.*

Proof. This follows from Algebra, Proposition 138.8 and Lemma 11.10. $\square$

0D0M **Lemma 13.6.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

1. *f is formally étale,*
2. *$H^{-1}(NL_{X/Y}) = H^0(NL_{X/Y}) = 0$.*

Proof. A formally étale morphism is formally smooth and hence we have $H^{-1}(NL_{X/Y}) = 0$ by Lemma 13.5. On the other hand, we have $\Omega_{X/Y} = 0$ by Lemma 8.6. Conversely, if (2) holds, then f is formally smooth by Lemma 13.5 and formally unramified by Lemma 6.7 and hence formally étale by Lemmas 11.4. $\square$

0D0N **Lemma 13.7.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

1. *f is smooth, and*
2. *f is locally of finite presentation, $H^{-1}(NL_{X/Y}) = 0$, and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is finite locally free.*

Proof. This follows from the definition of a smooth ring homomorphism (Algebra, Definition 137.1), Lemma 13.2, and the definition of a smooth morphism of schemes (Morphisms, Definition 34.1). We also use that finite locally free is the same as finite projective for modules over rings (Algebra, Lemma 78.2). $\square$

0G7Z **Lemma 13.8.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

1. *f is étale, and*
2. *f is locally of finite presentation and $H^{-1}(NL_{X/Y}) = H^0(NL_{X/Y}) = 0$.*

Proof. This follows from the definition of an étale ring homomorphism (Algebra, Definition 143.1), Lemma 13.2, and the definition of an étale morphism of schemes (Morphisms, Definition 36.1). $\square$

0FV2 **Lemma 13.9.** *Let $i : Z \to X$ be an immersion of schemes. Then $NL_{Z/X}$ is isomorphic to $\mathcal{C}_{Z/X}[1]$ in $D(\mathcal{O}_Z)$ where $\mathcal{C}_{Z/X}$ is the conormal sheaf of Z in X.*

Proof. This follows from Algebra, Lemma 134.6, Morphisms, Lemma 31.2, and Lemma 13.2. $\square$

0E44 **Lemma 13.10.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. There is a canonical six term exact sequence*

$$H^{-1}(f^* NL_{Y/Z}) \to H^{-1}(NL_{X/Z}) \to H^{-1}(NL_{X/Y}) \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

of cohomology sheaves.

Proof. Special case of Modules, Lemma 31.7. $\square$

0FV3 **Lemma 13.11.** *Let $f : X \to Y$ and $Y \to Z$ be morphisms of schemes. Assume $X \to Y$ is a complete intersection morphism. Then there is a canonical distinguished triangle*

$$f^* NL_{Y/Z} \to NL_{X/Z} \to NL_{X/Y} \to f^* NL_{Y/Z}[1]$$

in $D(\mathcal{O}_X)$ which recovers the 6-term exact sequence of Lemma 13.10.

Proof. It suffices to show the canonical map

$$f^* NL_{Y/Z} \to \mathrm{Cone}(NL_{X/Z} \to NL_{X/Y})[-1]$$

of Modules, Lemma 31.7 is an isomorphism in $D(\mathcal{O}_X)$. In order to show this, it suffices to show that the 6-term sequence has a zero on the left, i.e., that $H^{-1}(f^* NL_{Y/Z}) \to H^{-1}(NL_{X/Z})$ is injective. Affine locally this follows from the corresponding algebra result in More on Algebra, Lemma 33.6. To translate into algebra use Lemma 13.2. $\square$

0G80 **Lemma 13.12.** *Let $X \to Y \to Z$ be morphisms of schemes. Assume $X \to Z$ smooth and $Y \to Z$ étale. Then $X \to Y$ is smooth.*

Proof. The morphism $X \to Y$ is locally of finite presentation by Morphisms, Lemma 21.11. By Lemma 13.7 we have $H^{-1}(NL_{X/Z}) = 0$ and the module $\Omega_{X/Z}$ is finite locally free. By Lemma 13.8 we have $H^{-1}(NL_{Y/Z}) = H^0(NL_{Y/Z}) = 0$. By Lemma 13.10 we get $H^{-1}(NL_{X/Y}) = 0$ and $\Omega_{X/Y} \cong \Omega_{X/Z}$ is finite locally free. By Lemma 13.7 the morphism $X \to Y$ is smooth. $\square$

0FV4 **Lemma 13.13.** *Let $f : X \to Y$ be a morphism of schemes which factors as $f = g \circ i$ with i an immersion and $g : P \to Y$ formally smooth (for example smooth). Then there is a canonical isomorphism*

$$NL_{X/Y} \cong \left(\mathcal{C}_{X/P} \to i^*\Omega_{P/Y}\right)$$

in $D(\mathcal{O}_X)$ where the conormal sheaf $\mathcal{C}_{X/P}$ is placed in degree -1.

Proof. (For the parenthetical statement see Lemma 11.7.) By Lemmas 13.9 and 13.5 we have $NL_{X/P} = \mathcal{C}_{X/P}[1]$ and $NL_{P/Y} = \Omega_{P/Y}$ with $\Omega_{P/Y}$ locally projective. This implies that $i^* NL_{P/Y} \to i^*\Omega_{P/Y}$ is a quasi-isomorphism too (small detail omitted; the reason is that $i^* NL_{P/Y}$ is the same thing as $\tau_{\geq -1}Li^* NL_{P/Y}$, see More on Algebra, Lemma 85.1). Thus the canonical map

$$i^* NL_{P/Y} \to \mathrm{Cone}(NL_{X/Y} \to NL_{X/P})[-1]$$

of Modules, Lemma 31.7 is an isomorphism in $D(\mathcal{O}_X)$ because the cohomology group $H^{-1}(i^* NL_{P/Y})$ is zero by what we said above. In other words, we have a distinguished triangle

$$i^* NL_{P/Y} \to NL_{X/Y} \to NL_{X/P} \to i^* NL_{P/Y}[1]$$

Clearly, this means that $NL_{X/Y}$ is the cone on the map $NL_{X/P}[-1] \to i^* NL_{P/Y}$ which is equivalent to the statement of the lemma by our computation of the cohomology sheaves of these objects in the derived category given above. □

0FV5 **Lemma 13.14.** *Consider a cartesian diagram of schemes*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

The canonical map $(g')^* NL_{X/Y} \to NL_{X'/Y'}$ induces an isomorphism on H^0 and a surjection on H^{-1}.

Proof. Translated into algebra this is More on Algebra, Lemma 85.2. To do the translation use Lemma 13.2. □

0FJZ **Lemma 13.15.** *Consider a cartesian diagram of schemes*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

If $Y' \to Y$ is flat, then the canonical map $(g')^* NL_{X/Y} \to NL_{X'/Y'}$ is a quasi-isomorphism.

Proof. By Lemma 13.2 this follows from Algebra, Lemma 134.8. □

0FK0 **Lemma 13.16.** *Consider a cartesian diagram of schemes*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

If $X \to Y$ is flat, then the canonical map $(g')^* NL_{X/Y} \to NL_{X'/Y'}$ is a quasi-isomorphism. If in addition $NL_{X/Y}$ has tor-amplitude in $[-1, 0]$ then $L(g')^* NL_{X/Y} \to NL_{X'/Y'}$ is a quasi-isomorphism too.

Proof. Translated into algebra this is More on Algebra, Lemma 85.3. To do the translation use Lemma 13.2 and Derived Categories of Schemes, Lemmas 3.5 and 10.4. □

14. Pushouts in the category of schemes, I

07RS In this section we construct pushouts of $Y \leftarrow X \rightarrow X'$ where $X \rightarrow Y$ is affine and $X \rightarrow X'$ is a thickening. This will actually be an important case for us, hence a detailed discussion is merited. In Section 67 we discuss a more interesting and more difficult case. See Categories, Section 9 for a general discussion of pushouts in any category.

0ET0 **Lemma 14.1.** *Let $A' \to A$ be a surjection of rings and let $B \to A$ be a ring map. Let $B' = B \times_A A'$ be the fibre product of rings. Set $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$, $T = \mathrm{Spec}(B)$, and $T' = \mathrm{Spec}(B')$. Then*

$$\begin{tikzcd} S \arrow[r, "i"'] \arrow[d, "f"'] & S' \arrow[d, "f'"] \\ T \arrow[r, "i'"] & T' \end{tikzcd} \quad \textit{corresponding to} \quad \begin{tikzcd} A & A' \arrow[l] \\ B \arrow[u] & B' \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout of schemes.

Proof. By More on Algebra, Lemma 6.2 we have $T' = T \amalg_S S'$ as topological spaces, i.e., the diagram is a pushout in the category of topological spaces. Next, consider the map

$$((i')^\sharp, (f')^\sharp) : \mathcal{O}_{T'} \longrightarrow i'_*\mathcal{O}_T \times_{g_*\mathcal{O}_S} f'_*\mathcal{O}_{S'}$$

where $g = i' \circ f = f' \circ i$. We claim this map is an isomorphism of sheaves of rings. Namely, we can view both sides as quasi-coherent $\mathcal{O}_{T'}$-modules (use Schemes, Lemmas 24.1 for the right hand side) and the map is $\mathcal{O}_{T'}$-linear. Thus it suffices to show the map is an isomorphism on the level of global sections (Schemes, Lemma 7.5). On global sections we recover the identification $B' \to B \times_A A'$ from statement of the lemma (this is how we chose B').

Let X be a scheme. Suppose we are given morphisms of schemes $m' : S' \to X$ and $n : T \to X$ such that $m' \circ i = n \circ f$ (call this m). We get a unique map of topological spaces $n' : T' \to X$ compatible with m' and n as $T' = T \amalg_S S'$ (see above). By the description of $\mathcal{O}_{T'}$ in the previous paragraph we obtain a unique homomorphism of sheaves of rings

$$(n')^\sharp : \mathcal{O}_X \longrightarrow (n')_*\mathcal{O}_{T'} = m'_*\mathcal{O}_T \times_{m_*\mathcal{O}_T} n_*\mathcal{O}_S$$

given by $(m')^\sharp$ and $n^\sharp$. Thus $(n', (n')^\sharp)$ is the unique morphism of ringed spaces $T' \to X$ compatible with m' and n. To finish the proof it suffices to show that n' is a morphism of schemes, i.e., a morphism of locally ringed spaces.

Let $t' \in T'$ with image $x \in X$. We have to show that $\mathcal{O}_{X,x} \to \mathcal{O}_{T',t'}$ is local. If $t' \notin T$, then t' is the image of a unique point $s' \in S'$ and $\mathcal{O}_{T',t'} = \mathcal{O}_{S',s'}$. Namely, $S' \setminus S \to T' \setminus T$ is an isomorphism of schemes as $B' \to A'$ induces an isomorphism $\mathrm{Ker}(B' \to B) = \mathrm{Ker}(A' \to A)$. If t' is the image of $t \in T$, then we know that the composition $\mathcal{O}_{X,x} \to \mathcal{O}_{T',t'} \to \mathcal{O}_{T,t}$ is local and we conclude also. $\square$

0BMP **Lemma 14.2.** *Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of schemes. Let $(W, X_i \to W)$ be a cocone for the diagram in the category of schemes (Categories, Remark 14.5). If there exists a fpqc covering $\{W_a \to W\}_{a \in A}$ of schemes such that*

1. *for all $a \in A$ we have $W_a = \mathrm{colim}\, X_i \times_W W_a$ in the category of schemes, and*

(2) *for all* $a, b \in A$ *we have* $W_a \times_W W_b = \operatorname{colim} X_i \times_W W_a \times_W W_b$ *in the category of schemes,*

then $W = \operatorname{colim} X_i$ *in the category of schemes.*

Proof. Namely, for a scheme T a morphism $W \to T$ is the same thing as collection of morphism $W_a \to T$, $a \in A$ which agree on the overlaps $W_a \times_W W_b$, see Descent, Lemma 13.7. $\square$

07RT **Lemma 14.3.** *Let* $X \to X'$ *be a thickening of schemes and let* $X \to Y$ *be an affine morphism of schemes. Then there exists a pushout*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

in the category of schemes. Moreover, $Y \subset Y'$ *is a thickening,* $X = Y \times_{Y'} X'$*, and*

$$\mathcal{O}_{Y'} = \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'}$$

as sheaves on $|Y| = |Y'|$.

Proof. We first construct Y' as a ringed space. Namely, as topological space we take $Y' = Y$. Denote $f' : X' \to Y'$ the map of topological spaces which equals f. As structure sheaf $\mathcal{O}_{Y'}$ we take the right hand side of the equation of the lemma. To see that Y' is a scheme, we have to show that any point has an affine neighbourhood. Since the formation of the fibre product of sheaves commutes with restricting to opens, we may assume Y is affine. Then X is affine (as f is affine) and X' is affine as well (see Lemma 2.3). Say $Y \leftarrow X \to X'$ corresponds to $B \to A \leftarrow A'$. Set $B' = B \times_A A'$; this is the global sections of $\mathcal{O}_{Y'}$. As $A' \to A$ is surjective with locally nilpotent kernel we see that $B' \to B$ is surjective with locally nilpotent kernel. Hence $\operatorname{Spec}(B') = \operatorname{Spec}(B)$ (as topological spaces). We claim that $Y' = \operatorname{Spec}(B')$. To see this we will show for $g' \in B'$ with image $g \in B$ that $\mathcal{O}_{Y'}(D(g)) = B'_{g'}$. Namely, by More on Algebra, Lemma 5.3 we see that

$$(B')_{g'} = B_g \times_{A_h} A'_{h'}$$

where $h \in A$, $h' \in A'$ are the images of g'. Since B_g, resp. A_h, resp. $A'_{h'}$ is equal to $\mathcal{O}_Y(D(g))$, resp. $f_*\mathcal{O}_X(D(g))$, resp. $f'_*\mathcal{O}_{X'}(D(g))$ the claim follows.

It remains to show that Y' is the pushout. The discussion above shows the scheme Y' has an affine open covering $Y' = \bigcup W'_i$ such that the corresponding opens $U'_i \subset X'$, $W_i \subset Y$, and $U_i \subset X$ are affine open. Moreover, if A'_i, B_i, A_i are the rings corresponding to U'_i, W_i, U_i, then W'_i corresponds to $B_i \times_{A_i} A'_i$. Thus we can apply Lemmas 14.1 and 14.2 to conclude our construction is a pushout in the category of schemes. $\square$

In the following lemma we use the fibre product of categories as defined in Categories, Example 31.3.

07RV **Lemma 14.4.** *Let* $X \to X'$ *be a thickening of schemes and let* $X \to Y$ *be an affine morphism of schemes. Let* $Y' = Y \amalg_X X'$ *be the pushout (see Lemma 14.3). Base change gives a functor*

$$F : (Sch/Y') \longrightarrow (Sch/Y) \times_{(Sch/Y')} (Sch/X')$$

given by $V' \longmapsto (V' \times_{Y'} Y, V' \times_{Y'} X', 1)$ *which has a left adjoint*

$$G : (Sch/Y) \times_{(Sch/Y')} (Sch/X') \longrightarrow (Sch/Y')$$

which sends the triple (V, U', φ) *to the pushout* $V \amalg_{(V \times_Y X)} U'$. *Finally,* $F \circ G$ *is isomorphic to the identity functor.*

Proof. Let (V, U', φ) be an object of the fibre product category. Set $U = U' \times_{X'} X$. Note that $U \to U'$ is a thickening. Since $\varphi : V \times_Y X \to U' \times_{X'} X = U$ is an isomorphism we have a morphism $U \to V$ over $X \to Y$ which identifies U with the fibre product $X \times_Y V$. In particular $U \to V$ is affine, see Morphisms, Lemma 11.8. Hence we can apply Lemma 14.3 to get a pushout $V' = V \amalg_U U'$. Denote $V' \to Y'$ the morphism we obtain in virtue of the fact that V' is a pushout and because we are given morphisms $V \to Y$ and $U' \to X'$ agreeing on U as morphisms into Y'. Setting $G(V, U', \varphi) = V'$ gives the functor G.

Let us prove that G is a left adjoint to F. Let Z be a scheme over Y'. We have to show that

$$\mathrm{Mor}(V', Z) = \mathrm{Mor}((V, U', \varphi), F(Z))$$

where the morphism sets are taking in their respective categories. Let $g' : V' \to Z$ be a morphism. Denote $\tilde{g}$, resp. $\tilde{f}'$ the composition of g' with the morphism $V \to V'$, resp. $U' \to V'$. Base change $\tilde{g}$, resp. $\tilde{f}'$ by $Y \to Y'$, resp. $X' \to Y'$ to get a morphism $g : V \to Z \times_{Y'} Y$, resp. $f' : U' \to Z \times_{Y'} X'$. Then (g, f') is an element of the right hand side of the equation above (details omitted). Conversely, suppose that $(g, f') : (V, U', \varphi) \to F(Z)$ is an element of the right hand side. We may consider the composition $\tilde{g} : V \to Z$, resp. $\tilde{f}' : U' \to Z$ of g, resp. f by $Z \times_{Y'} X' \to Z$, resp. $Z \times_{Y'} Y \to Z$. Then $\tilde{g}$ and $\tilde{f}'$ agree as morphism from U to Z. By the universal property of pushout, we obtain a morphism $g' : V' \to Z$, i.e., an element of the left hand side. We omit the verification that these constructions are mutually inverse.

To prove that $F \circ G$ is isomorphic to the identity we have to show that the adjunction mapping $(V, U', \varphi) \to F(G(V, U', \varphi))$ is an isomorphism. To do this we may work affine locally. Say $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$, and $Y = \mathrm{Spec}(B)$. Then $A' \to A$ and $B \to A$ are ring maps as in More on Algebra, Lemma 6.4 and $Y' = \mathrm{Spec}(B')$ with $B' = B \times_A A'$. Next, suppose that $V = \mathrm{Spec}(D)$, $U' = \mathrm{Spec}(C')$ and φ is given by an A-algebra isomorphism $D \otimes_B A \to C' \otimes_{A'} A = C'/IC'$. Set $D' = D \times_{C'/IC'} C'$. In this case the statement we have to prove is that $D' \otimes_{B'} B \cong D$ and $D' \otimes_{B'} A' \cong C'$. This is a special case of More on Algebra, Lemma 6.4. □

08KU **Lemma 14.5.** *Let* $X \to X'$ *be a thickening of schemes and let* $X \to Y$ *be an affine morphism of schemes. Let* $Y' = Y \amalg_X X'$ *be the pushout (see Lemma 14.3). Let* $V' \to Y'$ *be a morphism of schemes. Set* $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$, *and* $U = X \times_{Y'} V'$. *There is an equivalence of categories between*

1. *quasi-coherent* $\mathcal{O}_{V'}$*-modules flat over* Y', *and*
2. *the category of triples* $(\mathcal{G}, \mathcal{F}', \varphi)$ *where*
   (a) $\mathcal{G}$ *is a quasi-coherent* $\mathcal{O}_V$*-module flat over* Y,
   (b) $\mathcal{F}'$ *is a quasi-coherent* $\mathcal{O}_{U'}$*-module flat over* X', *and*
   (c) $\varphi : (U \to V)^*\mathcal{G} \to (U \to U')^*\mathcal{F}'$ *is an isomorphism of* $\mathcal{O}_U$*-modules.*

The equivalence maps $\mathcal{G}'$ *to* $((V \to V')^*\mathcal{G}', (U' \to V')^*\mathcal{G}', can)$. *Suppose* $\mathcal{G}'$ *corresponds to the triple* $(\mathcal{G}, \mathcal{F}', \varphi)$. *Then*

(a) *$\mathcal{G}'$ is a finite type $\mathcal{O}_{V'}$-module if and only if $\mathcal{G}$ and $\mathcal{F}'$ are finite type $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules.*
(b) *if $V' \to Y'$ is locally of finite presentation, then $\mathcal{G}'$ is an $\mathcal{O}_{V'}$-module of finite presentation if and only if $\mathcal{G}$ and $\mathcal{F}'$ are $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules of finite presentation.*

Proof. A quasi-inverse functor assigns to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$ the fibre product

$$(V \to V')_*\mathcal{G} \times_{(U \to V')_*\mathcal{F}} (U' \to V')_*\mathcal{F}'$$

where $\mathcal{F} = (U \to U')^*\mathcal{F}'$. This works, because on affines we recover the equivalence of More on Algebra, Lemma 7.5. Some details omitted.

Parts (a) and (b) follow from More on Algebra, Lemmas 7.4 and 7.6. $\square$

07RX **Lemma 14.6.** *In the situation of Lemma 14.4. If $V' = G(V, U', \varphi)$ for some triple (V, U', φ), then*
(1) *$V' \to Y'$ is locally of finite type if and only if $V \to Y$ and $U' \to X'$ are locally of finite type,*
(2) *$V' \to Y'$ is flat if and only if $V \to Y$ and $U' \to X'$ are flat,*
(3) *$V' \to Y'$ is flat and locally of finite presentation if and only if $V \to Y$ and $U' \to X'$ are flat and locally of finite presentation,*
(4) *$V' \to Y'$ is smooth if and only if $V \to Y$ and $U' \to X'$ are smooth,*
(5) *$V' \to Y'$ is étale if and only if $V \to Y$ and $U' \to X'$ are étale, and*
(6) *add more here as needed.*

If W' is flat over Y', then the adjunction mapping $G(F(W')) \to W'$ is an isomorphism. Hence F and G define mutually quasi-inverse functors between the category of schemes flat over Y' and the category of triples (V, U', φ) with $V \to Y$ and $U' \to X'$ flat.

Proof. Looking over affine pieces the assertions of this lemma are equivalent to the corresponding assertions of More on Algebra, Lemma 7.7. $\square$

15. Openness of the flat locus

0398 This result takes some work to prove, and (perhaps) deserves its own section. Here it is.

0399 **Theorem 15.1.** *Let S be a scheme. Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is locally of finite presentation. Then* [DG67, IV Theorem 11.3.1]

$$U = \{x \in X \mid \mathcal{F} \text{ is flat over } S \text{ at } x\}$$

is open in X.

Proof. We may test for openness locally on X hence we may assume that f is a morphism of affine schemes. In this case the theorem is exactly Algebra, Theorem 129.4. $\square$

047C **Lemma 15.2.** *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x' \in X'$ with images $x = g'(x')$ and $s' = f'(x')$.

1. *If $\mathcal{F}$ is flat over S at x, then $(g')^*\mathcal{F}$ is flat over S' at x'.*
2. *If g is flat at s' and $(g')^*\mathcal{F}$ is flat over S' at x', then $\mathcal{F}$ is flat over S at x.*

In particular, if g is flat, f is locally of finite presentation, and $\mathcal{F}$ is locally of finite presentation, then formation of the open subset of Theorem 15.1 commutes with base change.

Proof. Consider the commutative diagram of local rings

$$\begin{tikzcd} \mathcal{O}_{X',x'} & \mathcal{O}_{X,x} \arrow[l] \\ \mathcal{O}_{S',s'} \arrow[u] & \mathcal{O}_{S,s} \arrow[l] \arrow[u] \end{tikzcd}$$

Note that $\mathcal{O}_{X',x'}$ is a localization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$, and that $((g')^*\mathcal{F})_{x'}$ is equal to $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$. Hence the lemma follows from Algebra, Lemma 100.1. $\square$

16. Critère de platitude par fibres

039A Consider a commutative diagram of schemes (left hand diagram)

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} X_s \arrow[rr, "f_s"] \arrow[dr] & & Y_s \arrow[dl] \\ & \mathrm{Spec}(\kappa(s)) & \end{tikzcd}$$

and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Given a point $x \in X$ lying over $s \in S$ with image $y = f(x)$ we consider the question: Is $\mathcal{F}$ flat over Y at x? If $\mathcal{F}$ is flat over S at x, then the theorem states this question is intimately related to the question of whether the restriction of $\mathcal{F}$ to the fibre

$$\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$$

is flat over Y_s at x. Below you will find a "Noetherian" version, a "finitely presented" version, and earlier we treated a "nilpotent" version, see Lemma 10.2.

039B **Theorem 16.1.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$. Set $y = f(x)$ and $s \in S$ the image of x in S. Assume S, X, Y locally Noetherian, $\mathcal{F}$ coherent, and $\mathcal{F}_x \neq 0$. Then the following are equivalent:*

1. *$\mathcal{F}$ is flat over S at x, and $\mathcal{F}_s$ is flat over Y_s at x, and*
2. *Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.*

Proof. Consider the ring maps

$$\mathcal{O}_{S,s} \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

and the module $\mathcal{F}_x$. The stalk of $\mathcal{F}_s$ at x is the module $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ and the local ring of Y_s at y is $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$. Thus the implication (1) $\Rightarrow$ (2) is Algebra, Lemma 99.15. If (2) holds, then the first ring map is faithfully flat and $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$ so by Algebra, Lemma 39.4 we see that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$. Moreover, $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ is the base change of the flat module $\mathcal{F}_x$ by $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$, hence flat by Algebra, Lemma 39.7. $\square$

Here is the non-Noetherian version.

039C **Theorem 16.2.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over S,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation, and*
(3) *Y is locally of finite type over S.*

Let $x \in X$. Set $y = f(x)$ and let $s \in S$ be the image of x in S. If $\mathcal{F}_x \neq 0$, then the following are equivalent:

(1) *$\mathcal{F}$ is flat over S at x, and $\mathcal{F}_s$ is flat over Y_s at x, and*
(2) *Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.*

Moreover, the set of points x where (1) and (2) hold is open in $Supp(\mathcal{F})$.

Proof. Consider the ring maps

$$\mathcal{O}_{S,s} \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

and the module $\mathcal{F}_x$. The stalk of $\mathcal{F}_s$ at x is the module $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ and the local ring of Y_s at y is $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$. Thus the implication (1) $\Rightarrow$ (2) is Algebra, Lemma 128.9. If (2) holds, then the first ring map is faithfully flat and $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$ so by Algebra, Lemma 39.4 we see that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$. Moreover, $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ is the base change of the flat module $\mathcal{F}_x$ by $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$, hence flat by Algebra, Lemma 39.7.

By Morphisms, Lemma 21.11 the morphism f is locally of finite presentation. Consider the set

05VI (16.2.1) $$U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over both } Y \text{ and } S\}.$$

This set is open in X by Theorem 15.1. Note that if $x \in U$, then $\mathcal{F}_s$ is flat at x over Y_s as a base change of a flat module under the morphism $Y_s \to Y$, see Morphisms, Lemma 25.7. Hence at every point of $U \cap \text{Supp}(\mathcal{F})$ condition (1) is satisfied. On the other hand, it is clear that if $x \in \text{Supp}(\mathcal{F})$ satisfies (1) and (2), then $x \in U$. Thus the open set we are looking for is $U \cap \text{Supp}(\mathcal{F})$. □

These theorems are often used in the following simplified forms. We give only the global statements – of course there are also pointwise versions.

039D **Lemma 16.3.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume*

(1) *S, X, Y are locally Noetherian,*
(2) *X is flat over S,*
(3) *for every $s \in S$ the morphism $f_s : X_s \to Y_s$ is flat.*

Then f is flat. If f is also surjective, then Y is flat over S.

Proof. This is a special case of Theorem 16.1. □

039E **Lemma 16.4.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume*

(1) *X is locally of finite presentation over S,*
(2) *X is flat over S,*
(3) *for every $s \in S$ the morphism $f_s : X_s \to Y_s$ is flat, and*
(4) *Y is locally of finite type over S.*

Then f is flat. If f is also surjective, then Y is flat over S.

Proof. This is a special case of Theorem 16.2. ☐

05VJ **Lemma 16.5.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

1. *X is locally of finite presentation over S,*
2. *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation,*
3. *$\mathcal{F}$ is flat over S, and*
4. *Y is locally of finite type over S.*

Then the set

$$U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } Y\}.$$

is open in X and its formation commutes with arbitrary base change: If $S' \to S$ is a morphism of schemes, and U' is the set of points of $X' = X \times_S S'$ where $\mathcal{F}' = \mathcal{F} \times_S S'$ is flat over $Y' = Y \times_S S'$, then $U' = U \times_S S'$.

Proof. By Morphisms, Lemma 21.11 the morphism f is locally of finite presentation. Hence U is open by Theorem 15.1. Because we have assumed that $\mathcal{F}$ is flat over S we see that Theorem 16.2 implies

$$U = \{x \in X \mid \mathcal{F}_s \text{ flat at } x \text{ over } Y_s\}.$$

where s always denotes the image of x in S. (This description also works trivially when $\mathcal{F}_x = 0$.) Moreover, the assumptions of the lemma remain in force for the morphism $f' : X' \to Y'$ and the sheaf $\mathcal{F}'$. Hence U' has a similar description. In other words, it suffices to prove that given $s' \in S'$ mapping to $s \in S$ we have

$$\{x' \in X'_{s'} \mid \mathcal{F}'_{s'} \text{ flat at } x' \text{ over } Y'_{s'}\}$$

is the inverse image of the corresponding locus in X_s. This is true by Lemma 15.2 because in the cartesian diagram

$$\begin{tikzcd} X'_{s'} \arrow[r] \arrow[d] & X_s \arrow[d] \\ Y'_{s'} \arrow[r] & Y_s \end{tikzcd}$$

the horizontal morphisms are flat as they are base changes by the flat morphism $\mathrm{Spec}(\kappa(s')) \to \mathrm{Spec}(\kappa(s))$. ☐

05VK **Lemma 16.6.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume*

1. *X is locally of finite presentation over S,*
2. *X is flat over S, and*
3. *Y is locally of finite type over S.*

Then the set

$$U = \{x \in X \mid X \text{ flat at } x \text{ over } Y\}.$$

is open in X and its formation commutes with arbitrary base change.

Proof. This is a special case of Lemma 16.5. ☐

The following lemma is a variant of Algebra, Lemma 99.4. Note that the hypothesis that $(\mathcal{F}_s)_x$ is a flat $\mathcal{O}_{X_s,x}$-module means that $(\mathcal{F}_s)_x$ is a free $\mathcal{O}_{X_s,x}$-module which is always the case if $x \in X_s$ is a generic point of an irreducible component of X_s and X_s is reduced (namely, in this case $\mathcal{O}_{X_s,x}$ is a field, see Algebra, Lemma 25.1).

080Q **Lemma 16.7.** *Let $f : X \to S$ be a morphism of schemes of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $x \in X$ with image $s \in S$. If $\mathcal{F}$ is flat at x over S and $(\mathcal{F}_s)_x$ is a flat $\mathcal{O}_{X_s,x}$-module, then $\mathcal{F}$ is finite free in a neighbourhood of x.*

Proof. If $\mathcal{F}_x \otimes \kappa(x)$ is zero, then $\mathcal{F}_x = 0$ by Nakayama's lemma (Algebra, Lemma 20.1) and hence $\mathcal{F}$ is zero in a neighbourhood of x (Modules, Lemma 9.5) and the lemma holds. Thus we may assume $\mathcal{F}_x \otimes \kappa(x)$ is not zero and we see that Theorem 16.2 applies with $f = \text{id} : X \to X$. We conclude that $\mathcal{F}_x$ is flat over $\mathcal{O}_{X,x}$. Hence $\mathcal{F}_x$ is free, see Algebra, Lemma 78.5 for example. Choose an open neighbourhood $x \in U \subset X$ and sections $s_1, \ldots, s_r \in \mathcal{F}(U)$ which map to a basis in $\mathcal{F}_x$. The corresponding map $\psi : \mathcal{O}_U^{\oplus r} \to \mathcal{F}|_U$ is surjective after shrinking U (Modules, Lemma 9.5). Then $\text{Ker}(\psi)$ is of finite type (see Modules, Lemma 11.3) and $\text{Ker}(\psi)_x = 0$. Whence after shrinking U once more ψ is an isomorphism. $\square$

0CZR **Lemma 16.8.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module flat over S. Then the set*

$$\{x \in X : \mathcal{F} \textit{ free in a neighbourhood of } x\}$$

is open in X and its formation commutes with arbitrary base change $S' \to S$.

Proof. Openness holds trivially. Let $x \in X$ mapping to $s \in S$. By Lemma 16.7 we see that x is in our set if and only if $\mathcal{F}|_{X_s}$ is flat at x over X_s. Clearly this is also equivalent to $\mathcal{F}$ being flat at x over X (because this statement is implied by freeness of $\mathcal{F}_x$ and implies flatness of $\mathcal{F}|_{X_s}$ at x over X_s). Thus the base change statement follows from Lemma 16.5 applied to $\text{id} : X \to X$ over S. $\square$

17. Closed immersions between smooth schemes

0H1G Some results that do not fit elsewhere very well.

0FUE **Lemma 17.1.** *Let S be a scheme. Let $Y \to X$ be a closed immersion of schemes smooth over S. For every $y \in Y$ there exist integers $0 \leq m, n$ and a commutative diagram*

$$\begin{tikzcd} Y \arrow[d] & V \arrow[l] \arrow[r] \arrow[d] & \mathbf{A}^m_S \arrow[d, "{(a_1,\ldots,a_m)\mapsto(a_1,\ldots,a_m,0\ldots,0)}"] \\ X & U \arrow[l] \arrow[r, "\pi"] & \mathbf{A}^{m+n}_S \end{tikzcd}$$

where $U \subset X$ is open, $V = Y \cap U$, π is étale, $V = \pi^{-1}(\mathbf{A}^m_S)$, and $y \in V$.

Proof. The question is local on X hence we may replace X by an open neighbourhood of y. Since $Y \to X$ is a regular immersion by Divisors, Lemma 22.11 we may assume $X = \text{Spec}(A)$ is affine and there exists a regular sequence $f_1, \ldots, f_n \in A$ such that $Y = V(f_1, \ldots, f_n)$. After shrinking X (and hence Y) further we may assume there exists an étale morphism $Y \to \mathbf{A}^m_S$, see Morphisms, Lemma 36.20. Let

$\overline{g}_1, \ldots, \overline{g}_m$ in $\mathcal{O}_Y(Y)$ be the coordinate functions of this étale morphism. Choose lifts $g_1, \ldots, g_m \in A$ of these functions and consider the morphism

$$(g_1, \ldots, g_m, f_1, \ldots, f_n) : X \longrightarrow \mathbf{A}^{m+n}_S$$

over S. This is a morphism of schemes locally of finite presentation over S and hence is locally of finite presentation (Morphisms, Lemma 21.11). The restriction of this morphism to $\mathbf{A}^m_S \subset \mathbf{A}^{m+n}_S$ is étale by construction. Thus, in order to show that $X \to \mathbf{A}^{m+n}_S$ is étale at y it suffices to show that $X \to \mathbf{A}^{m+n}_S$ is flat at y, see Morphisms, Lemma 36.15. Let $s \in S$ be the image of y. It suffices to check that $X_s \to \mathbf{A}^{m+n}_s$ is flat at y, see Theorem 16.2. Let $z \in \mathbf{A}^{m+n}_s$ be the image of y. The local ring map

$$\mathcal{O}_{\mathbf{A}^{m+n}_s, z} \longrightarrow \mathcal{O}_{X_s, y}$$

is flat by Algebra, Lemma 128.1. Namely, schemes smooth over fields are regular and regular rings are Cohen-Macaulay, see Varieties, Lemma 25.3 and Algebra, Lemma 106.3. Thus both source and target are regular local rings (and hence CM). The source and target have the same dimension: namely, we have $\dim(\mathcal{O}_{Y_s,y}) = \dim(\mathcal{O}_{\mathbf{A}^m_s,z})$ by More on Algebra, Lemma 44.2, we have $\dim(\mathcal{O}_{\mathbf{A}^{m+n}_s,z}) = n + \dim(\mathcal{O}_{\mathbf{A}^m_s,z})$, and we have $\dim(\mathcal{O}_{X_s,y}) = n + \dim(\mathcal{O}_{Y_s,y})$ because $\mathcal{O}_{Y_s,y}$ is the quotient of $\mathcal{O}_{X_s,y}$ by the regular sequence $f_1, \ldots, f_n$ of length n (see Divisors, Remark 22.5). Finally, the fibre ring of the displayed arrow is finite over $\kappa(z)$ since $Y_s \to \mathbf{A}^m_s$ is étale at y. This finishes the proof. □

0H1H **Remark 17.2.** We fix a ring R and we set $S = \mathrm{Spec}(R)$. Fix integers $0 \leq m$ and $1 \leq n$. Consider the closed immersion

$$Z = \mathbf{A}^m_S \longrightarrow \mathbf{A}^{m+n}_S = X, \quad (a_1, \ldots, a_m) \mapsto (a_1, \ldots, a_m, 0, \ldots 0).$$

We are going to consider the blowing up X' of X along the closed subscheme Z. Write

$$X = \mathrm{Spec}(A) \quad \text{with} \quad A = R[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

Then X' is the Proj of the Rees algebra of A with respect to the ideal $(y_1, \ldots, y_n)$. This Rees algebra is equal to $B = A[T_1, \ldots, T_n]/(y_iT_j - y_jT_i)$; details omitted. Hence $X' = \mathrm{Proj}(B)$ is smooth over S as it is covered by the affine opens

$$\begin{aligned} D_+(T_i) &= \mathrm{Spec}(B_{(T_i)}) \\ &= \mathrm{Spec}(A[t_1, \ldots, \hat{t}_i, \ldots t_n]/(y_j - y_it_j)) \\ &= \mathrm{Spec}(R[x_1, \ldots, x_m, y_i, t_1, \ldots, \hat{t}_i, \ldots, t_n]) \end{aligned}$$

which are isomorphic to $\mathbf{A}^{n+m}_S$. In this chart the exceptional divisor is cut out by setting $y_i = 0$ hence the exceptional divisor is smooth over S as well.

0FUT **Lemma 17.3.** *Let S be a scheme. Let $Z \to X$ be a closed immersion of schemes smooth over S. Let $b : X' \to X$ be the blowing up of Z with exceptional divisor $E \subset X'$. Then X' and E are smooth over S. The morphism $p : E \to Z$ is canonically isomorphic to the projective space bundle*

$$\mathbf{P}(\mathcal{I}/\mathcal{I}^2) \longrightarrow Z$$

where $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of Z. The relative $\mathcal{O}_E(1)$ coming from the projective space bundle structure is isomorphic to the restriction of $\mathcal{O}_{X'}(-E)$ to E.

Proof. By Divisors, Lemma 22.11 the immersion $Z \to X$ is a regular immmersion, hence the ideal sheaf $\mathcal{I}$ is of finite type, hence b is a projective morphism with relatively ample invertible sheaf $\mathcal{O}_{X'}(1) = \mathcal{O}_{X'}(-E)$, see Divisors, Lemmas 32.4 and 32.13. The canonical map $\mathcal{I} \to b_*\mathcal{O}_{X'}(1)$ gives a closed immersion

$$X' \longrightarrow \mathbf{P}\left(\bigoplus\nolimits_{n \geq 0} \text{Sym}^n_{\mathcal{O}_X}(\mathcal{I})\right)$$

by the very construction of the blowup. The restriction of this morphism to E gives a canonical map

$$E \longrightarrow \mathbf{P}\left(\bigoplus\nolimits_{n \geq 0} \text{Sym}^n_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2)\right)$$

over Z. Since $\mathcal{I}/\mathcal{I}^2$ is finite locally free if this canonical map is an isomorphism, then the final part of the lemma holds. Having said all of this, now the question is étale local on X. Namely, blowing up commutes with flat base change by Divisors, Lemma 32.3 and we can check smoothness after precomposing with a surjective étale morphism. Thus by the étale local structure of a closed immersion of schemes over S given in Lemma 17.1 this reduces us to the case discussed in Remark 17.2. $\square$

18. Flat modules and relative assassins

0GSF In this section we will prove that the support of a flat module is (in some sense) equidimensional over the base in geometric situations. For the Noetherian case we refer the reader to [DG67, IV Proposition 12.1.1.5]. First, we prove two helper lemmas.

0GSG **Lemma 18.1.** *Let A be a valuation ring. Let $A \to B$ is a local homomorphism of local rings which is essentially of finite type. Let $u : N \to M$ be a map of finite B-modules. Assume M is flat over A and $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective. Then u is injective and $M/u(N)$ is flat over A.*

Proof. We will deduce this lemma from Algebra, Lemma 128.4 (please note that we exchanged the roles of M and N). To do the reduction we will use More on Algebra, Lemma 25.7 to reduce to the finitely presented case.

By assumption we can write B as a quotient of the localization of a polynomial algebra $P = A[x_1, \ldots, x_n]$ at a prime ideal $\mathfrak{q}$. Then we can think of $u : N \to M$ as a map of finite $P_\mathfrak{q}$-modules. Hence we may and do assume that B is essentially of finite presentation over A.

Next, the B-module N is finite but perhaps not of finite presentation. Write $N = \text{colim}\, N_\lambda$ as a filtered colimit of finitely presented B-modules with surjective transition maps. For example choose a presentation $0 \to K \to B^{\oplus r} \to N \to 0$, write K as the union of its finite submodules K_λ, and set $N_\lambda = \text{Coker}(K_\lambda \to B^{\oplus r})$. The module $N/\mathfrak{m}_A N$ is of finite presentation over the Noetherian ring $B/\mathfrak{m}_A B$. Hence for λ large enough we have $N_\lambda/\mathfrak{m}_A N_\lambda = N/\mathfrak{m}_A N$. Now, if we can show the lemma for the composition $u_\lambda : N_\lambda \to M$, then we conclude that $N_\lambda = N$ and the result holds for u. Hence we may and do assume N is of finite presentation over B.

By More on Algebra, Lemma 25.7 the module M is of finite presentation over B. Thus all the assumptions of Algebra, Lemma 128.4 hold and we conclude. $\square$

0GSH **Lemma 18.2.** *Let $f : X \to S$ be a morphism of schemes. Let $y \in X$ be a point with image $t \in S$. Denote $Y \subset X$ the closure of $\{y\}$ viewed as an integral closed* This can be found in the proof of [DG67, IV Proposition 12.1.1.5] *subscheme of X. Let $s \in S$ and let $x \in Y_s$ be a generic point of an irreducible component of Y_s. There exists a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

with the following properties:

1. *S' is the spectrum of a valuation ring with generic point t' and closed point s',*
2. *$g(t') = t$ and $g(s') = s$,*
3. *there exists a point $y' \in X'_{t'}$ which is a generic point of an irreducible component of $(S' \times_S Y)_{t'} = Y_t \times_t t'$ and satisfies $g'(y') = y$,*
4. *denoting $Y' \subset X'$ the closure of $\{y'\}$ viewed as an integral closed subscheme of X' there exists a point $x' \in Y'_{s'}$ which is a generic point of an irreducible component of $Y'_{s'}$ with $g'(x') = x$.*

Proof. We choose a valuation ring R, we set $S' = \mathrm{Spec}(R)$ with generic point t' and closed point s', and we choose a morphism $h : S' \to X$ with $h(t') = y$ and $h(s') = x$. See Schemes, Lemma 20.4. Set $g = f \circ h$ so that $g(t') = t$ and $g(s') = s$. Consider the base change

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] \arrow[u, bend left, "\sigma"] & S \end{tikzcd}$$

We obtain a section σ of the base change such that $h = g' \circ \sigma$.

Of course σ factors through the base change $S' \times_S Y$ of Y as h factors through Y. Let $y' \in X'_{t'} \subset X'$ be the generic point of an irreducible component of the fibre

$$(S' \times_S Y)_{t'} = Y_t \times_t t'$$

containing the point $\sigma(t')$, i.e., such that $y' \rightsquigarrow \sigma(t')$. Since $g'(y') \in Y_t$ and $g(y') \rightsquigarrow g(\sigma(t')) = y$ we find that $g'(y') = y$ because y is the generic point of the fibre Y_t. Denote $Y' \subset X'$ the closure of $\{y'\}$ in X' viewed as an integral closed subscheme. Then σ factors through Y' as $\sigma(t') \in Y'$. Choose a generic point $x' \in Y'_{s'}$ of an irreducible component of $Y'_{s'}$ which contains $\sigma(s')$, i.e., we get $x' \rightsquigarrow \sigma(s')$ and hence $g'(x') \rightsquigarrow g'(\sigma(s')) = x$. Again as x is a generic point of an irreducible component of Y_s by assumption and as $g'(Y') \subset Y$ we conclude that $g'(x') = x$. $\square$

0GSI **Lemma 18.3.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent finite type $\mathcal{O}_X$-module. Let $y \in Ass_{X/S}(\mathcal{F})$ with image $t \in S$. Denote $Y \subset X$ the closure of $\{y\}$ in X viewed as an integral closed subscheme. Let $s \in S$ and let $x \in Y_s$ be a generic point of an irreducible component of Y_s. If $\mathcal{F}$ is flat over S at x, then $x \in Ass_{X/S}(\mathcal{F})$ and $\dim_x(Y_s) = \dim(Y_t)$.*

Proof. Choose a diagram as in Lemma 18.2. Set $\mathcal{F}' = (g')^*\mathcal{F}$. Divisors, Lemma 7.3 implies that $y' \in \mathrm{Ass}_{X'/S'}(\mathcal{F}')$. By our choice of y' we also see that $\dim(Y'_{t'}) = \dim(Y_t)$, see for example Algebra, Lemma 116.7. By Algebra, Lemma 125.9 we see that $Y'_{s'}$ is equidimensional of dimension equal to $\dim(Y_t)$. Since $\mathcal{F}$ is flat at x over S we see that $\mathcal{F}'$ is flat at x' over S', see Morphisms, Lemma 25.7.

Suppose that we can show $x' \in \mathrm{Ass}_{X'/S}(\mathcal{F}')$. Then Divisors, Lemma 7.3 implies that $x \in \mathrm{Ass}_{X/S}(\mathcal{F})$ and that the irreducible component C' of $Y'_{s'}$ containing x' is an irreducible component of $C \times_s s'$ where $C \subset Y_s$ is the irreducible component containing x. Whence $\dim(C) = \dim(C') = \dim(Y_t)$ (see above) and the proof is complete. This reduces us to the case discussed in the next paragraph.

Assume $S = \mathrm{Spec}(A)$ where A is a valuation ring and t and s are the generic and closed points of S. We will assume $x \notin \mathrm{Ass}_{X/S}(\mathcal{F})$ in order to get a contradiction. In other words, we assume $x \notin \mathrm{Ass}_{X_s}(\mathcal{F}_s)$ where $\mathcal{F}_s$ is the pullback of $\mathcal{F}$ to X_s. Consider the ring map

$$A \longrightarrow \mathcal{O}_{X,x} = B$$

and the module $N = \mathcal{F}_x$ over $B = \mathcal{O}_{X,x}$. Then $B/\mathfrak{m}_A B = \mathcal{O}_{X_s,x}$ and $N/\mathfrak{m}_A N$ is the stalk of $\mathcal{F}_s$ at the point x. Denote $\mathfrak{q} \subset B$ the prime ideal corresponding to the point y, see Schemes, Lemma 13.2. Since x is a generic point of Y_s we see that the radical of $\mathfrak{q} + \mathfrak{m}_A B$ is $\mathfrak{m}_B$. Then $\mathrm{Ass}_{B/\mathfrak{m}_A B}(N/\mathfrak{m}_A N)$ is a finite set of prime ideals (Algebra, Lemma 63.5) which doesn't contain the maximal ideal of $B/\mathfrak{m}_A B$ since $x \notin \mathrm{Ass}_{X/S}(\mathcal{F})$. Thus the image of of $\mathfrak{q}$ in $B/\mathfrak{m}_A B$ is not contained in any of those prime ideals. Hence by prime avoidance (Algebra, Lemma 15.2) we can find an element $g \in \mathfrak{q}$ whose image in $B/\mathfrak{m}_A B$ is a nonzerodivisor on $N/\mathfrak{m}_A N$ (this uses the description of zerodivisors in Algebra, Lemma 63.9). Since $N = \mathcal{F}_x$ is A-flat by Lemma 18.1 we see that

$$g : N \longrightarrow N$$

is injective. In particular, if $K = \mathrm{Frac}(A)$ is the fraction field of A, then we see that

$$g : N \otimes_A K \longrightarrow N \otimes_A K$$

is injective. Observe that $\mathfrak{q}$ corresponds to a prime ideal of $B \otimes_A K$. Denote $\mathcal{F}_t$ the restriction of $\mathcal{F}$ to the generic fibre X_t. We have $(B \otimes_A K)_\mathfrak{q} = \mathcal{O}_{X_t,y}$ and $(N \otimes_A K)_\mathfrak{q}$ is the stalk at y of $\mathcal{F}_t$. Hence we find that $g \in \mathfrak{m}_y \subset \mathcal{O}_{X_t,y}$ is a nonzerodivisor on the stalk $(\mathcal{F}_t)_y$ which contradicts our assumption that $y \in \mathrm{Ass}_{X/S}(\mathcal{F})$. □

0H3X **Lemma 18.4.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module flat over S. Assume S is irreducible with generic point η. If $\dim(Supp(\mathcal{F}_\eta)) \leq r$ then for all $s \in S$ we have $\dim(Supp(\mathcal{F}_s)) \leq r$.*

Proof. Let $x \in \mathrm{Supp}(\mathcal{F}_s)$ be a generic point of an irreducible component of $\mathrm{Supp}(\mathcal{F}_s)$. By Algebra, Lemma 41.12 we can find a specialization $y \rightsquigarrow x$ in $\mathrm{Supp}(\mathcal{F})$ with $f(y) = \eta$. Of course we may assume y is a generic point of an irreducible component of $\mathrm{Supp}(\mathcal{F}_\eta)$. We conclude from Lemma 18.3 that the dimension of $\overline{\{x\}}$ is at most r. □

0GSJ **Lemma 18.5.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $y \in Ass_{X/S}(\mathcal{F})$. Denote $Y \subset X$ the closure of $\{y\}$ in X viewed as an integral closed subscheme. Denote $T \subset S$ the closure of $\{f(y)\}$ viewed as an integral closed subscheme. We obtain a commutative diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

where $Y \to T$ is dominant. Assume $\mathcal{F}$ is flat over S at all generic points of irreducible components of fibres of $Y \to T$ (for example if $\mathcal{F}$ is flat over S). Then

(1) *if $s \in S$ and $x \in Y_s$ is the generic point of an irreducible component of Y_s, then $x \in Ass_{X/S}(\mathcal{F})$, and*
(2) *there is an integer $d \geq 0$ such that $Y \to T$ is of relative dimension d, see Morphisms, Definition 29.1.*

Proof. This follows immediately from the pointwise version Lemma 18.3. Note that to compute the dimension of the locally algebraic schemes Y_s it suffices to look near the generic points, see Varieties, Section 20. □

0GSK **Remark 18.6.** Here are some cases where the material above, especially Lemma 18.5, allows one to conclude that a morphism $f : X \to S$ of schemes has relative dimension d as defined in Morphisms, Definition 29.1. For example, this is true if

(1) X is integral with generic point ξ,
(2) the transcendence degree of $\kappa(\xi)$ over $\kappa(f(\xi))$ is d,
(3) f is locally of finite type, and
(4) there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type which is flat over S with $\text{Supp}(\mathcal{F}) = X$.

Another set of hypotheses that work are the following:

(1) S is irreducible with generic point η,
(2) X_η is dense in X,
(3) every irreducible component of X_η has dimension d,
(4) f is locally of finite type, and
(5) there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type which is flat over S with $\text{Supp}(\mathcal{F}) = X$.

Of course, we can relax the flatness condition on $\mathcal{F}$ and require only that $\mathcal{F}$ is flat over S in codimension 0, i.e., that $\mathcal{F}$ is flat over S at every generic point of every fibre. If we ever need these results, we will carefully state and prove them here.

19. Normalization revisited

081J Normalization commutes with smooth base change.

081K **Lemma 19.1.** *Let $f : Y \to X$ be a smooth morphism of schemes. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The integral closure of $\mathcal{O}_Y$ in $f^*\mathcal{A}$ is equal to $f^*\mathcal{A}'$ where $\mathcal{A}' \subset \mathcal{A}$ is the integral closure of $\mathcal{O}_X$ in $\mathcal{A}$.*

Proof. This is a translation of Algebra, Lemma 147.4 into the language of schemes. Details omitted. □

03GV **Lemma 19.2** (Normalization commutes with smooth base change)**.** *Let*

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d, "f_2"'] & Y_1 \arrow[d, "f_1"] \\ X_2 \arrow[r, "\varphi"] & X_1 \end{tikzcd}$$

be a fibre square in the category of schemes. Assume f_1 is quasi-compact and quasi-separated, and φ is smooth. Let $Y_i \to X_i' \to X_i$ be the normalization of X_i in Y_i. Then $X_2' \cong X_2 \times_{X_1} X_1'$.

Proof. The base change of the factorization $Y_1 \to X_1' \to X_1$ to X_2 is a factorization $Y_2 \to X_2 \times_{X_1} X_1' \to X_2$ and $X_2 \times_{X_1} X_1' \to X_2$ is integral (Morphisms, Lemma 44.6). Hence we get a morphism $h : X_2' \to X_2 \times_{X_1} X_1'$ by the universal property of Morphisms, Lemma 53.4. Observe that X_2' is the relative spectrum of the integral closure of $\mathcal{O}_{X_2}$ in $f_{2,*}\mathcal{O}_{Y_2}$. If $\mathcal{A}' \subset f_{1,*}\mathcal{O}_{Y_1}$ denotes the integral closure of $\mathcal{O}_{X_1}$, then $X_2 \times_{X_1} X_1'$ is the relative spectrum of $\varphi^*\mathcal{A}'$, see Constructions, Lemma 4.6. By Cohomology of Schemes, Lemma 5.2 we know that $f_{2,*}\mathcal{O}_{Y_2} = \varphi^* f_{1,*}\mathcal{O}_{Y_1}$. Hence the result follows from Lemma 19.1. $\square$

07TD **Lemma 19.3** (Normalization and smooth morphisms)**.** *Let $X \to Y$ be a smooth morphism of schemes. Assume every quasi-compact open of Y has finitely many irreducible components. Then the same is true for X and there is a unique isomorphism $X^\nu = X \times_Y Y^\nu$ over X where X^ν, Y^ν are the normalizations of X, Y.*

Proof. By Descent, Lemma 16.3 every quasi-compact open of X has finitely many irreducible components. Note that $X_{red} = X \times_Y Y_{red}$ as a scheme smooth over a reduced scheme is reduced, see Descent, Lemma 18.1. Hence we may assume that X and Y are reduced (as the normalization of a scheme is equal to the normalization of its reduction by definition). Next, note that $X' = X \times_Y Y^\nu$ is a normal scheme by Descent, Lemma 18.2. The morphism $X' \to Y^\nu$ is smooth (hence flat) thus the generic points of irreducible components of X' lie over generic points of irreducible components of Y^ν. Since $Y^\nu \to Y$ is birational we conclude that $X' \to X$ is birational too (because $X' \to Y^\nu$ induces an isomorphism on fibres over generic points of Y). We conclude that there exists a factorization $X^\nu \to X' \to X$, see Morphisms, Lemma 54.5 which is an isomorphism as X' is normal and integral over X. $\square$

0CBM **Lemma 19.4** (Normalization and henselization)**.** *Let X be a locally Noetherian scheme. Let $\nu : X^\nu \to X$ be the normalization morphism. Then for any point $x \in X$ the base change*

$$X^\nu \times_X \mathrm{Spec}(\mathcal{O}_{X,x}^h) \to \mathrm{Spec}(\mathcal{O}_{X,x}^h), \quad \textit{resp.} \quad X^\nu \times_X \mathrm{Spec}(\mathcal{O}_{X,x}^{sh}) \to \mathrm{Spec}(\mathcal{O}_{X,x}^{sh})$$

is the normalization of $\mathrm{Spec}(\mathcal{O}_{X,x}^h)$, *resp.* $\mathrm{Spec}(\mathcal{O}_{X,x}^{sh})$.

Proof. Let $\eta_1, \ldots, \eta_r$ be the generic points of the irreducible components of X passing through x. The base change of the normalization to $\mathrm{Spec}(\mathcal{O}_{X,x})$ is the spectrum of the integral closure of $\mathcal{O}_{X,x}$ in $\prod \kappa(\eta_i)$. This follows from our construction of the normalization of X in Morphisms, Definition 54.1 and Morphisms, Lemma 53.1; you can also use the description of the normalization in Morphisms, Lemma 54.3. Thus we reduce to the following algebra problem. Let A be a Noetherian local ring; recall that this implies the henselization A^h and strict henselization A^{sh} are Noetherian too (More on Algebra, Lemma 45.3). Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be its minimal primes. Let A' be the integral closure of A in $\prod \kappa(\mathfrak{p}_i)$. Problem: show that $A' \otimes_A A^h$, resp. $A' \otimes_A A^{sh}$ is constructed from the Noetherian local ring A^h, resp. A^{sh} in the same manner.

Since A^h, resp. A^{sh} are colimits of étale A-algebras, we see that the minimal primes of A and A^{sh} are exactly the primes of A^h, resp. A^{sh} lying over the minimal primes of A (by going down, see Algebra, Lemmas 39.19 and 30.7). Thus More on Algebra, Lemma 45.13 tells us that $A^h \otimes_A \prod \kappa(\mathfrak{p}_i)$, resp. $A^{sh} \otimes_A \prod \kappa(\mathfrak{p}_i)$ is the product of the residue fields at the minimal primes of A^h, resp. A^{sh}. We know that taking the integral closure in an overring commutes with étale base change, see Algebra, Lemma 147.2. Writing A^h and A^{sh} as a limit of étale A-algebras we see that the same thing is true for the base change to A^h and A^{sh} (you can also use the more general Algebra, Lemma 147.5). $\square$

20. Normal morphisms

038Z In the article [DM69] of Deligne and Mumford the notion of a normal morphism is mentioned. This is just one in a series of types[3] of morphisms that can all be defined similarly. Over time we will add these in their own sections as needed.

0390 **Definition 20.1.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *normal at x* if f is flat at x, and the scheme X_y is geometrically normal at x over $\kappa(y)$ (see Varieties, Definition 10.1).
(2) We say f is a *normal morphism* if f is normal at every point of X.

So the condition that the morphism $X \to Y$ is normal is stronger than just requiring all the fibres to be normal locally Noetherian schemes.

0391 **Lemma 20.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is normal, and*
(2) *f is flat and its fibres are geometrically normal schemes.*

Proof. This follows directly from the definitions. $\square$

056W **Lemma 20.3.** *A smooth morphism is normal.*

Proof. Let $f : X \to Y$ be a smooth morphism. As f is locally of finite presentation, see Morphisms, Lemma 34.8 the fibres X_y are locally of finite type over a field, hence locally Noetherian. Moreover, f is flat, see Morphisms, Lemma 34.9. Finally, the fibres X_y are smooth over a field (by Morphisms, Lemma 34.5) and hence geometrically normal by Varieties, Lemma 25.4. Thus f is normal by Lemma 20.2. $\square$

We want to show that this notion is local on the source and target for the smooth topology. First we deal with the property of having locally Noetherian fibres.

0392 **Lemma 20.4.** *The property $\mathcal{P}(f) =$"the fibres of f are locally Noetherian" is local in the fppf topology on the source and the target.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Moreover, as φ_i is of finite presentation the field extension $\kappa(y_i)/\kappa(y)$ is finitely generated. Hence in this situation we have that X_y is locally Noetherian if and

[3]The other types are coprof $\leq k$, Cohen-Macaulay, (S_k), regular, (R_k), and reduced. See [DG67, IV Definition 6.8.1.]. Gorenstein morphisms will be defined in Duality for Schemes, Section 24.

only if X_{i,y_i} is locally Noetherian, see Varieties, Lemma 11.1. This fact implies locality on the target.

Let $\{X_i \to X\}$ be an fppf covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is an fppf covering of the fibre. Hence the locality on the source follows from Descent, Lemma 16.1. $\square$

0393 **Lemma 20.5.** *The property $\mathcal{P}(f) =$"the fibres of f are locally Noetherian and f is normal" is local in the fppf topology on the target and local in the smooth topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f) \wedge \mathcal{P}_3(f)$ where $\mathcal{P}_1(f) =$"the fibres of f are locally Noetherian", $\mathcal{P}_2(f) =$"f is flat", and $\mathcal{P}_3(f) =$"the fibres of f are geometrically normal". We have already seen that $\mathcal{P}_1$ and $\mathcal{P}_2$ are local in the fppf topology on the source and the target, see Lemma 20.4, and Descent, Lemmas 23.15 and 27.1. Thus we have to deal with $\mathcal{P}_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i\in I}$ be an fpqc covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Hence in this situation we have that X_y is geometrically normal if and only if X_{i,y_i} is geometrically normal, see Varieties, Lemma 10.4. This fact implies $\mathcal{P}_3$ is fpqc local on the target.

Let $\{X_i \to X\}$ be a smooth covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $\mathcal{P}_3$ for the smooth topology on the source follows from Descent, Lemma 18.2. Combining the above the lemma follows. $\square$

21. Regular morphisms

07R6 Compare with Section 20. The algebraic version of this notion is discussed in More on Algebra, Section 41.

07R7 **Definition 21.1.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *regular at x* if f is flat at x, and the scheme X_y is geometrically regular at x over $\kappa(y)$ (see Varieties, Definition 12.1).
(2) We say f is a *regular morphism* if f is regular at every point of X.

The condition that the morphism $X \to Y$ is regular is stronger than just requiring all the fibres to be regular locally Noetherian schemes.

07R8 **Lemma 21.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is regular,*
(2) *f is flat and its fibres are geometrically regular schemes,*
(3) *for every pair of affine opens $U \subset X$, $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is regular,*
(4) *there exists an open covering $Y = \bigcup_{j\in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i\in I_j} U_i$ such that each of the morphisms $U_i \to V_j$ is regular, and*

(5) *there exists an affine open covering* $Y = \bigcup_{j \in J} V_j$ *and affine open coverings* $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ *such that the ring maps* $\mathcal{O}_Y(V_j) \to \mathcal{O}_X(U_i)$ *are regular.*

Proof. The equivalence of (1) and (2) is immediate from the definitions. Let $x \in X$ with $y = f(x)$. By definition f is flat at x if and only if $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a flat ring map, and X_y is geometrically regular at x over $\kappa(y)$ if and only if $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is a geometrically regular algebra over $\kappa(y)$. Hence Whether or not f is regular at x depends only on the local homomorphism of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Thus the equivalence of (1) and (4) is clear.

Recall (More on Algebra, Definition 41.1) that a ring map $A \to B$ is regular if and only if it is flat and the fibre rings $B \otimes_A \kappa(\mathfrak{p})$ are Noetherian and geometrically regular for all primes $\mathfrak{p} \subset A$. By Varieties, Lemma 12.3 this is equivalent to $\text{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ being a geometrically regular scheme over $\kappa(\mathfrak{p})$. Thus we see that (2) implies (3). It is clear that (3) implies (5). Finally, assume (5). This implies that f is flat (see Morphisms, Lemma 25.3). Moreover, if $y \in Y$, then $y \in V_j$ for some j and we see that $X_y = \bigcup_{i \in I_j} U_{i,y}$ with each $U_{i,y}$ geometrically regular over $\kappa(y)$ by Varieties, Lemma 12.3. Another application of Varieties, Lemma 12.3 shows that X_y is geometrically regular. Hence (2) holds and the proof of the lemma is finished. $\square$

07R9 **Lemma 21.3.** *A smooth morphism is regular.*

Proof. Let $f : X \to Y$ be a smooth morphism. As f is locally of finite presentation, see Morphisms, Lemma 34.8 the fibres X_y are locally of finite type over a field, hence locally Noetherian. Moreover, f is flat, see Morphisms, Lemma 34.9. Finally, the fibres X_y are smooth over a field (by Morphisms, Lemma 34.5) and hence geometrically regular by Varieties, Lemma 25.4. Thus f is regular by Lemma 21.2. $\square$

07RA **Lemma 21.4.** *The property $\mathcal{P}(f) =$"the fibres of f are locally Noetherian and f is regular" is local in the fppf topology on the target and local in the smooth topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f) \wedge \mathcal{P}_3(f)$ where $\mathcal{P}_1(f) =$"the fibres of f are locally Noetherian", $\mathcal{P}_2(f) =$"f is flat", and $\mathcal{P}_3(f) =$"the fibres of f are geometrically regular". We have already seen that $\mathcal{P}_1$ and $\mathcal{P}_2$ are local in the fppf topology on the source and the target, see Lemma 20.4, and Descent, Lemmas 23.15 and 27.1. Thus we have to deal with $\mathcal{P}_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fpqc covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \text{Spec}(\kappa(y_i)) \times_{\text{Spec}(\kappa(y))} X_y.$$

Hence in this situation we have that X_y is geometrically regular if and only if X_{i,y_i} is geometrically regular, see Varieties, Lemma 12.4. This fact implies $\mathcal{P}_3$ is fpqc local on the target.

Let $\{X_i \to X\}$ be a smooth covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $\mathcal{P}_3$ for the smooth topology on the source follows from Descent, Lemma 18.4. Combining the above the lemma follows. $\square$

22. Cohen-Macaulay morphisms

045Q Compare with Section 20. Note that, as pointed out in Algebra, Section 167 and Varieties, Section 13 "geometrically Cohen-Macaulay" is the same as plain Cohen-Macaulay.

045R **Definition 22.1.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *Cohen-Macaulay at x* if f is flat at x, and the local ring of the scheme X_y at x is Cohen-Macaulay.
(2) We say f is a *Cohen-Macaulay morphism* if f is Cohen-Macaulay at every point of X.

Here is a translation.

045S **Lemma 22.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is Cohen-Macaulay, and*
(2) *f is flat and its fibres are Cohen-Macaulay schemes.*

Proof. This follows directly from the definitions. □

0AFG **Lemma 22.3.** *Let $f : X \to Y$ be a morphism of locally Noetherian schemes which is locally of finite type and Cohen-Macaulay. For every point x in X with image y in Y,*

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y),$$

where X_y denotes the fiber over y.

Proof. After replacing X by an open neighborhood of x, there is a natural number d such that all fibers of $X \to Y$ have dimension d at every point, see Morphisms, Lemma 29.4. Then f is flat, locally of finite type and of relative dimension d. Hence the result follows from Morphisms, Lemma 29.6. □

0C0W **Lemma 22.4.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Assume that the fibres of f, g, and $g \circ f$ are locally Noetherian. Let $x \in X$ with images $y \in Y$ and $z \in Z$.*

(1) *If f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$, then $g \circ f$ is Cohen-Macaulay at x.*
(2) *If f and g are Cohen-Macaulay, then $g \circ f$ is Cohen-Macaulay.*
(3) *If $g \circ f$ is Cohen-Macaulay at x and f is flat at x, then f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$.*
(4) *If $g \circ f$ is Cohen-Macaulay and f is flat, then f is Cohen-Macaulay and g is Cohen-Macaulay at every point in the image of f.*

Proof. Consider the map of Noetherian local rings

$$\mathcal{O}_{Y_z,y} \to \mathcal{O}_{X_z,x}$$

and observe that its fibre is

$$\mathcal{O}_{X_z,x}/\mathfrak{m}_{Y_z,y}\mathcal{O}_{X_z,x} = \mathcal{O}_{X_y,x}$$

Thus the lemma this follows from Algebra, Lemma 163.3. □

0C0X **Lemma 22.5.** *Let $f : X \to Y$ be a flat morphism of locally Noetherian schemes. If X is Cohen-Macaulay, then f is Cohen-Macaulay and $\mathcal{O}_{Y,f(x)}$ is Cohen-Macaulay for all $x \in X$.*

Proof. After translating into algebra this follows from Algebra, Lemma 163.3. $\square$

045T **Lemma 22.6.** *Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of f. Let $x' \in X'$ be a point with image $x \in X$.*

1. *If f is Cohen-Macaulay at x, then $f' : X' \to Y'$ is Cohen-Macaulay at x'.*
2. *If f is flat at x and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.*
3. *If $Y' \to Y$ is flat at $f'(x')$ and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.*

Proof. Note that the assumption on $Y' \to Y$ implies that for $y' \in Y'$ mapping to $y \in Y$ the field extension $\kappa(y')/\kappa(y)$ is finitely generated. Hence also all the fibres $X'_{y'} = (X_y)_{\kappa(y')}$ are locally Noetherian, see Varieties, Lemma 11.1. Thus the lemma makes sense. Set $y' = f'(x')$ and $y = f(x)$. Hence we get the following commutative diagram of local rings

$$\begin{tikzcd} \mathcal{O}_{X',x'} & \mathcal{O}_{X,x} \arrow[l] \\ \mathcal{O}_{Y',y'} \arrow[u] & \mathcal{O}_{Y,y} \arrow[l] \arrow[u] \end{tikzcd}$$

where the upper left corner is a localization of the tensor product of the upper right and lower left corners over the lower right corner.

Assume f is Cohen-Macaulay at x. The flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ implies the flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$, see Algebra, Lemma 100.1. The fact that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Cohen-Macaulay implies that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay, see Varieties, Lemma 13.1. Hence we see that f' is Cohen-Macaulay at x'.

Assume f is flat at x and f' is Cohen-Macaulay at x'. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Cohen-Macaulay, see Varieties, Lemma 13.1. Hence we see that f is Cohen-Macaulay at x.

Assume $Y' \to Y$ is flat at y' and f' is Cohen-Macaulay at x'. The flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y',y'}$ implies the flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, see Algebra, Lemma 100.1. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Cohen-Macaulay, see Varieties, Lemma 13.1. Hence we see that f is Cohen-Macaulay at x. $\square$

045U **Lemma 22.7.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let* [DG67, IV Corollary 12.1.7(iii)]

$$W = \{x \in X \mid f \text{ is Cohen-Macaulay at } x\}$$

Then

1. *$W = \{x \in X \mid \mathcal{O}_{X_{f(x)},x} \text{ is Cohen-Macaulay}\}$,*
2. *W is open in X,*
3. *W is dense in every fibre of $X \to S$,*

(4) *the formation of W commutes with arbitrary base change of f: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set W' for the morphism f' is equal to $W' = (g')^{-1}(W)$.*

Proof. As f is flat with locally Noetherian fibres the equality in (1) holds by definition. Parts (2) and (3) follow from Algebra, Lemma 130.5. Part (4) follows either from Algebra, Lemma 130.7 or Varieties, Lemma 13.1. $\square$

0BUU **Lemma 22.8.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $x \in X$ with image $s \in S$. Set $d = \dim_x(X_s)$. The following are equivalent*

(1) *f is Cohen-Macaulay at x,*
(2) *there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite morphism $U \to \mathbf{A}^d_S$ over S which is flat at x,*
(3) *there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite flat morphism $U \to \mathbf{A}^d_S$ over S,*
(4) *for any S-morphism $g : U \to \mathbf{A}^d_S$ of an open neighbourhood $U \subset X$ of x we have: g is quasi-finite at $x \Rightarrow g$ is flat at x.*

Proof. Openness of flatness shows (2) and (3) are equivalent, see Theorem 15.1.

Choose affine open $U = \text{Spec}(A) \subset X$ with $x \in U$ and $V = \text{Spec}(R) \subset S$ with $f(U) \subset V$. Then $R \to A$ is a flat ring map of finite presentation. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. After replacing A by a principal localization we may assume there exists a quasi-finite map $R[x_1, \ldots, x_d] \to A$, see Algebra, Lemma 125.2. Thus there exists at least one pair (U, g) consisting of an open neighbourhood $U \subset X$ of x and a locally[4] quasi-finite morphism $g : U \to \mathbf{A}^d_S$.

Claim: Given $R \to A$ flat and of finite presentation, a prime $\mathfrak{p} \subset A$ and $\varphi : R[x_1, \ldots, x_d] \to A$ quasi-finite at $\mathfrak{p}$ we have: $\text{Spec}(\varphi)$ is flat at $\mathfrak{p}$ if and only if $\text{Spec}(A) \to \text{Spec}(R)$ is Cohen-Macaulay at $\mathfrak{p}$. Namely, by Theorem 16.2 flatness may be checked on fibres. The same is true for being Cohen-Macaulay (as A is already assumed flat over R). Thus the claim follows from Algebra, Lemma 130.1.

The claim shows that (1) is equivalent to (4) and combined with the fact that we have constructed a suitable (U, g) in the second paragraph, the claim also shows that (1) is equivalent to (2). $\square$

054T **Lemma 22.9.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. For $d \geq 0$ there exist opens $U_d \subset X$ with the following properties*

(1) *$W = \bigcup_{d \geq 0} U_d$ is dense in every fibre of f, and*
(2) *$U_d \to S$ is of relative dimension d (see Morphisms, Definition 29.1).*

Proof. This follows by combining Lemma 22.7 with Morphisms, Lemma 29.4. $\square$

054U **Lemma 22.10.** *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Suppose $x' \rightsquigarrow x$ is a specialization of points of X with image $s' \rightsquigarrow s$ in S. If x is a generic point of an irreducible component of X_s then $\dim_{x'}(X_{s'}) = \dim_x(X_s)$.*

[4] If S is quasi-separated, then g will be quasi-finite.

Proof. The point x is contained in U_d for some d, where U_d as in Lemma 22.9. $\square$

045V **Lemma 22.11.** *The property $\mathcal{P}(f) =$"the fibres of f are locally Noetherian and f is Cohen-Macaulay" is local in the fppf topology on the target and local in the syntomic topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f)$ where $\mathcal{P}_1(f) =$"f is flat", and $\mathcal{P}_2(f) =$"the fibres of f are locally Noetherian and Cohen-Macaulay". We know that $\mathcal{P}_1$ is local in the fppf topology on the source and the target, see Descent, Lemmas 23.15 and 27.1. Thus we have to deal with $\mathcal{P}_2$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

and that $\kappa(y_i)/\kappa(y)$ is a finitely generated field extension. Hence if X_y is locally Noetherian, then X_{i,y_i} is locally Noetherian, see Varieties, Lemma 11.1. And if in addition X_y is Cohen-Macaulay, then X_{i,y_i} is Cohen-Macaulay, see Varieties, Lemma 13.1. Thus $\mathcal{P}_2$ is fppf local on the target.

Let $\{X_i \to X\}$ be a syntomic covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a syntomic covering of the fibre. Hence the locality of $\mathcal{P}_2$ for the syntomic topology on the source follows from Descent, Lemma 17.2. Combining the above the lemma follows. $\square$

23. Slicing Cohen-Macaulay morphisms

056X The results in this section eventually lead to the assertion that the fppf topology is the same as the "finitely presented, flat, quasi-finite" topology. The following lemma is very closely related to Divisors, Lemma 18.9.

056Y **Lemma 23.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Assume*

1. *f is locally of finite presentation,*
2. *f is flat at x, and*
3. *the image $\overline{h}$ of h in $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is a nonzerodivisor.*

Then there exists an affine open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ flat and locally of finite presentation.

Proof. We are going to prove this by reducing to the Noetherian case. By openness of flatness (see Theorem 15.1) we may assume, after replacing X by an open neighbourhood of x, that $X \to S$ is flat. We may also assume that X and S are affine. After possible shrinking X a bit we may assume that there exists an $h \in \Gamma(X, \mathcal{O}_X)$ which maps to our given h.

We may write $S = \mathrm{Spec}(A)$ and we may write $A = \mathrm{colim}_i A_i$ as a directed colimit of finite type $\mathbf{Z}$ algebras. Then by Algebra, Lemma 168.1 or Limits, Lemmas 10.1, 8.2, and 10.1 we can find a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X_0 \arrow[d, "f_0"] \\ S \arrow[r] & S_0 \end{tikzcd}$$

with f_0 flat and of finite presentation, X_0 affine, and S_0 affine and Noetherian. Let $x_0 \in X_0$, resp. $s_0 \in S_0$ be the image of x, resp. s. We may also assume there exists an element $h_0 \in \Gamma(X_0, \mathcal{O}_{X_0})$ which restricts to h on X. (If you used the algebra reference above then this is clear; if you used the references to the chapter on limits then this follows from Limits, Lemma 10.1 by thinking of h as a morphism $X \to \mathbf{A}^1_S$.) Note that $\mathcal{O}_{X_s,x}$ is a localization of $\mathcal{O}_{(X_0)_{s_0},x_0} \otimes_{\kappa(s_0)} \kappa(s)$, so that $\mathcal{O}_{(X_0)_{s_0},x_0} \to \mathcal{O}_{X_s,x}$ is a flat local ring map, in particular faithfully flat. Hence the image $\overline{h}_0 \in \mathcal{O}_{(X_0)_{s_0},x_0}$ is contained in $\mathfrak{m}_{(X_0)_{s_0},x_0}$ and is a nonzerodivisor. We claim that after replacing X_0 by a principal open neighbourhood of x_0 the element h_0 is a nonzerodivisor in $B_0 = \Gamma(X_0, \mathcal{O}_{X_0})$ such that B_0/h_0B_0 is flat over $A_0 = \Gamma(S_0, \mathcal{O}_{S_0})$. If so then

$$0 \to B_0 \xrightarrow{h_0} B_0 \to B_0/h_0B_0 \to 0$$

is a short exact sequence of flat A_0-modules. Hence this remains exact on tensoring with A (by Algebra, Lemma 39.12) and the lemma follows.

It remains to prove the claim above. The corresponding algebra statement is the following (we drop the subscript $_0$ here): Let $A \to B$ be a flat, finite type ring map of Noetherian rings. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$. Assume $h \in \mathfrak{q}$ maps to a nonzerodivisor in $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$. Goal: show that after possible replacing B by B_g for some $g \in B$, $g \notin \mathfrak{q}$ the element h becomes a nonzerodivisor and B/hB becomes flat over A. By Algebra, Lemma 99.2 we see that h is a nonzerodivisor in $B_\mathfrak{q}$ and that $B_\mathfrak{q}/hB_\mathfrak{q}$ is flat over A. By openness of flatness, see Algebra, Theorem 129.4 or Theorem 15.1 we see that B/hB is flat over A after replacing B by B_g for some $g \in B$, $g \notin \mathfrak{q}$. Finally, let $I = \{b \in B \mid hb = 0\}$ be the annihilator of h. Then $IB_\mathfrak{q} = 0$ as h is a nonzerodivisor in $B_\mathfrak{q}$. Also I is finitely generated as B is Noetherian. Hence there exists a $g \in B$, $g \notin \mathfrak{q}$ such that $IB_g = 0$. After replacing B by B_g we see that h is a nonzerodivisor. ☐

06LI **Lemma 23.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h_1, \ldots, h_r \in \mathcal{O}_{X,x}$. Assume*

1. *f is locally of finite presentation,*
2. *f is flat at x, and*
3. *the images of $h_1, \ldots, h_r$ in $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ form a regular sequence.*

Then there exists an affine open neighbourhood $U \subset X$ of x such that $h_1, \ldots, h_r$ come from $h_1, \ldots, h_r \in \Gamma(U, \mathcal{O}_U)$ and such that $Z = V(h_1, \ldots, h_r) \to U$ is a regular immersion with $x \in Z$ and $Z \to S$ flat and locally of finite presentation. Moreover, the base change $Z_{S'} \to U_{S'}$ is a regular immersion for any scheme S' over S.

Proof. (Our conventions on regular sequences imply that $h_i \in \mathfrak{m}_x$ for each i.) The case $r = 1$ follows from Lemma 23.1 combined with Divisors, Lemma 18.1 to see that $V(h_1)$ remains an effective Cartier divisor after base change. The case $r > 1$ follows from a straightforward induction on r (applying the result for $r = 1$ exactly r times; details omitted).

Another way to prove the lemma is using the material from Divisors, Section 22. Namely, first by openness of flatness (see Theorem 15.1) we may assume, after replacing X by an open neighbourhood of x, that $X \to S$ is flat. We may also assume that X and S are affine. After possible shrinking X a bit we may assume that we have $h_1, \ldots, h_r \in \Gamma(X, \mathcal{O}_X)$. Set $Z = V(h_1, \ldots, h_r)$. Note that X_s is a Noetherian scheme (because it is an algebraic $\kappa(s)$-scheme, see Varieties, Section 20) and that the topology on X_s is induced from the topology on X (see Schemes, Lemma 18.5). Hence after shrinking X a bit more we may assume that $Z_s \subset X_s$ is a regular immersion cut out by the r elements $h_i|_{X_s}$, see Divisors, Lemma 20.8 and its proof. It is also clear that $r = \dim_x(X_s) - \dim_x(Z_s)$ because

$$\begin{aligned} \dim_x(X_s) &= \dim(\mathcal{O}_{X_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim_x(Z_s) &= \dim(\mathcal{O}_{Z_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim(\mathcal{O}_{X_s,x}) &= \dim(\mathcal{O}_{Z_s,x}) + r \end{aligned}$$

the first two equalities by Algebra, Lemma 116.3 and the second by r times applying Algebra, Lemma 60.13. Hence Divisors, Lemma 22.7 part (3) applies to show that (after Zariski shrinking X) the morphism $Z \to X$ is a regular immersion to which Divisors, Lemma 22.4 applies (which gives the flatness and the statement on base change). $\square$

056Z **Lemma 23.3.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

1. *f is locally of finite presentation,*
2. *f is flat at x, and*
3. *$\mathcal{O}_{X_s,x}$ has depth ≥ 1.*

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is flat and of finite presentation.

Proof. Pick any $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ which maps to a nonzerodivisor in $\mathcal{O}_{X_s,x}$ and apply Lemma 23.1. $\square$

0570 **Lemma 23.4.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume* [DG67, IV Proposition 17.16.1]

1. *f is locally of finite presentation,*
2. *f is Cohen-Macaulay at x, and*
3. *x is a closed point of X_s.*

Then there exists a regular immersion $Z \to X$ containing x such that

(a) *$Z \to S$ is flat and locally of finite presentation,*
(b) *$Z \to S$ is locally quasi-finite, and*
(c) *$Z_s = \{x\}$ set theoretically.*

Proof. We may and do replace S by an affine open neighbourhood of s. We will prove the lemma for affine S by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take Z to be an open neighbourhood of x. (Note that an open immersion is a regular immersion.) Namely, if $d = 0$, then $X \to S$ is quasi-finite at x, see Morphisms, Lemma 29.5. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma 56.2. Thus after replacing X by U we see that the fibre X_s is a finite discrete set. Hence after replacing X by a further affine open neighbourhood of X we see that $f^{-1}(\{s\}) = \{x\}$ (because the topology on X_s is induced from the topology on X, see Schemes, Lemma 18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because x is a closed point of its fibre the extension $\kappa(x)/\kappa(s)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 20.3). Thus we see

$$\text{depth}(\mathcal{O}_{X_s,x}) = \dim(\mathcal{O}_{X_s,x}) = d > 0$$

the first equality as $\mathcal{O}_{X_s,x}$ is Cohen-Macaulay and the second by Morphisms, Lemma 28.1. Thus we may apply Lemma 23.3 to find a diagram

$$\begin{tikzcd} D \arrow[r] \arrow[rrd] & U \arrow[r] \arrow[rd] & X \arrow[d] \\ & & S \end{tikzcd}$$

with $x \in D$. Note that $\mathcal{O}_{D_s,x} = \mathcal{O}_{X_s,x}/(\overline{h})$ for some nonzerodivisor $\overline{h}$, see Divisors, Lemma 18.1. Hence $\mathcal{O}_{D_s,x}$ is Cohen-Macaulay of dimension one less than the dimension of $\mathcal{O}_{X_s,x}$, see Algebra, Lemma 104.2 for example. Thus the morphism $D \to S$ is flat, locally of finite presentation, and Cohen-Macaulay at x with $\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1$. By induction hypothesis we can find a regular immersion $Z \to D$ having properties (a), (b), (c). As $Z \to D \to U$ are both regular immersions, we see that also $Z \to U$ is a regular immersion by Divisors, Lemma 21.7. This finishes the proof. □

0571 **Lemma 23.5.** *Let $f : X \to S$ be a flat morphism of schemes which is locally of finite presentation. Let $s \in S$ be a point in the image of f. Then there exists a commutative diagram*

$$\begin{tikzcd} S' \arrow[rr] \arrow[rd, "g"'] & & X \arrow[ld, "f"] \\ & S & \end{tikzcd}$$

where $g : S' \to S$ is flat, locally of finite presentation, locally quasi-finite, and $s \in g(S')$.

Proof. The fibre X_s is not empty by assumption. Hence there exists a closed point $x \in X_s$ where f is Cohen-Macaulay, see Lemma 22.7. Apply Lemma 23.4 and set $S' = S$. □

The following lemma shows that sheaves for the fppf topology are the same thing as sheaves for the "quasi-finite, flat, finite presentation" topology.

0572 **Lemma 23.6.** *Let S be a scheme. Let $\mathcal{U} = \{S_i \to S\}_{i \in I}$ be an fppf covering of S, see Topologies, Definition 7.1. Then there exists an fppf covering $\mathcal{V} = \{T_j \to S\}_{j \in J}$ which refines (see Sites, Definition 8.1) $\mathcal{U}$ such that each $T_j \to S$ is locally quasi-finite.*

Proof. For every $s \in S$ there exists an $i \in I$ such that s is in the image of $S_i \to S$. By Lemma 23.5 we can find a morphism $g_s : T_s \to S$ such that $s \in g_s(T_s)$ which is flat, locally of finite presentation and locally quasi-finite and such that g_s factors through $S_i \to S$. Hence $\{T_s \to S\}$ is the desired covering of S that refines $\mathcal{U}$. □

24. Generic fibres

054V Some results on the relationship between generic fibres and nearby fibres.

054W **Lemma 24.1.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $X_\eta = \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X = \emptyset$.*

Proof. Follows immediately from the more general Morphisms, Lemma 8.5. □

05F5 **Lemma 24.2.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $X_\eta \neq \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X \to V$ is surjective.*

Proof. This follows, upon taking affine opens, from Algebra, Lemma 30.2. (Of course it also follows from generic flatness.) □

054X **Lemma 24.3.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $Z \subset X$ is a closed subset with Z_η nowhere dense in X_η, then there exists a nonempty open $V \subset Y$ such that Z_y is nowhere dense in X_y for all $y \in V$.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Set $X' = Y' \times_Y X$ and $Z' = Y' \times_Y Z$. As $Y' \to Y$ is a universal homeomorphism by Morphisms, Lemma 45.6 we see that it suffices to prove the lemma for $Z' \subset X' \to Y'$. Thus we may assume that Y is integral, see Properties, Lemma 3.4. By Morphisms, Proposition 27.1 there exists a nonempty affine open $V \subset Y$ such that $X_V \to V$ and $Z_V \to V$ are flat and of finite presentation. We claim that V works. Pick $y \in V$. If Z_y has a nonempty interior, then Z_y contains a generic point ξ of an irreducible component of X_y. Note that $\eta \rightsquigarrow f(\xi)$. Since $Z_V \to V$ is flat we can choose a specialization $\xi' \rightsquigarrow \xi$, $\xi' \in Z$ with $f(\xi') = \eta$, see Morphisms, Lemma 25.9. By Lemma 22.10 we see that

$$\dim_{\xi'}(Z_\eta) = \dim_\xi(Z_y) = \dim_\xi(X_y) = \dim_{\xi'}(X_\eta).$$

Hence some irreducible component of Z_η passing through ξ' has dimension $\dim_{\xi'}(X_\eta)$ which contradicts the assumption that Z_η is nowhere dense in X_η and we win. □

0573 **Lemma 24.4.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. Let $U \subset X$ be an open subscheme such that U_η is scheme theoretically dense in X_η. Then there exists a nonempty open $V \subset Y$ such that U_y is scheme theoretically dense in X_y for all $y \in V$.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' = Y' \times_Y X$ and $U' = Y' \times_Y U$. As $Y' \to Y$ induces a bijection on points, and as $U' \to U$ and $X' \to X$ induce isomorphisms of scheme theoretic fibres, we may replace Y by Y' and X by X'. Thus we may assume that Y is integral, see Properties, Lemma 3.4. We may also replace Y by a nonempty affine open. In other words we may assume that $Y = \text{Spec}(A)$ where A is a domain with fraction field K.

As f is of finite type we see that X is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ for some affine opens X_i. By Morphisms, Definition 7.1 we see that $U_i = X_i \cap U$ is an open subscheme of X_i such that $U_{i,\eta}$ is scheme theoretically dense in $X_{i,\eta}$. Thus it suffices to prove the result for the pairs (X_i, U_i), in other words we may assume that X is affine.

Write $X = \mathrm{Spec}(B)$. Note that B_K is Noetherian as it is a finite type K-algebra. Hence U_η is quasi-compact. Thus we can find finitely many $g_1, \ldots, g_m \in B$ such that $D(g_j) \subset U$ and such that $U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta$. The fact that U_η is scheme theoretically dense in X_η means that $B_K \to \bigoplus_j (B_K)_{g_j}$ is injective, see Morphisms, Example 7.4. By Algebra, Lemma 24.4 this is equivalent to the injectivity of $B_K \to \bigoplus_{j=1,\ldots,m} B_K$, $b \mapsto (g_1 b, \ldots, g_m b)$. Let M be the cokernel of this map over A, i.e., such that we have an exact sequence

$$0 \to I \to B \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} B \to M \to 0$$

After replacing A by A_h for some nonzero h we may assume that B is a flat, finitely presented A-algebra, and that M is flat over A, see Algebra, Lemma 118.3. The flatness of B over A implies that B is torsion free as an A-module, see More on Algebra, Lemma 22.9. Hence $B \subset B_K$. By assumption $I_K = 0$ which implies that $I = 0$ (as $I \subset B \subset B_K$ is a subset of I_K). Hence now we have a short exact sequence

$$0 \to B \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} B \to M \to 0$$

with M flat over A. Hence for every homomorphism $A \to \kappa$ where κ is a field, we obtain a short exact sequence

$$0 \to B \otimes_A \kappa \xrightarrow{(g_1\otimes 1,\ldots,g_m\otimes 1)} \bigoplus\nolimits_{j=1,\ldots,m} B \otimes_A \kappa \to M \otimes_A \kappa \to 0$$

see Algebra, Lemma 39.12. Reversing the arguments above this means that $\bigcup D(g_j \otimes 1)$ is scheme theoretically dense in $\mathrm{Spec}(B \otimes_A \kappa)$. As $\bigcup D(g_j \otimes 1) = \bigcup D(g_j)_\kappa \subset U_\kappa$ we obtain that U_κ is scheme theoretically dense in X_κ which is what we wanted to prove. $\square$

Suppose given a morphism of schemes $f : X \to Y$ and a point $y \in Y$. Recall that the fibre X_y is homeomorphic to the subset $f^{-1}(\{y\})$ of X with induced topology, see Schemes, Lemma 18.5. Suppose given a closed subset $T(y) \subset X_y$. Let T be the closure of $T(y)$ in X. Endow T with the induced reduced scheme structure. Then T is a closed subscheme of X with the property that $T_y = T(y)$ set-theoretically. In fact T is the smallest closed subscheme of X with this property. Thus it is "harmless" to denote a closed subset of X_y by T_y if we so desire. In the following lemma we apply this to the generic fibre of f.

054Y **Lemma 24.5.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. Let $X_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta}$ be a covering of the generic fibre by closed subsets of X_η. Let Z_i be the closure of $Z_{i,\eta}$ in X (see discussion above). Then there exists a nonempty open $V \subset Y$ such that $X_y = Z_{1,y} \cup \ldots \cup Z_{n,y}$ for all $y \in V$.*

Proof. If Y is Noetherian then $U = X \setminus (Z_1 \cup \ldots \cup Z_n)$ is of finite type over Y and we can directly apply Lemma 24.1 to get that $U_V = \emptyset$ for a nonempty open $V \subset Y$. In general we argue as follows. As the question is topological we may replace Y by its reduction. Thus Y is integral, see Properties, Lemma 3.4. After shrinking Y we may assume that $X \to Y$ is flat, see Morphisms, Proposition 27.1. In this case every point x in X_y is a specialization of a point $x' \in X_\eta$ by Morphisms, Lemma 25.9. As the Z_i are closed in X and cover the generic fibre this implies that $X_y = \bigcup Z_{i,y}$ for $y \in Y$ as desired. $\square$

The following lemma says that generic fibres of morphisms whose source is reduced are reduced.

054Z **Lemma 24.6.** *Let $f : X \to Y$ be a morphism of schemes. Let $\eta \in Y$ be a generic point of an irreducible component of Y. Then $(X_\eta)_{red} = (X_{red})_\eta$.*

Proof. Choose an affine neighbourhood $\mathrm{Spec}(A) \subset Y$ of η. Choose an affine open $\mathrm{Spec}(B) \subset X$ mapping into $\mathrm{Spec}(A)$ via the morphism f. Let $\mathfrak{p} \subset A$ be the minimal prime corresponding to η. Let B_{red} be the quotient of B by the nilradical $\sqrt{(0)}$. The algebraic content of the lemma is that $C = B_{red} \otimes_A \kappa(\mathfrak{p})$ is reduced. Denote $I \subset A$ the nilradical so that $A_{red} = A/I$. Denote $\mathfrak{p}_{red} = \mathfrak{p}/I$ which is a minimal prime of A_{red} with $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}_{red})$. Since $A \to B_{red}$ and $A \to \kappa(\mathfrak{p})$ both factor through $A \to A_{red}$ we have $C = B_{red} \otimes_{A_{red}} \kappa(\mathfrak{p}_{red})$. Now $\kappa(\mathfrak{p}_{red}) = (A_{red})_{\mathfrak{p}_{red}}$ is a localization by Algebra, Lemma 25.1. Hence C is a localization of B_{red} (Algebra, Lemma 12.15) and hence reduced. □

0550 **Lemma 24.7.** *Let $f : X \to Y$ be a morphism of schemes. Assume that Y is irreducible and f is of finite type. There exists a diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

where

1. *V is a nonempty open of Y,*
2. *$X_V = V \times_Y X$,*
3. *$g : Y' \to V$ is a finite universal homeomorphism,*
4. *$X' = (Y' \times_Y X)_{red} = (Y' \times_V X_V)_{red}$,*
5. *g' is a finite universal homeomorphism,*
6. *Y' is an integral affine scheme,*
7. *f' is flat and of finite presentation, and*
8. *the generic fibre of f' is geometrically reduced.*

Proof. Let $V = \mathrm{Spec}(A)$ be a nonempty affine open of Y. By assumption the Jacobson radical of A is a prime ideal $\mathfrak{p}$. Let $K = \kappa(\mathfrak{p})$. Let p be the characteristic of K if positive and 1 if the characteristic is zero. By Varieties, Lemma 6.11 there exists a finite purely inseparable field extension K'/K such that $(X_{K'})_{red}$ is geometrically reduced over K'. Choose elements $x_1, \ldots, x_n \in K'$ which generate K' over K and such that some p-power of x_i is in $A/\mathfrak{p}$. Let $A' \subset K'$ be the finite A-subalgebra of K' generated by $x_1, \ldots, x_n$. Note that A' is a domain with fraction field K'. By Algebra, Lemma 46.7 we see that $A \to A'$ induces a universal homeomorphism on spectra. Set $Y' = \mathrm{Spec}(A')$. Set $X' = (Y' \times_Y X)_{red}$. The generic fibre of $X' \to Y'$ is $(X_{K'})_{red}$ by Lemma 24.6 which is geometrically reduced by construction. Note that $X' \to X_V$ is a finite universal homeomorphism as the composition of the reduction morphism $X' \to Y' \times_Y X$ (see Morphisms, Lemma 45.6) and the base change of g. At this point all of the properties of the lemma hold except for possibly (7). This can be achieved by shrinking Y' and hence V, see Morphisms, Proposition 27.1. □

0551 **Lemma 24.8.** *Let $f : X \to Y$ be a morphism of schemes. Assume that Y is irreducible and f is of finite type. There exists a diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

where

(1) *V is a nonempty open of Y,*
(2) *$X_V = V \times_Y X$,*
(3) *$g : Y' \to V$ is surjective finite étale,*
(4) *$X' = Y' \times_Y X = Y' \times_V X_V$,*
(5) *g' is surjective finite étale,*
(6) *Y' is an irreducible affine scheme, and*
(7) *all irreducible components of the generic fibre of f' are geometrically irreducible.*

Proof. Let $V = \mathrm{Spec}(A)$ be a nonempty affine open of Y. By assumption the Jacobson radical of A is a prime ideal $\mathfrak p$. Let $K = \kappa(\mathfrak p)$. By Varieties, Lemma 8.15 there exists a finite separable field extension K'/K such that all irreducible components of $X_{K'}$ are geometrically irreducible over K'. Choose an element $\alpha \in K'$ which generates K' over K, see Fields, Lemma 19.1. Let $P(T) \in K[T]$ be the minimal polynomial for α over K. After replacing α by $f\alpha$ for some $f \in A$, $f \not\in \mathfrak p$ we may assume that there exists a monic polynomial $T^d + a_1T^{d-1} + \ldots + a_d \in A[T]$ which maps to $P(T) \in K[T]$ under the map $A[T] \to K[T]$. Set $A' = A[T]/(P)$. Then $A \to A'$ is a finite free ring map such that there exists a unique prime $\mathfrak q$ lying over $\mathfrak p$, such that $K = \kappa(\mathfrak p) \subset \kappa(\mathfrak q) = K'$ is finite separable, and such that $\mathfrak pA'_{\mathfrak q}$ is the maximal ideal of $A'_{\mathfrak q}$. Hence $g : Y' = \mathrm{Spec}(A') \to V = \mathrm{Spec}(A)$ is étale at $\mathfrak q$, see Algebra, Lemma 143.7. This means that there exists an open $W \subset \mathrm{Spec}(A')$ such that $g|_W : W \to \mathrm{Spec}(A)$ is étale. Since g is finite and since $\mathfrak q$ is the only point lying over $\mathfrak p$ we see that $Z = g(Y' \setminus W)$ is a closed subset of V not containing $\mathfrak p$. Hence after replacing V by a principal affine open of V which does not meet Z we obtain that g is finite étale. □

25. Relative assassins

05KM

05F1 **Lemma 25.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal F$ be a quasi-coherent $\mathcal O_X$-module. Let $\xi \in Ass_{X/S}(\mathcal F)$ and set $Z = \overline{\{\xi\}} \subset X$. If f is locally of finite type and $\mathcal F$ is a finite type $\mathcal O_X$-module, then there exists a nonempty open $V \subset Z$ such that for every $s \in f(V)$ the generic points of V_s are elements of $Ass_{X/S}(\mathcal F)$.*

Proof. We may replace S by an affine open neighbourhood of $f(\xi)$ and X by an affine open neighbourhood of ξ. Hence we may assume $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and that f is given by the finite type ring map $A \to B$, see Morphisms, Lemma 15.2. Moreover, we may write $\mathcal F = \widetilde M$ for some finite B-module M, see Properties, Lemma 16.1. Let $\mathfrak q \subset B$ be the prime corresponding to ξ and let $\mathfrak p \subset A$ be the corresponding prime of A. By assumption $\mathfrak q \in \mathrm{Ass}_B(M \otimes_A \kappa(\mathfrak p))$, see Algebra, Remark 65.6 and Divisors, Lemma 2.2. With this notation $Z = V(\mathfrak q) \subset \mathrm{Spec}(B)$.

In particular $f(Z) \subset V(\mathfrak{p})$. Hence clearly it suffices to prove the lemma after replacing A, B, and M by $A/\mathfrak{p}A$, $B/\mathfrak{p}B$, and $M/\mathfrak{p}M$. In other words we may assume that A is a domain with fraction field K and $\mathfrak{q} \subset B$ is an associated prime of $M \otimes_A K$.

At this point we can use generic flatness. Namely, by Algebra, Lemma 118.3 there exists a nonzero $g \in A$ such that M_g is flat as an A_g-module. After replacing A by A_g we may assume that M is flat as an A-module.

In this case, by Algebra, Lemma 65.4 we see that $\mathfrak{q}$ is also an associated prime of M. Hence we obtain an injective B-module map $B/\mathfrak{q} \to M$. Let Q be the cokernel so that we obtain a short exact sequence

$$0 \to B/\mathfrak{q} \to M \to Q \to 0$$

of finite B-modules. After applying generic flatness Algebra, Lemma 118.3 once more, this time to the B-module Q, we may assume that Q is a flat A-module. In particular we may assume the short exact sequence above is universally injective, see Algebra, Lemma 39.12. In this situation $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}') \subset M \otimes_A \kappa(\mathfrak{p}')$ for any prime $\mathfrak{p}'$ of A. The lemma follows as a minimal prime $\mathfrak{q}'$ of the support of $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}')$ is an associated prime of $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}')$ by Divisors, Lemma 2.9. □

05KN **Lemma 25.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be an open subscheme. Assume*

1. *f is of finite type,*
2. *$\mathcal{F}$ is of finite type,*
3. *Y is irreducible with generic point η, and*
4. *$Ass_{X_\eta}(\mathcal{F}_\eta)$ is not contained in U_η.*

Then there exists a nonempty open subscheme $V \subset Y$ such that for all $y \in V$ the set $Ass_{X_y}(\mathcal{F}_y)$ is not contained in U_y.

Proof. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition 5.5. Then Z_η is the scheme theoretic support of $\mathcal{F}_\eta$ (Morphisms, Lemma 25.14). Hence the generic points of irreducible components of Z_η are contained in $\text{Ass}_{X_\eta}(\mathcal{F}_\eta)$ by Divisors, Lemma 2.9. Hence we see that $Z_\eta \cap U_\eta = \emptyset$. Thus $T = Z \setminus U$ is a closed subset of Z with $T_\eta = \emptyset$. If we endow T with the induced reduced scheme structure then $T \to Y$ is a morphism of finite type. By Lemma 24.1 there is a nonempty open $V \subset Y$ with $T_V = \emptyset$. Then V works. □

05KP **Lemma 25.3.** *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be an open subscheme. Assume*

1. *f is of finite type,*
2. *$\mathcal{F}$ is of finite type,*
3. *Y is irreducible with generic point η, and*
4. *$Ass_{X_\eta}(\mathcal{F}_\eta) \subset U_\eta$.*

Then there exists a nonempty open subscheme $V \subset Y$ such that for all $y \in V$ we have $Ass_{X_y}(\mathcal{F}_y) \subset U_y$.

Proof. (This proof is the same as the proof of Lemma 24.4. We urge the reader to read that proof first.) Since the statement is about fibres it is clear that we may replace Y by its reduction. Hence we may assume that Y is integral, see Properties, Lemma 3.4. We may also assume that $Y = \mathrm{Spec}(A)$ is affine. Then A is a domain with fraction field K.

As f is of finite type we see that X is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ for some affine opens X_i and set $\mathcal{F}_i = \mathcal{F}|_{X_i}$. By assumption the generic fibre of $U_i = X_i \cap U$ contains $\mathrm{Ass}_{X_{i,\eta}}(\mathcal{F}_{i,\eta})$. Thus it suffices to prove the result for the triples $(X_i, \mathcal{F}_i, U_i)$, in other words we may assume that X is affine.

Write $X = \mathrm{Spec}(B)$. Let N be a finite B-module such that $\mathcal{F} = \widetilde{N}$. Note that B_K is Noetherian as it is a finite type K-algebra. Hence U_η is quasi-compact. Thus we can find finitely many $g_1, \ldots, g_m \in B$ such that $D(g_j) \subset U$ and such that $U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta$. Since $\mathrm{Ass}_{X_\eta}(\mathcal{F}_\eta) \subset U_\eta$ we see that $N_K \to \bigoplus_j (N_K)_{g_j}$ is injective. By Algebra, Lemma 24.4 this is equivalent to the injectivity of $N_K \to \bigoplus_{j=1,\ldots,m} N_K$, $n \mapsto (g_1 n, \ldots, g_m n)$. Let I and M be the kernel and cokernel of this map over A, i.e., such that we have an exact sequence

$$0 \to I \to N \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} N \to M \to 0$$

After replacing A by A_h for some nonzero h we may assume that B is a flat, finitely presented A-algebra and that both M and N are flat over A, see Algebra, Lemma 118.3. The flatness of N over A implies that N is torsion free as an A-module, see More on Algebra, Lemma 22.9. Hence $N \subset N_K$. By construction $I_K = 0$ which implies that $I = 0$ (as $I \subset N \subset N_K$ is a subset of I_K). Hence now we have a short exact sequence

$$0 \to N \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} N \to M \to 0$$

with M flat over A. Hence for every homomorphism $A \to \kappa$ where κ is a field, we obtain a short exact sequence

$$0 \to N \otimes_A \kappa \xrightarrow{(g_1\otimes 1,\ldots,g_m\otimes 1)} \bigoplus\nolimits_{j=1,\ldots,m} N \otimes_A \kappa \to M \otimes_A \kappa \to 0$$

see Algebra, Lemma 39.12. Reversing the arguments above this means that $\bigcup D(g_j \otimes 1)$ contains $\mathrm{Ass}_{B\otimes_A \kappa}(N \otimes_A \kappa)$. As $\bigcup D(g_j \otimes 1) = \bigcup D(g_j)_\kappa \subset U_\kappa$ we obtain that U_κ contains $\mathrm{Ass}_{X\otimes\kappa}(\mathcal{F} \otimes \kappa)$ which is what we wanted to prove. $\square$

05KQ **Lemma 25.4.** *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $U \subset X$ be an open subscheme. Let $g : S' \to S$ be a morphism of schemes, let $f' : X' = X_{S'} \to S'$ be the base change of f, let $g' : X' \to X$ be the projection, set $\mathcal{F}' = (g')^*\mathcal{F}$, and set $U' = (g')^{-1}(U)$. Finally, let $s' \in S'$ with image $s = g(s')$. In this case*

$$Ass_{X_s}(\mathcal{F}_s) \subset U_s \Leftrightarrow Ass_{X'_{s'}}(\mathcal{F}'_{s'}) \subset U'_{s'}.$$

Proof. This follows immediately from Divisors, Lemma 7.3. See also Divisors, Remark 7.4. $\square$

05KR **Lemma 25.5.** *Let $f : X \to Y$ be a morphism of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Let $U \subset X$ be an open subscheme such that $U \to Y$ is quasi-compact. Then the set*

$$E = \{y \in Y \mid Ass_{X_y}(\mathcal{F}_y) \subset U_y\}$$

is locally constructible in Y.

Proof. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. After possibly increasing i we may assume there exists a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation whose pullback to X is isomorphic to $\mathcal{F}$, see Limits, Lemma 10.2. After possibly increasing i one more time we may assume there exists an open subscheme $U_i \subset X_i$ whose inverse image in X is U, see Limits, Lemma 4.11. By Lemma 25.4 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that E is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E \cap Z$ either contains a nonempty open subset or is not dense in Z. This follows from Lemmas 25.2 and 25.3 applied to the base change $(X, \mathcal{F}, U) \times_Y Z$ over Z. $\square$

26. Reduced fibres

0574

0575 **Lemma 26.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η is nonreduced, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y is nonreduced.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to Y'$ be the base change of f. Note that $Y' \to Y$ induces a bijection on points and that $X' \to X$ identifies fibres. Hence we may assume that Y' is reduced, i.e., integral, see Properties, Lemma 3.4. We may also replace Y by an affine open. Hence we may assume that $Y = \mathrm{Spec}(A)$ with A a domain. Denote K the fraction field of A. Pick an affine open $\mathrm{Spec}(B) = U \subset X$ and a section $h_\eta \in \Gamma(U_\eta, \mathcal{O}_{U_\eta}) = B_K$ which is nonzero and nilpotent. After shrinking Y we may assume that h comes from $h \in \Gamma(U, \mathcal{O}_U) = B$. After shrinking Y a bit more we may assume that h is nilpotent. Let $I = \{b \in B \mid hb = 0\}$ be the annihilator of h. Then $C = B/I$ is a finite type A-algebra whose generic fiber $(B/I)_K$ is nonzero (as $h_\eta \neq 0$). We apply generic flatness to $A \to C$ and $A \to B/hB$, see Algebra, Lemma 118.3, and we obtain a $g \in A$, $g \neq 0$ such that C_g is free as an A_g-module and $(B/hB)_g$ is flat as an A_g-module. Replace Y by $D(g) \subset Y$. Now we have the short exact sequence

$$0 \to C \to B \to B/hB \to 0.$$

with B/hB flat over A and with C nonzero free as an A-module. It follows that for any homomorphism $A \to \kappa$ to a field the ring $C \otimes_A \kappa$ is nonzero and the sequence

$$0 \to C \otimes_A \kappa \to B \otimes_A \kappa \to B/hB \otimes_A \kappa \to 0$$

is exact, see Algebra, Lemma 39.12. Note that $B/hB \otimes_A \kappa = (B \otimes_A \kappa)/h(B \otimes_A \kappa)$ by right exactness of tensor product. Thus we conclude that multiplication by h is not zero on $B \otimes_A \kappa$. This clearly means that for any point $y \in Y$ the element h restricts to a nonzero element of U_y, whence X_y is nonreduced. $\square$

0576 **Lemma 26.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then*

$$\{y' \in Y' \mid X'_{y'} \text{ is geometrically reduced}\}$$
$$= g^{-1}(\{y \in Y \mid X_y \text{ is geometrically reduced}\}).$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically reduced over $\kappa(y')$ if and only if X_y is geometrically reduced over $\kappa(y)$. This follows from Varieties, Lemma 6.6. $\square$

0577 **Lemma 26.3.** *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η is not geometrically reduced, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y is not geometrically reduced.*

Proof. Apply Lemma 24.7 to get

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

with all the properties mentioned in that lemma. Let η' be the generic point of Y'. Consider the morphism $X' \to X_{Y'}$ (which is the reduction morphism) and the resulting morphism of generic fibres $X'_{\eta'} \to X_{\eta'}$. Since $X'_{\eta'}$ is geometrically reduced, and X_η is not this cannot be an isomorphism, see Varieties, Lemma 6.6. Hence $X_{\eta'}$ is nonreduced. Hence by Lemma 26.1 the fibres of $X_{Y'} \to Y'$ are nonreduced at all points $y' \in V'$ of a nonempty open $V' \subset Y'$. Since $g : Y' \to V$ is a homeomorphism Lemma 26.2 proves that $g(V')$ is the open we are looking for. $\square$

0578 **Lemma 26.4.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

1. *Y is irreducible with generic point η,*
2. *X_η is geometrically reduced, and*
3. *f is of finite type.*

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically reduced fibres.

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to Y'$ be the base change of f. Note that $Y' \to Y$ induces a bijection on points and that $X' \to X$ identifies fibres. Hence we may assume that Y' is reduced, i.e., integral, see Properties, Lemma 3.4. We may also replace Y by an affine open. Hence we may assume that $Y = \mathrm{Spec}(A)$ with A a domain. Denote K the fraction field of A. After shrinking Y a bit we may also assume that $X \to Y$ is flat and of finite presentation, see Morphisms, Proposition 27.1.

As X_η is geometrically reduced there exists an open dense subset $V \subset X_\eta$ such that $V \to \mathrm{Spec}(K)$ is smooth, see Varieties, Lemma 25.7. Let $U \subset X$ be the set of points where f is smooth. By Morphisms, Lemma 34.15 we see that $V \subset U_\eta$. Thus the generic fibre of U is dense in the generic fibre of X. Since X_η is reduced, it follows that U_η is scheme theoretically dense in X_η, see Morphisms, Lemma 7.8. We note that as $U \to Y$ is smooth all the fibres of $U \to Y$ are geometrically reduced. Thus it suffices to show that, after shrinking Y, for all $y \in Y$ the scheme U_y is scheme theoretically dense in X_y, see Morphisms, Lemma 7.9. This follows from Lemma 24.4. $\square$

0579 **Lemma 26.5.** *Let $f : X \to Y$ be a morphism which is quasi-compact and locally of finite presentation. Then the set*

$$E = \{y \in Y \mid X_y \text{ is geometrically reduced}\}$$

is locally constructible in Y.

Proof. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E \cap V$ is constructible in V. Thus we may assume that Y is affine. Then X is quasi-compact. Choose a finite affine open covering $X = U_1 \cup \ldots \cup U_n$. Then the fibres of $U_i \to Y$ at y form an affine open covering of the fibre of $X \to Y$ at y. Hence we may assume X is affine as well. Write $Y = \text{Spec}(A)$. Write $A = \text{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.1 we can find an i and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 26.2 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that E is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E \cap Z$ either contains a nonempty open subset or is not dense in Z. If X_ξ is geometrically reduced, then Lemma 26.4 (applied to the morphism $X_Z \to Z$) implies that all fibres X_y are geometrically reduced for a nonempty open $V \subset Z$. If X_ξ is not geometrically reduced, then Lemma 26.3 (applied to the morphism $X_Z \to Z$) implies that all fibres X_y are geometrically reduced for a nonempty open $V \subset Z$. Thus we win. $\square$

0C0D **Lemma 26.6.** *Let $X \to \text{Spec}(R)$ be a proper flat morphism where R is a discrete valuation ring. If the special fibre is reduced, then both X and the generic fibre X_η are reduced.*

Proof. Assume the special fibre X_s is reduced. Let $x \in X$ be any point, and let us show that $\mathcal{O}_{X,x}$ is reduced; this will prove that X and X_η are reduced. Let $x \rightsquigarrow x'$ be a specialization with x' in the special fibre; such a specialization exists as a proper morphism is closed. Consider the local ring $A = \mathcal{O}_{X,x'}$. Then $\mathcal{O}_{X,x}$ is a localization of A, so it suffices to show that A is reduced. Let $\pi \in R$ be a uniformizer. If $a \in A$ then there exists an $n \geq 0$ and an element $a' \in A$ such that $a = \pi^n a'$ and $a' \not\in \pi A$. This follows from Krull intersection theorem (Algebra, Lemma 51.4). If a is nilpotent, so is a', because π is a nonzerodivisor by flatness of A over R. But a' maps to a nonzero element of the reduced ring $A/\pi A = \mathcal{O}_{X_s,x'}$. This is a contradiction unless A is reduced, which is what we wanted to show. $\square$

0C0E **Lemma 26.7.** *Let $f : X \to Y$ be a flat proper morphism of finite presentation. Then the set $\{y \in Y \mid X_y \text{ is geometrically reduced}\}$ is open in Y.*

Proof. We may assume Y is affine. Then Y is a cofiltered limit of affine schemes of finite type over $\mathbf{Z}$. Hence we can assume $X \to Y$ is the base change of $X_0 \to Y_0$ where Y_0 is the spectrum of a finite type $\mathbf{Z}$-algebra and $X_0 \to Y_0$ is flat and proper. See Limits, Lemma 10.1, 8.7, and 13.1. Since the formation of the set of points where the fibres are geometrically reduced commutes with base change (Lemma 26.2), we may assume the base is Noetherian.

Assume Y is Noetherian. The set is constructible by Lemma 26.5. Hence it suffices to show the set is stable under generalization (Topology, Lemma 19.10). By Properties, Lemma 5.10 we reduce to the case where $Y = \text{Spec}(R)$, R is a discrete valuation ring, and the closed fibre X_y is geometrically reduced. To show: the generic fibre X_η is geometrically reduced.

If not then there exists a finite extension L of the fraction field of R such that X_L is not reduced, see Varieties, Lemma 6.4. There exists a discrete valuation ring $R' \subset L$ with fraction field L dominating R, see Algebra, Lemma 120.18. After replacing R by R' we reduce to Lemma 26.6. □

27. Irreducible components of fibres

0553

0554 **Lemma 27.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has n irreducible components, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has at least n irreducible components.*

Proof. As the question is purely topological we may replace X and Y by their reductions. In particular this implies that Y is integral, see Properties, Lemma 3.4. Let $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ be the decomposition of X_η into irreducible components. Let $X_i \subset X$ be the reduced closed subscheme whose generic fibre is $X_{i,\eta}$. Note that $Z_{i,j} = X_i \cap X_j$ is a closed subset of X_i whose generic fibre $Z_{i,j,\eta}$ is nowhere dense in $X_{i,\eta}$. Hence after shrinking Y we may assume that $Z_{i,j,y}$ is nowhere dense in $X_{i,y}$ for every $y \in Y$, see Lemma 24.3. After shrinking Y some more we may assume that $X_y = \bigcup X_{i,y}$ for $y \in Y$, see Lemma 24.5. Moreover, after shrinking Y we may assume that each $X_i \to Y$ is flat and of finite presentation, see Morphisms, Proposition 27.1. The morphisms $X_i \to Y$ are open, see Morphisms, Lemma 25.10. Thus there exists an open neighbourhood V of η which is contained in $f(X_i)$ for each i. For each $y \in V$ the schemes $X_{i,y}$ are nonempty closed subsets of X_y, we have $X_y = \bigcup X_{i,y}$ and the intersections $Z_{i,j,y} = X_{i,y} \cap X_{j,y}$ are not dense in $X_{i,y}$. Clearly this implies that X_y has at least n irreducible components. □

0555 **Lemma 27.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then*

$$\begin{aligned} & \{y' \in Y' \mid X'_{y'} \text{ is geometrically irreducible}\} \\ & = g^{-1}(\{y \in Y \mid X_y \text{ is geometrically irreducible}\}). \end{aligned}$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically irreducible over $\kappa(y')$ if and only if X_y is geometrically irreducible over $\kappa(y)$. This follows from Varieties, Lemma 8.2. □

0556 **Lemma 27.3.** *Let $f : X \to Y$ be a morphism of schemes. Let*

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of irreducible components of $(X_y)_K$ where K is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram

$$\begin{tikzcd} K \arrow[r] & K'' & K' \arrow[l] \\ \kappa(y) \arrow[u] \arrow[rr] & & \kappa(y') \arrow[u] \end{tikzcd}$$

of fields. The result follows as the morphisms of schemes

$$(X'_{y'})_{K'} \longleftarrow (X'_{y'})_{K''} = (X_y)_{K''} \longrightarrow (X_y)_K$$

induce bijections between irreducible components, see Varieties, Lemma 8.7. □

0557 **Lemma 27.4.** *Let A be a domain with fraction field K. Let $P \in A[x_1, \ldots, x_n]$. Denote $\overline{K}$ the algebraic closure of K. Assume P is irreducible in $\overline{K}[x_1, \ldots, x_n]$. Then there exists a $f \in A$ such that $P^\varphi \in \kappa[x_1, \ldots, x_n]$ is irreducible for all homomorphisms $\varphi : A_f \to \kappa$ into fields.*

Proof. There exists an automorphism Ψ of $A[x_1, \ldots, x_n]$ over A such that $\Psi(P) = ax_n^d +$ lower order terms in x_n with $a \neq 0$, see Algebra, Lemma 115.2. We may replace P by $\Psi(P)$ and we may replace A by A_a. Thus we may assume that P is monic in x_n of degree $d > 0$. For $i = 1, \ldots, n-1$ let d_i be the degree of P in x_i. Note that this implies that P^φ is monic of degree d in x_n and has degree $\leq d_i$ in x_i for every homomorphism $\varphi : A \to \kappa$ where κ is a field. Thus if P^φ is reducible, then we can write

$$P^\varphi = Q_1 Q_2$$

with Q_1, Q_2 monic of degree $e_1, e_2 \geq 0$ in x_n with $e_1 + e_2 = d$ and having degree $\leq d_i$ in x_i for $i = 1, \ldots, n-1$. In other words we can write

0558 (27.4.1) $$Q_j = x_n^{e_j} + \sum\nolimits_{0 \leq l < e_j} \left(\sum\nolimits_{L \in \mathcal{L}} a_{j,l,L} x^L \right) x_n^l$$

where the sum is over the set $\mathcal{L}$ of multi-indices L of the form $L = (l_1, \ldots, l_{n-1})$ with $0 \leq l_i \leq d_i$. For any $e_1, e_2 \geq 0$ with $e_1 + e_2 = d$ we consider the A-algebra

$$B_{e_1,e_2} = A[\{a_{1,l,L}\}_{0 \leq l < e_1, L \in \mathcal{L}}, \{a_{2,l,L}\}_{0 \leq l < e_2, L \in \mathcal{L}}]/(\text{relations})$$

where the (relations) is the ideal generated by the coefficients of the polynomial

$$P - Q_1 Q_2 \in A[\{a_{1,l,L}\}_{0 \leq l < e_1, L \in \mathcal{L}}, \{a_{2,l,L}\}_{0 \leq l < e_2, L \in \mathcal{L}}][x_1, \ldots, x_n]$$

with Q_1 and Q_2 defined as in (27.4.1). OK, and the assumption that P is irreducible over $\overline{K}$ implies that there does not exist any A-algebra homomorphism $B_{e_1,e_2} \to \overline{K}$. By the Hilbert Nullstellensatz, see Algebra, Theorem 34.1 this means that $B_{e_1,e_2} \otimes_A K = 0$. As B_{e_1,e_2} is a finitely generated A-algebra this signifies that we can find an $f_{e_1,e_2} \in A$ such that $(B_{e_1,e_2})_{f_{e_1,e_2}} = 0$. By construction this means that if $\varphi : A_{f_{e_1,e_2}} \to \kappa$ is a homomorphism to a field, then P^φ does not have a factorization $P^\varphi = Q_1 Q_2$ with Q_1 of degree e_1 in x_n and Q_2 of degree e_2 in x_n. Thus taking $f = \prod_{e1, e_2 \geq 0, e_1 + e_2 = d} f_{e_1,e_2}$ we win. □

0559 **Lemma 27.5.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

1. *Y is irreducible with generic point η,*
2. *X_η is geometrically irreducible, and*
3. *f is of finite type.*

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically irreducible fibres.

First proof of Lemma 27.5. We give two proofs of the lemma. These are essentially equivalent; the second is more self contained but a bit longer. Choose a diagram

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

as in Lemma 24.7. Note that the generic fibre of f' is the reduction of the generic fibre of f (see Lemma 24.6) and hence is geometrically irreducible. Suppose that the lemma holds for the morphism f'. Then after shrinking V all the fibres of f' are geometrically irreducible. As $X' = (Y' \times_V X_V)_{red}$ this implies that all the fibres of $Y' \times_V X_V$ are geometrically irreducible. Hence by Lemma 27.2 all the fibres of $X_V \to V$ are geometrically irreducible and we win. In this way we see that we may assume that the generic fibre is geometrically reduced as well as geometrically irreducible and we may assume $Y = \mathrm{Spec}(A)$ with A a domain.

Let $x \in X_\eta$ be the generic point. As X_η is geometrically irreducible and reduced we see that $L = \kappa(x)$ is a finitely generated extension of $K = \kappa(\eta)$ which is geometrically reduced and geometrically irreducible, see Varieties, Lemmas 6.2 and 8.6. In particular the field extension L/K is separable, see Algebra, Lemma 44.2. Hence we can find $x_1, \ldots, x_{r+1} \in L$ which generate L over K and such that $x_1, \ldots, x_r$ is a transcendence basis for L over K, see Algebra, Lemma 42.3. Let $P \in K(x_1, \ldots, x_r)[T]$ be the minimal polynomial for x_{r+1}. Clearing denominators we may assume that P has coefficients in $A[x_1, \ldots, x_r]$. Note that as L is geometrically reduced and geometrically irreducible over K, the polynomial P is irreducible in $\overline{K}[x_1, \ldots, x_r, T]$ where $\overline{K}$ is the algebraic closure of K. Denote

$$B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1}))$$

and set $X' = \mathrm{Spec}(B')$. By construction the fraction field of B' is isomorphic to $L = \kappa(x)$ as K-extensions. Hence there exists an open $U \subset X$, and open $U' \subset X'$ and a Y-isomorphism $U \to U'$, see Morphisms, Lemma 50.7. Here is a diagram:

$$\begin{tikzcd} X \arrow[rd] & U \arrow[l] \arrow[r, equal] \arrow[d] & U' \arrow[r] \arrow[d] & X' \arrow[r, equal] \arrow[ld] & \mathrm{Spec}(B') \\ & Y \arrow[r, equal] & Y & & \end{tikzcd}$$

Note that $U_\eta \subset X_\eta$ and $U'_\eta \subset X'_\eta$ are dense opens. Thus after shrinking Y by applying Lemma 24.3 we obtain that U_y is dense in X_y and U'_y is dense in X'_y for all $y \in Y$. Thus it suffices to prove the lemma for $X' \to Y$ which is the content of Lemma 27.4. $\square$

Second proof of Lemma 27.5. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to X$ be the reduction of X. Note that $X' \to X \to Y$ factors through Y', see Schemes, Lemma 12.7. As $Y' \to Y$ and $X' \to X$ are universal homeomorphisms by Morphisms, Lemma 45.6 we see that it suffices to prove the lemma for $X' \to Y'$. Thus we may assume that X and Y are reduced. In particular Y is integral, see Properties, Lemma 3.4. Thus by Morphisms, Proposition 27.1 there exists a nonempty affine open $V \subset Y$ such that $X_V \to V$ is flat and of finite presentation. After replacing Y by V we may assume, in addition to (1), (2), (3) that Y is integral affine, X is reduced, and f is flat and of finite presentation. In particular f is universally open, see Morphisms, Lemma 25.10.

Pick a nonempty affine open $U \subset X$. Then $U \to Y$ is flat and of finite presentation with geometrically irreducible generic fibre. The complement $X_\eta \setminus U_\eta$ is nowhere dense. Thus after shrinking Y we may assume $U_y \subset X_y$ is open dense for all $y \in Y$, see Lemma 24.3. Thus we may replace X by U and we reduce to the case where Y is integral affine and X is reduced affine, flat and of finite presentation over Y with geometrically irreducible generic fibre X_η.

Write $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. Then A is a domain, B is reduced, $A \to B$ is flat of finite presentation, and B_K is geometrically irreducible over the fraction field K of A. In particular we see that B_K is a domain. Let L be the fraction field of B_K. Note that L is a finitely generated field extension of K as B is an A-algebra of finite presentation. Let K'/K be a finite purely inseparable extension such that $(L \otimes_K K')_{red}$ is a separably generated field extension, see Algebra, Lemma 45.3. Choose $x_1, \ldots, x_n \in K'$ which generate the field extension K' over K, and such that $x_i^{q_i} \in A$ for some prime power q_i (proof existence x_i omitted). Let A' be the A-subalgebra of K' generated by $x_1, \ldots, x_n$. Then A' is a finite A-subalgebra $A' \subset K'$ whose fraction field is K'. Note that $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is a universal homeomorphism, see Algebra, Lemma 46.7. Hence it suffices to prove the result after base changing to $\mathrm{Spec}(A')$. We are going to replace A by A' and B by $(B \otimes_A A')_{red}$ to arrive at the situation where L is a separably generated field extension of K. Of course it may happen that $(B \otimes_A A')_{red}$ is no longer flat, or of finite presentation over A', but this can be remedied by replacing A' by A'_f for a suitable $f \in A'$, see Algebra, Lemma 118.3.

At this point we know that A is a domain, B is reduced, $A \to B$ is flat and of finite presentation, B_K is a domain whose fraction field L is a separably generated field extension of the fraction field K of A. By Algebra, Lemma 42.3 we may write $L = K(x_1, \ldots, x_{r+1})$ where $x_1, \ldots, x_r$ are algebraically independent over K, and x_{r+1} is separable over $K(x_1, \ldots, x_r)$. After clearing denominators we may assume that the minimal polynomial $P \in K(x_1, \ldots, x_r)[T]$ of x_{r+1} over $K(x_1, \ldots, x_r)$ has coefficients in $A[x_1, \ldots, x_r]$. Note that since L/K is separable and since L is geometrically irreducible over K, the polynomial P is irreducible over the algebraic closure $\overline{K}$ of K. Denote

$$B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1})).$$

By construction the fraction fields of B and B' are isomorphic as K-extensions. Hence there exists an isomorphism of A-algebras $B_h \cong B'_{h'}$ for suitable $h \in B$ and $h' \in B'$, see Morphisms, Lemma 50.7. In other words X and $X' = \mathrm{Spec}(B')$ have a common affine open U. Here is a diagram:

$$\begin{tikzcd}
X = \mathrm{Spec}(B) \arrow[rd] & U \arrow[l] \arrow[r] \arrow[d] & \mathrm{Spec}(B') = X' \arrow[ld] \\
 & Y = \mathrm{Spec}(A) &
\end{tikzcd}$$

After shrinking Y once more (by applying Lemma 24.3 to $Z = X \setminus U$ in X and $Z' = X' \setminus U$ in X') we see that U_y is dense in X_y and U_y is dense in X'_y for all $y \in Y$. Thus it suffices to prove the lemma for $X' \to Y$ which is the content of Lemma 27.4. □

055A **Lemma 27.6.** *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically irreducible components of fibres of f introduced in Lemma 27.3. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that $n_{X/Y}|_V$ is constant.*

Proof. Let Z be the reduced induced scheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ be the base change of f. Clearly it suffices to prove the lemma for f_Z and the generic point of Z. Hence we may assume that Y is an integral scheme, see Properties, Lemma 3.4. Our goal in this case is to produce a nonempty open $V \subset Y$ such that $n_{X/Y}|_V$ is constant.

We apply Lemma 24.8 to $f : X \to Y$ and we get $g : Y' \to V \subset Y$. As $g : Y' \to V$ is surjective finite étale, in particular open (see Morphisms, Lemma 36.13), it suffices to prove that there exists an open $V' \subset Y'$ such that $n_{X'/Y'}|_{V'}$ is constant, see Lemma 27.3. Thus we see that we may assume that all irreducible components of the generic fibre X_η are geometrically irreducible over $\kappa(\eta)$.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. In particular $n_{X/Y}(\eta) = n$. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 24.5. After shrinking Y some more we see that each fibre of f has at least n irreducible components, see Lemma 27.1. Hence $n_{X/Y}(y) \geq n$ for all $y \in Y$. After shrinking Y some more we obtain that $X_{i,y}$ is geometrically irreducible for each i and all $y \in Y$, see Lemma 27.5. Since $X_y = \bigcup X_{i,y}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. □

055B **Lemma 27.7.** *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically irreducible components of fibres of f introduced in Lemma 27.3. Assume f of finite presentation. Then the level sets*

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \text{Spec}(A)$ and $A = \text{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.1 we can find an i and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 27.3 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 27.6 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This clearly implies what we want. □

28. Connected components of fibres

055C

055D **Lemma 28.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has n connected components, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has at least n connected components.*

Proof. As the question is purely topological we may replace X and Y by their reductions. In particular this implies that Y is integral, see Properties, Lemma 3.4. Let $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ be the decomposition of X_η into connected components. Let $X_i \subset X$ be the reduced closed subscheme whose generic fibre is $X_{i,\eta}$. Note that $Z_{i,j} = X_i \cap X_j$ is a closed subset of X whose generic fibre $Z_{i,j,\eta}$ is empty. Hence after shrinking Y we may assume that $Z_{i,j} = \emptyset$, see Lemma 24.1. After shrinking Y some more we may assume that $X_y = \bigcup X_{i,y}$ for $y \in Y$, see Lemma 24.5. Moreover, after shrinking Y we may assume that each $X_i \to Y$ is flat and of finite presentation, see Morphisms, Proposition 27.1. The morphisms $X_i \to Y$ are open, see Morphisms, Lemma 25.10. Thus there exists an open neighbourhood V of η which is contained in $f(X_i)$ for each i. For each $y \in V$ the schemes $X_{i,y}$ are nonempty closed subsets of X_y, we have $X_y = \bigcup X_{i,y}$ and the intersections $Z_{i,j,y} = X_{i,y} \cap X_{j,y}$ are empty! Clearly this implies that X_y has at least n connected components. □

055E **Lemma 28.2.** *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then*

$$\begin{aligned} &\{y' \in Y' \mid X'_{y'} \text{ is geometrically connected}\} \\ &= g^{-1}(\{y \in Y \mid X_y \text{ is geometrically connected}\}). \end{aligned}$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically connected over $\kappa(y')$ if and only if X_y is geometrically connected over $\kappa(y)$. This follows from Varieties, Lemma 7.3. □

055F **Lemma 28.3.** *Let $f : X \to Y$ be a morphism of schemes. Let*

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of connected components of $(X_y)_K$ where K is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram

$$\begin{tikzcd} K \arrow[r] & K'' & K' \arrow[l] \\ \kappa(y) \arrow[u] \arrow[rr] & & \kappa(y') \arrow[u] \end{tikzcd}$$

of fields. The result follows as the morphisms of schemes

$$(X'_{y'})_{K'} \longleftarrow (X'_{y'})_{K''} = (X_y)_{K''} \longrightarrow (X_y)_K$$

induce bijections between connected components, see Varieties, Lemma 7.6. $\square$

055G **Lemma 28.4.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

1. *Y is irreducible with generic point η,*
2. *X_η is geometrically connected, and*
3. *f is of finite type.*

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically connected fibres.

Proof. Choose a diagram

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

as in Lemma 24.8. Note that the generic fibre of f' is geometrically connected (for example by Lemma 28.3). Suppose that the lemma holds for the morphism f'. This means that there exists a nonempty open $W \subset Y'$ such that every fibre of $X' \to Y'$ over W is geometrically connected. Then, as g is an open morphism by Morphisms, Lemma 36.13 all the fibres of f at points of the nonempty open $V = g(W)$ are geometrically connected, see Lemma 28.3. In this way we see that we may assume that the irreducible components of the generic fibre X_η are geometrically irreducible.

Let Y' be the reduction of Y, and set $X' = Y' \times_Y X$. Then it suffices to prove the lemma for the morphism $X' \to Y'$ (for example by Lemma 28.3 once again). Since the generic fibre of $X' \to Y'$ is the same as the generic fibre of $X \to Y$ we see that we may assume that Y is irreducible and reduced (i.e., integral, see Properties, Lemma 3.4) and that the irreducible components of the generic fibre X_η are geometrically irreducible.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 24.5. Let $Z_{i,j} = X_i \cap X_j$. Let

$$\{1, \ldots, n\} \times \{1, \ldots, n\} = I \amalg J$$

where $(i,j) \in I$ if $Z_{i,j,\eta} = \emptyset$ and $(i,j) \in J$ if $Z_{i,j,\eta} \neq \emptyset$. After shrinking Y we may assume that $Z_{i,j} = \emptyset$ for all $(i,j) \in I$, see Lemma 24.1. After shrinking Y we obtain that $X_{i,y}$ is geometrically irreducible for each i and all $y \in Y$, see Lemma 27.5. After shrinking Y some more we achieve the situation where each $Z_{i,j} \to Y$ is flat and of finite presentation for all $(i,j) \in J$, see Morphisms, Proposition 27.1. This means that $f(Z_{i,j}) \subset Y$ is open, see Morphisms, Lemma 25.10. We claim that

$$V = \bigcap\nolimits_{(i,j) \in J} f(Z_{i,j})$$

works, i.e., that X_y is geometrically connected for each $y \in V$. Namely, the fact that X_η is connected implies that the equivalence relation generated by the pairs in J has only one equivalence class. Now if $y \in V$ and $K \supset \kappa(y)$ is a separably closed extension, then the irreducible components of $(X_y)_K$ are the fibres $(X_{i,y})_K$. Moreover, we see by construction and $y \in V$ that $(X_{i,y})_K$ meets $(X_{j,y})_K$ if and only if $(i, j) \in J$. Hence the remark on equivalence classes shows that $(X_y)_K$ is connected and we win. $\square$

055H **Lemma 28.5.** *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically connected components of fibres of f introduced in Lemma 28.3. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that $n_{X/Y}|_V$ is constant.*

Proof. Let Z be the reduced induced scheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ be the base change of f. Clearly it suffices to prove the lemma for f_Z and the generic point of Z. Hence we may assume that Y is an integral scheme, see Properties, Lemma 3.4. Our goal in this case is to produce a nonempty open $V \subset Y$ such that $n_{X/Y}|_V$ is constant.

We apply Lemma 24.8 to $f : X \to Y$ and we get $g : Y' \to V \subset Y$. As $g : Y' \to V$ is surjective finite étale, in particular open (see Morphisms, Lemma 36.13), it suffices to prove that there exists an open $V' \subset Y'$ such that $n_{X'/Y'}|_{V'}$ is constant, see Lemma 27.3. Thus we see that we may assume that all irreducible components of the generic fibre X_η are geometrically irreducible over $\kappa(\eta)$. By Varieties, Lemma 8.16 this implies that also the connected components of X_η are geometrically connected.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) connected components. In particular $n_{X/Y}(\eta) = n$. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 24.5. After shrinking Y some more we see that each fibre of f has at least n connected components, see Lemma 28.1. Hence $n_{X/Y}(y) \geq n$ for all $y \in Y$. After shrinking Y some more we obtain that $X_{i,y}$ is geometrically connected for each i and all $y \in Y$, see Lemma 28.4. Since $X_y = \bigcup X_{i,y}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. $\square$

055I **Lemma 28.6.** *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometric connected components of fibres of f introduced in Lemma 28.3. Assume f of finite presentation. Then the level sets*

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \text{Spec}(A)$ and $A = \text{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.1 we can find an i and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 28.3 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 28.5 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This clearly implies what we want. $\square$

055J **Lemma 28.7.** *Let $f : X \to S$ be a morphism of schemes. Assume that*

(1) *S is the spectrum of a discrete valuation ring,*
(2) *f is flat,*
(3) *X is connected,*
(4) *the closed fibre X_s is reduced.*

Then the generic fibre X_η is connected.

Proof. Write $S = \text{Spec}(R)$ and let $\pi \in R$ be a uniformizer. To get a contradiction assume that X_η is disconnected. This means there exists a nontrivial idempotent $e \in \Gamma(X_\eta, \mathcal{O}_{X_\eta})$. Let $U = \text{Spec}(A)$ be any affine open in X. Note that π is a nonzerodivisor on A as A is flat over R, see More on Algebra, Lemma 22.9 for example. Then $e|_{U_\eta}$ corresponds to an element $e \in A[1/\pi]$. Let $z \in A$ be an element such that $e = z/\pi^n$ with $n \geq 0$ minimal. Note that $z^2 = \pi^n z$. This means that $z \bmod \pi A$ is nilpotent if $n > 0$. By assumption $A/\pi A$ is reduced, and hence minimality of n implies $n = 0$. Thus we conclude that $e \in A$! In other words $e \in \Gamma(X, \mathcal{O}_X)$. As X is connected it follows that e is a trivial idempotent which is a contradiction. $\square$

29. Connected components meeting a section

055K The results in this section are in particular applicable to a group scheme $G \to S$ and its neutral section $e : S \to G$.

055L **Situation 29.1.** Here $f : X \to Y$ be a morphism of schemes, and $s : Y \to X$ is a section of f. For every $y \in Y$ we denote X_y^0 the connected component of X_y containing $s(y)$. Finally, we set $X^0 = \bigcup_{y \in Y} X_y^0$.

055M **Lemma 29.2.** *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 29.1. If $g : Y' \to Y$ is any morphism, consider the base change diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"'] \\ Y' \arrow[r, "g"] \arrow[u, bend left, "s'"] & Y \arrow[u, bend right, "s"'] \end{tikzcd}$$

so that we obtain $(X')^0 \subset X'$. Then $(X')^0 = (g')^{-1}(X^0)$.

Proof. Let $y' \in Y'$ with image $y \in Y$. We may think of X_y^0 as a closed subscheme of X_y, see for example Morphisms, Definition 26.3. As $s(y) \in X_y^0$ we conclude from Varieties, Lemma 7.14 that X_y^0 is a geometrically connected scheme over $\kappa(y)$. Hence $X_y^0 \times_y y' \to X'_{y'}$ is a connected closed subscheme which contains $s'(y')$. Thus $X_y^0 \times_y y' \subset (X'_{y'})^0$. The other inclusion $X_y^0 \times_y y' \supset (X'_{y'})^0$ is clear as the image of $(X'_{y'})^0$ in X_y is a connected subset of X_y which contains $s(y)$. $\square$

055N **Lemma 29.3.** *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 29.1. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that the inverse image of X^0 in the base change X_V is open and closed in X_V.*

Proof. Let $Z \subset Y$ be the induced reduced closed subscheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ and $s_Z : Z \to X_Z$ be the base changes of f and s. By Lemma 29.2 we have $(X_Z)^0 = (X^0)_Z$. Hence it suffices to prove the lemma for the morphism $X_Z \to Z$ and the point $x \in X_Z$ which maps to the generic point of Z. In other words we have reduced the problem to the case where Y is an integral scheme (see Properties, Lemma 3.4) with generic point η. Our goal is to show that after shrinking Y the subset X^0 becomes an open and closed subset of X.

Note that the scheme X_η is of finite type over a field, hence Noetherian. Thus its connected components are open as well as closed. Hence we may write $X_\eta = X_\eta^0 \amalg T_\eta$ for some open and closed subset T_η of X_η. Next, let $T \subset X$ be the closure of T_η and let $X^{00} \subset X$ be the closure of X_η^0. Note that T_η, resp. X_η^0 is the generic fibre of T, resp. X^{00}, see discussion preceding Lemma 24.5. Moreover, that lemma implies that after shrinking Y we may assume that $X = X^{00} \cup T$ (set theoretically). Note that $(T \cap X^{00})_\eta = T_\eta \cap X_\eta^0 = \emptyset$. Hence after shrinking Y we may assume that $T \cap X^{00} = \emptyset$, see Lemma 24.1. In particular X^{00} is open in X. Note that X_η^0 is connected and has a rational point, namely $s(\eta)$, hence it is geometrically connected, see Varieties, Lemma 7.14. Thus after shrinking Y we may assume that all fibres of $X^{00} \to Y$ are geometrically connected, see Lemma 28.4. At this point it follows that the fibres X_y^{00} are open, closed, and connected subsets of X_y containing $\sigma(y)$. It follows that $X^0 = X^{00}$ and we win. ☐

055P **Lemma 29.4.** *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 29.1. If f is of finite presentation then X^0 is locally constructible in X.*

Proof. Let $x \in X$. We have to show that there exists an open neighbourhood U of x such that $X^0 \cap U$ is constructible in U. This reduces us to the case where Y is affine. Write $Y = \text{Spec}(A)$ and $A = \text{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.1 we can find an i and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation, endowed with a section $s_i : \text{Spec}(A_i) \to X_i$ whose base change to Y recovers f and the section s. By Lemma 29.2 it suffices to prove the lemma for f_i, s_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

Assume Y is a Noetherian affine scheme. Since f is of finite presentation, i.e., of finite type, we see that X is a Noetherian scheme too, see Morphisms, Lemma 15.6. In order to prove the lemma in this case it suffices to show that for every irreducible closed subset $Z \subset X$ the intersection $Z \cap X^0$ either contains a nonempty open of Z or is not dense in Z, see Topology, Lemma 16.3. Let $x \in Z$ be the generic point, and let $y = f(x)$. By Lemma 29.3 there exists a nonempty open subset $V \subset \overline{\{y\}}$ such that $X^0 \cap X_V$ is open and closed in X_V. Since $f(Z) \subset \overline{\{y\}}$ and $f(x) = y \in V$ we see that $W = f^{-1}(V) \cap Z$ is a nonempty open subset of Z. It follows that $X^0 \cap W$ is open and closed in W. Since W is irreducible we see that $X^0 \cap W$ is either empty or equal to W. This proves the lemma. ☐

055Q **Lemma 29.5.** *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 29.1. Let $y \in Y$ be a point. Assume*

(1) *f is of finite presentation and flat, and*
(2) *the fibre X_y is geometrically reduced.*

Then X^0 is a neighbourhood of X_y^0 in X.

Proof. We may replace Y with an affine open neighbourhood of y. Write $Y = \text{Spec}(A)$ and $A = \text{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.1 we can find an i and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation, endowed with a section $s_i : \text{Spec}(A_i) \to X_i$ whose base change to Y recovers f

and the section s. After possibly increasing i we may also assume that f_i is flat, see Limits, Lemma 8.7. Let y_i be the image of y in Y_i. Note that $X_y = (X_{i,y_i}) \times_{y_i} y$. Hence X_{i,y_i} is geometrically reduced, see Varieties, Lemma 6.6. By Lemma 29.2 it suffices to prove the lemma for the system $f_i, s_i, y_i \in Y_i$. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

Assume Y is the spectrum of a Noetherian ring. Since f is of finite presentation, i.e., of finite type, we see that X is a Noetherian scheme too, see Morphisms, Lemma 15.6. Let $x \in X^0$ be a point lying over y. By Topology, Lemma 16.4 it suffices to prove that for any irreducible closed $Z \subset X$ passing through x the intersection $X^0 \cap Z$ is dense in Z. In particular it suffices to prove that the generic point $x' \in Z$ is in X^0. By Properties, Lemma 5.10 we can find a discrete valuation ring R and a morphism $\text{Spec}(R) \to X$ which maps the special point to x and the generic point to x'. We are going to think of $\text{Spec}(R)$ as a scheme over Y via the composition $\text{Spec}(R) \to X \to Y$. By Lemma 29.2 we have that $(X_R)^0$ is the inverse image of X^0. By construction we have a second section $t : \text{Spec}(R) \to X_R$ (besides the base change s_R of s) of the structure morphism $X_R \to \text{Spec}(R)$ such that $t(\eta_R)$ is a point of X_R which maps to x' and $t(0_R)$ is a point of X_R which maps to x. Note that $t(0_R)$ is in $(X_R)^0$ and that $t(\eta_R) \rightsquigarrow t(0_R)$. Thus it suffices to prove that this implies that $t(\eta_R) \in (X_R)^0$. Hence it suffices to prove the lemma in the case where Y is the spectrum of a discrete valuation ring and y its closed point.

Assume Y is the spectrum of a discrete valuation ring and y is its closed point. Our goal is to prove that X^0 is a neighbourhood of X^0_y. Note that X^0_y is open and closed in X_y as X_y has finitely many irreducible components. Hence the complement $C = X_y \setminus X^0_y$ is closed in X. Thus $U = X \setminus C$ is an open neighbourhood of X^0_y and $U^0 = X^0$. Hence it suffices to prove the result for the morphism $U \to Y$. In other words, we may assume that X_y is connected. Suppose that X is disconnected, say $X = X_1 \amalg \ldots \amalg X_n$ is a decomposition into connected components. Then $s(Y)$ is completely contained in one of the X_i. Say $s(Y) \subset X_1$. Then $X^0 \subset X_1$. Hence we may replace X by X_1 and assume that X is connected. At this point Lemma 28.7 implies that X_η is connected, i.e., $X^0 = X$ and we win. □

055R **Lemma 29.6.** *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 29.1. Assume*

1. *f is of finite presentation and flat, and*
2. *all fibres of f are geometrically reduced.*

Then X^0 is open in X.

Proof. This is an immediate consequence of Lemma 29.5. □

30. Dimension of fibres

05F6

05F7 **Lemma 30.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has dimension n, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has dimension n.*

Proof. Let $Z = \{x \in X \mid \dim_x(X_{f(x)}) > n\}$. By Morphisms, Lemma 28.4 this is a closed subset of X. By assumption $Z_\eta = \emptyset$. Hence by Lemma 24.1 we may shrink Y and assume that $Z = \emptyset$. Let $Z' = \{x \in X \mid \dim_x(X_{f(x)}) > n - 1\} = \{x \in X \mid \dim_x(X_{f(x)}) = n\}$. As before this is a closed subset of X. By assumption we have

$Z'_\eta \neq \emptyset$. Hence after shrinking Y we may assume that $Z' \to Y$ is surjective, see Lemma 24.2. Hence we win. □

05F8 **Lemma 30.2.** *Let $f : X \to Y$ be a morphism of finite type. Let*

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the dimension of X_y. If $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. This follows from Morphisms, Lemma 28.3. □

05F9 **Lemma 30.3.** *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 30.2. Assume f of finite presentation. Then the level sets*

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 30.2 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 30.1 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This implies what we want. □

0D4H **Lemma 30.4.** *Let $f : X \to Y$ be a flat morphism of schemes of finite presentation. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 30.2. Then $n_{X/Y}$ is lower semi-continuous.*

Proof. Let $W \subset X$, $W = \coprod_{d \geq 0} U_d$ be the open constructed in Lemmas 22.7 and 22.9. Let $y \in Y$ be a point. If $n_{X/Y}(y) = \dim(X_y) = n$, then y is in the image of $U_n \to Y$. By Morphisms, Lemma 25.10 we see that $f(U_n)$ is open in Y. Hence there is an open neighbourhood of y where $n_{X/Y}$ is $\geq n$. □

0D4I **Lemma 30.5.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 30.2. Then $n_{X/Y}$ is upper semi-continuous.*

Proof. Let $Z_d = \{x \in X \mid \dim_x(X_{f(x)}) > d\}$. Then Z_d is a closed subset of X by Morphisms, Lemma 28.4. Since f is proper $f(Z_d)$ is closed. Since $y \in f(Z_d) \Leftrightarrow n_{X/Y}(y) > d$ we see that the lemma is true. □

0D4J **Lemma 30.6.** *Let $f : X \to Y$ be a proper, flat morphism of schemes of finite presentation. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 30.2. Then $n_{X/Y}$ is locally constant.*

Proof. Immediate consequence of Lemmas 30.4 and 30.5. $\square$

31. Weak relative Noether normalization

0GTD The goal of this section is to prove Lemma 31.3.

0GTE **Lemma 31.1.** *Let R be a ring. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be prime ideals of R with $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ if $i \neq j$. Let $k_i \subset \kappa(\mathfrak{p}_i)$ be subfields such that the extensions $\kappa(\mathfrak{p}_i)/k_i$ are not algebraic. Let $J \subset R$ be an ideal not contained in any of the $\mathfrak{p}_i$. Then there exists an element $x \in J$ such that the image of x in $\kappa(\mathfrak{p}_i)$ is transcendental over k_i for $i = 1, \ldots, r$.*

Proof. The ideal $J_i = J\mathfrak{p}_1 \ldots \hat{\mathfrak{p}}_i \ldots \mathfrak{p}_r$ is not contained in $\mathfrak{p}_i$, see Algebra, Lemma 15.1. It follows that every element ξ of $\kappa(\mathfrak{p}_i) = \text{Frac}(B/\mathfrak{p}_i)$ is of the form $\xi = a/b$ with $a, b \in J_i$ and $b \notin \mathfrak{p}_i$. Choosing ξ transcendental over k_i we see that either a or b maps to an element of $\kappa(\mathfrak{p}_i)$ transcendental over k_i. We conclude that for every $i = 1, \ldots, r$ we can find an element $x_i \in J_i = J\mathfrak{p}_1 \ldots \hat{\mathfrak{p}}_i \ldots \mathfrak{p}_r$ which maps to an element of $\kappa(\mathfrak{p}_i)$ transcendental over k_i. Then $x = x_1 + \ldots + x_r$ works. $\square$

0GTF **Lemma 31.2.** *Let $R \to S$ be a finite type ring map. Let $d \geq 0$. Let $a, b \in S$. Assume that the fibres of*

$$f_a : \text{Spec}(S) \longrightarrow \mathbf{A}^1_R$$

given by the R-algebra map $R[x] \to S$ sending x to a have dimension $\leq d$. Then there exists an n_0 such that for $n \geq n_0$ the fibres of

$$f_{a^n+b} : \text{Spec}(S) \longrightarrow \mathbf{A}^1_R$$

given by the R-algebra map $R[x] \to S$ sending x to $a^n + b$ have dimension $\leq d$.

Proof. In this paragraph we reduce to the case where $R \to S$ is of finite presentation. Namely, write $S = R[A, B, x_1, \ldots, x_n]/J$ for some ideal $J \subset R[x_1, \ldots, x_n]$ where A and B map to a and b in S. Then J is the union of its finitely generated ideals $J_\lambda \subset J$. Set $S_\lambda = R[A, B, x_1, \ldots, x_n]/J_\lambda$ and denote $a_\lambda, b_\lambda \in S_\lambda$ the images of A and B. Then for some λ the fibres of

$$f_{a_\lambda} : \text{Spec}(S_\lambda) \longrightarrow \mathbf{A}^1_R$$

have dimension $\leq d$, see Limits, Lemma 18.1. Fix such a λ. If we can find n_0 which works for $R \to S_\lambda$, a_λ, b_λ, then n_0 works for $R \to S$. Namely, the fibres of $f_{a_\lambda^n+b_\lambda} : \text{Spec}(S_\lambda) \to \mathbf{A}^1_R$ contain the fibres of $f_{a^n+b} : \text{Spec}(S) \to \mathbf{A}^1_R$. This reduces us to the case discussed in the next paragraph.

Assume $R \to S$ is of finite presentation. In this paragraph we reduce to the case where R is of finite type over $\mathbf{Z}$. By Algebra, Lemma 127.18 we can find a directed set Λ and a system of ring maps $R_\lambda \to S_\lambda$ over Λ whose colimit is $R \to S$ such that $S_\mu = S_\lambda \otimes_{R_\lambda} R_\mu$ for $\mu \geq \lambda$ and such that each R_λ and S_λ is of finite type over $\mathbf{Z}$. Choose $\lambda_0 \in \Lambda$ and elements $a_{\lambda_0}, b_{\lambda_0} \in S_{\lambda_0}$ mapping to $a, b \in S$. For $\lambda \geq \lambda_0$ denote $a_\lambda, b_\lambda \in S_\lambda$ the image of $a_{\lambda_0}, b_{\lambda_0}$. Then for $\lambda \geq \lambda_0$ large enough the fibres of

$$f_{a_\lambda} : \text{Spec}(S_\lambda) \longrightarrow \mathbf{A}^1_{R_\lambda}$$

have dimension $\leq d$, see Limits, Lemma 18.4. Fix such a λ. If we can find n_0 which works for $R_\lambda \to S_\lambda$, a_λ, b_λ, then n_0 works for $R \to S$. Namely, any fibre of $f_{a^n+b} : \text{Spec}(S) \to \mathbf{A}^1_R$ has the same dimension as a fibre of $f_{a_\lambda^n+b_\lambda} : \text{Spec}(S_\lambda) \to \mathbf{A}^1_{R_\lambda}$ by Morphisms, Lemma 28.3. This reduces us the the case discussed in the next paragraph.

Assume R and S are of finite type over $\mathbf{Z}$. In particular the dimension of R is finite, and we may use induction on $\dim(R)$. Thus we may assume the result holds for all situations with $R' \to S'$, a, b as in the lemma with R' and S' of finite type over $\mathbf{Z}$ but with $\dim(R') < \dim(R)$.

Since the statement is about the topology of the spectrum of S we may assume S is reduced. Let S^ν be the normalization of S. Then $S \subset S^\nu$ is a finite extension as S is excellent, see Algebra, Proposition 162.16 and Morphisms, Lemma 54.11. Thus $\mathrm{Spec}(S^\nu) \to \mathrm{Spec}(S)$ is surjective and finite (Algebra, Lemma 36.17). It follows that if the result holds for $R \to S^\nu$ and the images of a, b in S^ν, then the result holds for $R \to S$, a, b. (Small detail omitted.) This reduces us to the case discussed in the next paragraph.

Assume R and S are of finite type over $\mathbf{Z}$ and S normal. Then $S = S_1 \times \ldots \times S_r$ for some normal domains S_i. If the result holds for each $R \to S_i$ and the images of a, b in S_i, then the result holds for $R \to S$, a, b. (Small detail omitted.) This reduces us to the case discussed in the next paragraph.

Assume R and S are of finite type over $\mathbf{Z}$ and S a normal domain. We may replace R by the image of R in S (this does not increase the dimension of R). This reduces us to the case discussed in the next paragraph.

Assume $R \subset S$ are of finite type over $\mathbf{Z}$ and S a normal domain. Consider the morphism

$$f_a : \mathrm{Spec}(S) \to \mathbf{A}^1_R$$

The assumption tells us that f_a has fibres of dimension $\leq d$. Hence the fibres of $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ have dimension $\leq d + 1$ (Morphisms, Lemma 28.2). Consider the morphism of integral schemes

$$\phi : \mathrm{Spec}(S) \to \mathbf{A}^2_R = \mathrm{Spec}(R[x, y])$$

corresponding to the R-algebra map $R[x, y] \to S$ sending x to a and y to b. There are two cases to consider

(1) ϕ is dominant, and
(2) ϕ is not dominant.

We claim that in both cases there exists an integer n_0 and a nonempty open $V \subset \mathrm{Spec}(R)$ such that for $n \geq n_0$ the fibres of f_{a^n+b} at points $q \in \mathbf{A}^1_V$ have dimension $\leq d$.

Proof of the claim in case (1). We have $f_{a^n+b} = \pi_n \circ \phi$ where

$$\pi_n : \mathbf{A}^2_R \to \mathbf{A}^1_R$$

is the flat morphism corresponding to the R-algebra map $R[x] \to R[x, y]$ sending x to $x^n + y$. Since ϕ is dominant there is a dense open $U \subset \mathrm{Spec}(S)$ such that $\phi|_U : U \to \mathbf{A}^2_R$ is flat (this follows for example from generic flatness, see Morphisms, Proposition 27.1). Then the composition

$$f_{a^n+b}|_U : U \xrightarrow{\phi|_U} \mathbf{A}^2_R \xrightarrow{\pi_n} \mathbf{A}^1_R$$

is flat as well. Hence the fibres of this morphism have at least codimension 1 in the fibres of $f|_U : U \to \mathrm{Spec}(R)$ by Morphisms, Lemma 28.2. In other words, the fibres of $f_{a^n+b}|_U$ have dimension $\leq d$. On the other hand, since U is dense in $\mathrm{Spec}(S)$, we can find a nonempty open $V \subset \mathrm{Spec}(R)$ such that $U \cap f^{-1}(p) \subset f^{-1}(p)$ is dense for all $p \in V$ (see for example Lemma 24.3). Thus $\dim(f^{-1}(p) \setminus U \cap f^{-1}(p)) \leq d$

and we conclude that our claim is true (as any fibres of $f_{a^n+b} : \text{Spec}(S) \to \mathbf{A}^1_R$ is contained in a fibre of f).

Case (2). In this case we can find a nonzero $g = \sum c_{ij} x^i y^j$ in $R[x, y]$ such that $\text{Im}(\phi) \subset V(g)$. In fact, we may assume g is irreducible over $\text{Frac}(R)$. If $g \in R[x]$, say with leading coefficient c, then over $V = D(c) \subset \text{Spec}(R)$ the fibres of f already have dimension $\leq d$ (because the image of f_a is contained in $V(g) \subset \mathbf{A}^1_R$ which has finite fibres over V). Hence we may assume g is not contained in $R[x]$. Let $s \geq 1$ be the degree of g as a polynomial in y and let t be the degree of $\sum c_{is} x^i$ as a polynomial in x. Then c_{ts} is nonzero and

$$g(x, -x^n) = (-1)^s c_{ts} x^{t+sn} + l.o.t.$$

provided that n is bigger than the degree of g as a polynomial in x (small detail omitted). For such n the polynomial $g(x, -x^n)$ is a nonzero polynomial in x and maps to a nonzero polynomial in $\kappa(\mathfrak{p})[x]$ for all $\mathfrak{p} \subset R$, $c_{st} \notin \mathfrak{p}$. We conclude that our claim is true for V equal to the principal open $D(c_{ts})$ of $\text{Spec}(R)$.

OK, and now we can use induction on $\dim(R)$. Namely, let $I \subset R$ be an ideal such that $V(I) = \text{Spec}(R) \setminus V$. Observe that $\dim(R/I) < \dim(R)$ as R is a domain. Let n'_0 be the integer we have by induction on $\dim(R)$ for $R/I \to S/IS$ and the images of a and b in S/IS. Then $\max(n_0, n'_0)$ works. $\square$

0GTG **Lemma 31.3.** *Let $R \to S$ be a finite type ring map. Let d be the maximum of the dimensions of fibres of* $\text{Spec}(S) \to \text{Spec}(R)$*. Then there exists a quasi-finite ring map $R[t_1, \ldots, t_d] \to S$.*

Proof. In this paragraph we reduce to the case where $R \to S$ is of finite presentation. Namely, write $S = R[x_1, \ldots, x_n]/J$ for some ideal $J \subset R[x_1, \ldots, x_n]$. Then J is the union of its finitely generated ideals $J_\lambda \subset J$. Set $S_\lambda = R[x_1, \ldots, x_n]/J_\lambda$. Then for some λ the fibres of $\text{Spec}(S_\lambda) \to \text{Spec}(R)$ have dimension $\leq d$, see Limits, Lemma 18.1. Fix such a λ. If we can find a quasi-finite $R[t_1, \ldots, t_d] \to S_\lambda$, then of course the composition $R[t_1, \ldots, t_d] \to S$ is quasi-finite. This reduces us to the case discussed in the next paragraph.

Assume $R \to S$ is of finite presentation. In this paragraph we reduce to the case where R is of finite type over $\mathbf{Z}$. By Algebra, Lemma 127.18 we can find a directed set Λ and a system of ring maps $R_\lambda \to S_\lambda$ over Λ whose colimit is $R \to S$ such that $S_\mu = S_\lambda \otimes_{R_\lambda} R_\mu$ for $\mu \geq \lambda$ and such that each R_λ and S_λ is of finite type over $\mathbf{Z}$. Then for λ large enough the fibres of $\text{Spec}(S_\lambda) \to \text{Spec}(R_\lambda)$ have dimension $\leq d$, see Limits, Lemma 18.4. Fix such a λ. If we can find a quasi-finite ring map $R_\lambda[t_1, \ldots, t_d] \to S_\lambda$, then the base change $R[t_1, \ldots, t_d] \to S$ is quasi-finite too (Algebra, Lemma 122.8). This reduces us the the case discussed in the next paragraph.

Assume R and S are of finite type over $\mathbf{Z}$. If $d = 0$, then the ring map is quasi-finite and we are done. Assume $d > 0$. We will find an element $a \in S$ such that the fibres of the R-algebra map $R[x] \to S$, $x \mapsto a$ have dimension $< d$. This will finish the proof by induction.

We will prove the existence of a by induction on $\dim(R)$.

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r \subset S$ be those among the minimal primes of S such that $\dim_{\mathfrak{q}_i}(S/R) = d$. For notation, see Algebra, Definition 125.1. Say $\mathfrak{q}_i$ lies over the prime $\mathfrak{p}_i \subset R$. We have $\text{trdeg}_{\kappa(\mathfrak{p}_i)}(\kappa(\mathfrak{q}_i)) = d$ as $\mathfrak{q}_i$ is a generic point of its fibre; for example apply

Algebra, Lemma 116.3 to $S \otimes_R \kappa(\mathfrak{p}_i)$. Hence by Lemma 31.1 we can find an element $a \in S$ such that the image of a in $\kappa(\mathfrak{q}_i)$ is transcendental over $\kappa(\mathfrak{p}_i)$ for $i = 1, \ldots, r$. Consider the morphism

$$f_a : \mathrm{Spec}(S) \longrightarrow \mathbf{A}^1_R$$

corresponding the R-algebra homomorphism $R[x] \to S$ to mapping x to a. Let $U \subset \mathrm{Spec}(S)$ be the open subset where the fibres have dimension $\leq d-1$, see Morphisms, Lemma 28.4. By construction U contains all the generic points of $\mathrm{Spec}(S)$. In particular we see that U contains all generic points of all the generic fibres of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ as such points are necessarily generic points of $\mathrm{Spec}(S)$. Set $T = \mathrm{Spec}(S) \setminus U$ viewed as a reduced closed subscheme of $\mathrm{Spec}(S)$. It follows from what we just said and the assumption that $\dim(S/R) \leq d$ that the generic fibres of $T \to \mathrm{Spec}(R)$ have dimension $\leq d-1$. Hence by Lemma 30.1, applied several times to produce open neighbourhoods of the generic points of $\mathrm{Spec}(R)$, we can find a dense open $V \subset \mathrm{Spec}(R)$ such that $T_V \to V$ has fibres of dimension $\leq d-1$. We conclude that for $q \in \mathbf{A}^1_V$ the fibre of f_a over q has dimension $\leq d-1$ (as we have bounded the dimension of the fibre of $f_a|_U$ and of the fibre of $f_a|_T$).

By prime avoidance, we may assume that $V = D(f)$ for some $f \in R$. Then we see that the ring map $R_f[x] \to S_f$, $x \mapsto a$ has fibres of dimension $\leq d-1$. We may replace a by fa and assume $a \in (f)$. By induction on $\dim(R)$ we can find an element $\overline{b} \in S/fS$ such that the fibres of $\mathrm{Spec}(S/fS) \to \mathrm{Spec}(R/fR[x])$, $x \mapsto \overline{b}$ have dimension $\leq d-1$. Let $b \in S$ be a lift of $\overline{b}$. By Lemma 31.2 there exists an $n > 0$ such that $a^n + b$ still works for $R_f \to S_f$. On the other hand, the image of $a^n + b$ in S/fS is $\overline{b}$ and the proof is complete. □

32. Bertini theorems

0G4C We continue the discussion started in Varieties, Section 47. In this section we prove that general hyperplane sections of geometrically irreducible varieties are geometrically irreducible following the remarkable argument given in [Jou83].

0G4D **Lemma 32.1.** *Let K/k be a geometrically irreducible and finitely generated field extension. Let $n \geq 1$. Let $g_1, \ldots, g_n \in K$ be elements such that there exist $c_1, \ldots, c_n \in k$ such that the elements* See pages 71 and 72 of [Jou83]

$$x_1, \ldots, x_n, \sum g_i x_i, \sum c_i g_i \in K(x_1, \ldots, x_n)$$

are algebraically independent over k. Then $K(x_1, \ldots, x_n)$ is geometrically irreducible over $k(x_1, \ldots, x_n, \sum g_i x_i)$.

Proof. Let $c_1, \ldots, c_n \in k$ be as in the statement of the lemma. Write $\xi = \sum g_i x_i$ and $\delta = \sum c_i g_i$. For $a \in k$ consider the automorphism σ_a of $K(x_1, \ldots, x_n)$ given by the identity on K and the rules

$$\sigma_a(x_i) = x_i + ac_i$$

Observe that $\sigma_a(\xi) = \xi + a\delta$ and $\sigma_a(\delta) = \delta$. Consider the tower of fields

$$K_0 = k(x_1, \ldots, x_n) \subset K_1 = K_0(\xi) \subset K_2 = K_0(\xi, \delta) \subset K(x_1, \ldots, x_n) = \Omega$$

Observe that $\sigma_a(K_0) = K_0$ and $\sigma_a(K_2) = K_2$. Let $\theta \in \Omega$ be separable algebraic over K_1. We have to show $\theta \in K_1$, see Algebra, Lemma 47.12.

Denote K_2' the separable algebraic closure of K_2 in Ω. Since K_2'/K_2 is finite (Algebra, Lemma 47.13) and separable there are only a finite number of fields in between

K_2' and K_2 (Fields, Lemma 19.1). If k is infinite[5], then we can find distinct elements a_1, a_2 of k such that

$$K_2(\sigma_{a_1}(\theta)) = K_2(\sigma_{a_2}(\theta))$$

as subfields of Ω. Write $\theta_i = \sigma_{a_i}(\theta)$ and $\xi_i = \sigma_{a_i}(\xi) = \xi + a_i\delta$. Observe that

$$K_2 = K_0(\xi_1, \xi_2)$$

as we have $\xi_i = \xi + a_i\delta$, $\xi = (a_2\xi_1 - a_1\xi_2)/(a_2 - a_1)$, and $\delta = (\xi_1 - \xi_2)/(a_1 - a_2)$. Since K_2/K_0 is purely transcendental of degree 2 we conclude that ξ_1 and ξ_2 are algebraically independent over K_0. Since θ_1 is algebraic over $K_0(\xi_1)$ we conclude that ξ_2 is transcendental over $K_0(\xi_1, \theta_1)$.

By assumption K/k is geometrically irreducible. This implies that $K(x_1, \ldots, x_n)/K_0$ is geometrically irreducible (Algebra, Lemma 47.10). This in turn implies that $K_0(\xi_1, \theta_1)/K_0$ is geometrically irreducible as a subextension (Algebra, Lemma 47.6). Since ξ_2 is transcendental over $K_0(\xi_1, \theta_1)$ we conclude that $K_0(\xi_1, \xi_2, \theta_1)/K_0(\xi_2)$ is geometrically irreducible (Algebra, Lemma 47.11). By our choice of a_1, a_2 above we have

$$K_0(\xi_1, \xi_2, \theta_1) = K_2(\sigma_{a_1}(\theta)) = K_2(\sigma_{a_2}(\theta)) = K_0(\xi_1, \xi_2, \theta_2)$$

Since θ_2 is separably algebraic over $K_0(\xi_2)$ we conclude by Algebra, Lemma 47.12 again that $\theta_2 \in K_0(\xi_2)$. Taking $\sigma_{a_2}^{-1}$ of this relation givens $\theta \in K_0(\xi) = K_1$ as desired.

This finishes the proof in case k is infinite. If k is finite, then we can choose a variable t and consider the extension $K(t)/k(t)$ which is geometrically irreducible by Algebra, Lemma 47.10. Since it is still be true that $x_1, \ldots, x_n, \sum g_i x_i, \sum c_i g_i$ in $K(t, x_1, \ldots, x_n)$ are algebraically independent over $k(t)$ we conclude that $K(t, x_1, \ldots, x_n)$ is geometrically irreducible over $k(t, x_1, \ldots, x_n, \sum g_i x_i)$ by the argument already given. Then using Algebra, Lemma 47.10 once more finishes the job. $\square$

0G4E **Lemma 32.2.** *Let A be a domain of finite type over a field k. Let $n \geq 2$. Let $g_1, \ldots, g_n \in A$ be elements such that $V(g_1, g_2)$ has an irreducible component of dimension $\dim(A) - 2$. Then there exist $c_1, \ldots, c_n \in k$ such that the elements*

$$x_1, \ldots, x_n, \sum g_i x_i, \sum c_i g_i \in \mathit{Frac}(A)(x_1, \ldots, x_n)$$

are algebraically independent over k.

Proof. The algebraic independence over k means that the morphism

$$T = \text{Spec}(A[x_1, \ldots, x_n]) \longrightarrow \text{Spec}(k[x_1, \ldots, x_n, y, z]) = S$$

given by $y = \sum g_i x_i$ and $z = \sum c_i g_i$ is dominant. Set $d = \dim(A)$. If $T \to S$ is not dominant, then the image has dimension $< n + 2$ and hence every irreducible component of every fibre has dimension $> d + n - (n + 2) = d - 2$, see Varieties, Lemma 20.4. Choose a closed point $u \in V(g_1, g_2)$ contained in an irreducible component of dimension $d - 2$ and in no other component of $V(g_1, g_2)$. Consider the closed point $t = (u, 1, 0, \ldots 0)$ of T lying over u. Set $(c_1, \ldots, c_n) = (0, 1, 0, \ldots, 0)$. Then t maps to the point $s = (1, 0, \ldots, 0)$ of S. The fibre of $T \to S$ over s is cut out by

$$x_1 - 1, x_2, \ldots, x_n, \sum x_i g_i, g_2$$

[5]We will deal with the finite field case in the last paragraph of the proof.

and hence equivalently is cut out by

$$x_1 - 1, x_2, \ldots, x_n, g_1, g_2$$

By our condition on g_1, g_2 this subscheme has an irreducible component of dimension $d - 2$. $\square$

0G4F **Lemma 32.3.** *In Varieties, Situation 47.2 assume* [Jou83, Theorem 6.3 part 4)]

1. *X is of finite type over k,*
2. *X is geometrically irreducible over k,*
3. *there exist $v_1, v_2, v_3 \in V$ and an irreducible component Z of $H_{v_2} \cap H_{v_3}$ such that $Z \not\subset H_{v_1}$ and $codim(Z, X) = 2$, and*
4. *every irreducible component Y of $\bigcap_{v \in V} H_v$ has $codim(Y, X) \geq 2$.*

Then for general $v \in V \otimes_k k'$ the scheme H_v is geometrically irreducible over k'.

Proof. In order for assumption (3) to hold, the elements v_1, v_2, v_3 must be k-linearly independent in V (small detail omitted). Thus we may choose a basis $v_1, \ldots, v_r$ of V incorporating these elements as the first 3. Recall that $H_{univ} \subset \mathbf{A}^r_k \times_k X$ is the "universal divisor". Consider the projection $q : H_{univ} \to \mathbf{A}^r_k$ whose scheme theoretic fibres are the divisors H_v. By Lemma 27.5 it suffices to show that the generic fibre of q is geometrically irreducible. To prove this we may replace X by its reduction, hence we may assume X is an integral scheme of finite type over k.

Let $U \subset X$ be a nonempty affine open such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Write $U = \text{Spec}(A)$. Denote $f_i \in A$ the element corresponding to section $\psi(v_i)|_U$ via the isomorphism $\mathcal{L}|_U \cong \mathcal{O}_U$. Then $H_{univ} \cap (\mathbf{A}^r_k \times_k U)$ is given by

$$H_U = \text{Spec}(A[x_1, \ldots, x_r]/(x_1 f_1 + \ldots + x_r f_r))$$

By our choice of basis we see that f_1 cannot be zero because this would mean $v_1 = 0$ and hence $H_{v_1} = X$ which contradicts assumption (3). Hence $\sum x_i f_i$ is a nonzerodivisor in $A[x_1, \ldots, x_r]$. It follows that every irreducible component of H_U has dimension $d + r - 1$ where $d = \dim(X) = \dim(A)$. If $U' = U \cap D(f_1)$ then we see that

$$H_{U'} = \text{Spec}(A_{f_1}[x_1, \ldots, x_r]/(x_1 f_1 + \ldots + x_r f_r)) \cong \text{Spec}(A_{f_1}[x_2, \ldots x_r]) = \mathbf{A}^{r-1}_k \times_k U'$$

is irreducible. On the other hand, we have

$$H_U \setminus H_{U'} = \text{Spec}(A/(f_1)[x_1, \ldots, x_r]/(x_2 f_2 + \ldots + x_r f_r))$$

which has dimension at most $d + r - 2$. Namely, for $i \neq 1$ the scheme $(H_U \setminus H_{U'}) \times_U D(f_i)$ is either empty (if $f_i = 0$) or by the same argument as above isomorphic to an $r - 1$ dimensional affine space over an open of $\text{Spec}(A/(f_1))$ and hence has dimension at most $d + r - 2$. On the other hand, $(H_U \setminus H_{U'}) \times_U V(f_2, \ldots, f_r)$ is an r dimensional affine space over $\text{Spec}(A/(f_1, \ldots, f_r))$ and hence assumption (4) tells us this has dimension at most $d + r - 2$. We conclude that H_U is irreducible for every U as above. It follows that H_{univ} is irreducible.

Thus it suffices to show that the generic point of H_{univ} is geometrically irreducible over the generic point of $\mathbf{A}^r_k$, see Varieties, Lemma 8.6. Choose a nonempty affine open $U = \text{Spec}(A)$ of X contained in $X \setminus H_{v_1}$ which meets the irreducible component

Z of $H_{v_2} \cap H_{v_3}$ whose existence is asserted in assumption (3). With notation as above we have to prove that the field extension

$$\text{Frac}(A[x_1, \ldots, x_r]/(x_1 f_1 + \ldots + x_r f_r))/k(x_1, \ldots, x_r)$$

is geometrically irreducible. Observe that f_1 is invertible in A by our choice of U. Set $K = \text{Frac}(A)$ equal to the fraction field of A. Eliminating the variable x_1 as above, we find that we have to show that the field extension

$$K(x_2, \ldots, x_r)/k(x_2, \ldots, x_r, -\sum\nolimits_{i=2,\ldots,r} f_1^{-1} f_i x_i)$$

is geometrically irreducible. By Lemma 32.1 it suffices to show that for some $c_2, \ldots, c_r \in k$ the elements

$$x_2, \ldots, x_r, \sum\nolimits_{i=2,\ldots,r} f_1^{-1} f_i x_i, \sum\nolimits_{i=2,\ldots,r} c_i f_1^{-1} f_i$$

are algebraically independent over k in the fraction field of $A[x_2, \ldots, x_r]$. This follows from Lemma 32.2 and the fact that $Z \cap U$ is an irreducible component of $V(f_1^{-1} f_2, f_1^{-1} f_3) \subset U$. $\square$

0G4G **Remark 32.4.** Let us sketch a "geometric" proof of a special case of Lemma 32.3. Namely, say k is an algebraically closed field and $X \subset \mathbf{P}^n_k$ is smooth and irreducible of dimension ≥ 2. Then we claim there is a hyperplane $H \subset \mathbf{P}^n_k$ such that $X \cap H$ is smooth and irreducible. Namely, by Varieties, Lemma 47.3 for a general $v \in V = kT_0 \oplus \ldots \oplus kT_n$ the corresponding hyperplane section $X \cap H_v$ is smooth. On the other hand, by Enriques-Severi-Zariski the scheme $X \cap H_v$ is connected, see Varieties, Lemma 48.3. Hence $X \cap H_v$ is smooth and irreducible.

33. Theorem of the cube

0BEZ The following lemma tells us that the diagonal of the Picard functor is representable by locally closed immersions under the assumptions made in the lemma.

0BDP **Lemma 33.1.** *Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. For a morphism $g : T \to S$ consider the base change diagram*

$$\begin{tikzcd} X_T \arrow[r, "q"] \arrow[d, "p"] & X \arrow[d, "f"] \\ T \arrow[r, "g"] & S \end{tikzcd}$$

Assume $\mathcal{O}_T \to p_*\mathcal{O}_{X_T}$ is an isomorphism for all $g : T \to S$. Then there exists an immersion $j : Z \to S$ of finite presentation such that a morphism $g : T \to S$ factors through Z if and only if there exists a finite locally free $\mathcal{O}_T$-module $\mathcal{N}$ with $p^*\mathcal{N} \cong q^*\mathcal{E}$.

Proof. Observe that the fibres X_s of f are connected by our assumption that $H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$. Thus the rank of $\mathcal{E}$ is constant on the fibres. Since f is open (Morphisms, Lemma 25.10) and closed we conclude that there is a decomposition $S = \coprod S_r$ of S into open and closed subschemes such that $\mathcal{E}$ has constant rank r on the inverse image of S_r. Thus we may assume $\mathcal{E}$ has constant rank r. We will denote $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ the dual rank r module.

By cohomology and base change (more precisely by Derived Categories of Schemes, Lemma 30.4) we see that $E = Rf_*\mathcal{E}$ is a perfect object of the derived category of

S and that its formation commutes with arbitrary change of base. Similarly for $E' = Rf_*\mathcal{E}^\vee$. Since there is never any cohomology in degrees < 0, we see that E and E' have (locally) tor-amplitude in $[0, b]$ for some b. Observe that for any $g : T \to S$ we have $p_*(q^*\mathcal{E}) = H^0(Lg^*E)$ and $p_*(q^*\mathcal{E}^\vee) = H^0(Lg^*E')$. Let $j : Z \to S$ and $j' : Z' \to S$ be immersions of finite presentation constructed in Derived Categories of Schemes, Lemma 31.4 for E and E' with $a = 0$ and $r = r$; these are roughly speaking characterized by the property that $H^0(Lj^*E)$ and $H^0((j')^*E')$ are finite locally free modules compatible with pullback.

Let $g : T \to S$ be a morphism. If there exists an $\mathcal{N}$ as in the lemma, then, using the projection formula Cohomology, Lemma 54.2, we see that the modules

$$p_*(q^*\mathcal{L}) \cong p_*(p^*\mathcal{N}) \cong \mathcal{N} \otimes_{\mathcal{O}_T} p_*\mathcal{O}_{X_T} \cong \mathcal{N} \quad \text{and similarly} \quad p_*(q^*\mathcal{E}^\vee) \cong \mathcal{N}^\vee$$

are finite locally free modules of rank r and remain so after any further base change $T' \to T$. Hence in this case $T \to S$ factors through j and through j'. Thus we may replace S by $Z \times_S Z'$ and assume that $f_*\mathcal{E}$ and $f_*\mathcal{E}^\vee$ are finite locally free $\mathcal{O}_S$-modules of rank r whose formation commutes with arbitrary change of base (small detail omitted).

In this situation if $g : T \to S$ be a morphism and there exists an $\mathcal{N}$ as in the lemma, then the map (cup product in degree 0)

$$p_*(q^*\mathcal{E}) \otimes_{\mathcal{O}_T} p_*(q^*\mathcal{E}^\vee) \longrightarrow \mathcal{O}_T$$

is a perfect pairing. Conversely, if this cup product map is a perfect pairing, then we see that locally on T we may choose a basis of sections $\sigma_1, \ldots, \sigma_r$ in $p_*(q^*\mathcal{E})$ and $\tau_1, \ldots, \tau_r$ in $p_*(q^*\mathcal{E}^\vee)$ whose products satisfy $\sigma_i\tau_j = \delta_{ij}$. Thinking of σ_i as a section of $q^*\mathcal{E}$ on X_T and τ_j as a section of $q^*\mathcal{E}^\vee$ on X_T, we conclude that

$$\sigma_1, \ldots, \sigma_r : \mathcal{O}_{X_T}^{\oplus r} \longrightarrow q^*\mathcal{E}$$

is an isomorphism with inverse given by

$$\tau_1, \ldots, \tau_r : q^*\mathcal{E} \longrightarrow \mathcal{O}_{X_T}^{\oplus r}$$

In other words, we see that $p^*p_*q^*\mathcal{E} \cong q^*\mathcal{E}$. But the condition that the cupproduct is nondegenerate picks out a retrocompact open subscheme (namely, the locus where a suitable determinant is nonzero) and the proof is complete. $\square$

The lemma above in particular tells us, that if a vector bundle is trivial on fibres for a proper flat family of proper spaces, then it is the pull back of a vector bundle. Let's spell this out a bit.

0EX7 **Lemma 33.2.** *Let $f : X \to S$ be a flat, proper morphism of finite presentation such that $f_*\mathcal{O}_X = \mathcal{O}_S$ and this remains true after arbitrary base change. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Assume*

1. *$\mathcal{E}|_{X_s}$ is isomorphic to $\mathcal{O}_{X_s}^{\oplus r_s}$ for all $s \in S$, and*
2. *S is reduced.*

Then $\mathcal{E} = f^*\mathcal{N}$ for some finite locally free $\mathcal{O}_S$-module $\mathcal{N}$.

Proof. Namely, in this case the locally closed immersion $j : Z \to S$ of Lemma 33.1 is bijective and hence a closed immersion. But since S is reduced, j is an isomorphism. $\square$

0EX8 **Lemma 33.3.** *Let $f : X \to S$ be a proper flat morphism of finite presentation. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume*

(1) *S is the spectrum of a valuation ring,*
(2) *$\mathcal{L}$ is trivial on the generic fibre X_η of f,*
(3) *the closed fibre X_0 of f is integral,*
(4) *$H^0(X_\eta, \mathcal{O}_{X_\eta})$ is equal to the function field of S.*

Then $\mathcal{L}$ is trivial.

Proof. Write $S = \text{Spec}(A)$. We will first prove the lemma when A is a discrete valuation ring (as this is the case most often used in practice). Let $\pi \in A$ be a uniformizer. Take a trivializing section $s \in \Gamma(X_\eta, \mathcal{L}_\eta)$. After replacing s by $\pi^n s$ if necessary we can assume that $s \in \Gamma(X, \mathcal{L})$. If $s|_{X_0} = 0$, then we see that s is divisible by π (because X_0 is the scheme theoretic fibre and X is flat over A). Thus we may assume that $s|_{X_0}$ is nonzero. Then the zero locus $Z(s)$ of s is contained in X_0 but does not contain the generic point of X_0 (because X_0 is integral). This means that the $Z(s)$ has codimension ≥ 2 in X which contradicts Divisors, Lemma 15.3 unless $Z(s) = \emptyset$ as desired.

Proof in the general case. Since the valuation ring A is coherent (Algebra, Example 90.2) we see that $H^0(X, \mathcal{L})$ is a coherent A-module, see Derived Categories of Schemes, Lemma 33.1. Equivalently, $H^0(X, \mathcal{L})$ is a finitely presented A-module (Algebra, Lemma 90.4). Since $H^0(X, \mathcal{L})$ is torsion free (by flatness of X over A), we see from More on Algebra, Lemma 124.3 that $H^0(X, \mathcal{L}) = A^{\oplus n}$ for some n. By flat base change (Cohomology of Schemes, Lemma 5.2) we have

$$K = H^0(X_\eta, \mathcal{O}_{X_\eta}) \cong H^0(X_\eta, \mathcal{L}_\eta) = H^0(X, \mathcal{L}) \otimes_A K$$

where K is the fraction field of A. Thus $n = 1$. Pick a generator $s \in H^0(X, \mathcal{L})$. Let $\mathfrak{m} \subset A$ be the maximal ideal. Then $\kappa = A/\mathfrak{m} = \text{colim}\, A/\pi$ where this is a filtered colimit over nonzero $\pi \in \mathfrak{m}$ (here we use that A is a valuation ring). Thus $X_0 = \lim X \times_S \text{Spec}(A/\pi)$. If $s|_{X_0}$ is zero, then for some π we see that s restricts to zero on $X \times_S \text{Spec}(A/\pi)$, see Limits, Lemma 4.7. But if this happens, then $\pi^{-1}s$ is a global section of $\mathcal{L}$ which contradicts the fact that s is a generator of $H^0(X, \mathcal{L})$. Thus $s|_{X_0}$ is not zero. Let $Z(s) \subset X$ be the zero scheme of s. Since $s|_{X_0}$ is not zero and since X_0 is integral, we see that $Z(s)_0 \subset X_0$ is an effective Cartier divisor. Since f is proper and S is local, every point of $Z(s)$ specializes to a point of $Z(s)_0$. Thus by Divisors, Lemma 18.9 part (3) we see that $Z(s)$ is a relative effective Cartier divisor, in particular $Z(s) \to S$ is flat. Hence if $Z(s)$ were nonemtpy, then $Z(s)_\eta$ would be nonempty which contradicts the fact that $s|_{X_\eta}$ is a trivialization of $\mathcal{L}_\eta$. Thus $Z(s) = \emptyset$ as desired. □

0BF0 **Lemma 33.4.** *Let $f : X \to S$ and $\mathcal{E}$ be as in Lemma 33.1 and in addition assume $\mathcal{E}$ is an invertible $\mathcal{O}_X$-module. If moreover the geometric fibres of f are integral, then Z is closed in S.*

Proof. Since $j : Z \to S$ is of finite presentation, it suffices to show: for any morphism $g : \text{Spec}(A) \to S$ where A is a valuation ring with fraction field K such that $g(\text{Spec}(K)) \in j(Z)$ we have $g(\text{Spec}(A)) \subset j(Z)$. See Morphisms, Lemma 6.5. This follows from Lemma 33.3 and the characterization of $j : Z \to S$ in Lemma 33.1. □

0BF1 **Lemma 33.5.** *Consider a commutative diagram of schemes*

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

with $f' : X' \to S$ and $f : X \to S$ satisfying the hypotheses of Lemma 33.1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and let $\mathcal{L}'$ be the pullback to X'. Let $Z \subset S$, resp. $Z' \subset S$ be the locally closed subscheme constructed in Lemma 33.1 for $(f, \mathcal{L})$, resp. $(f', \mathcal{L}')$ so that $Z \subset Z'$. If $s \in Z$ and

$$H^1(X_s, \mathcal{O}) \longrightarrow H^1(X'_s, \mathcal{O})$$

is injective, then $Z \cap U = Z' \cap U$ for some open neighbourhood U of s.

Proof. We may replace S by Z'. After shrinking S to an affine open neighbourhood of s we may assume that $\mathcal{L}' = \mathcal{O}_{X'}$. Let $E = Rf_*\mathcal{L}$ and $E' = Rf'_*\mathcal{L}' = Rf'_*\mathcal{O}_{X'}$. These are perfect complexes whose formation commutes with arbitrary change of base (Derived Categories of Schemes, Lemma 30.4). In particular we see that

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s) = R\Gamma(X_s, \mathcal{L}_s) = R\Gamma(X_s, \mathcal{O}_{X_s})$$

The second equality because $s \in Z$. Set $h_i = \dim_{\kappa(s)} H^i(X_s, \mathcal{O}_{X_s})$. After shrinking S we can represent E by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h_1} \to \mathcal{O}_S^{\oplus h_2} \to \ldots$$

see More on Algebra, Lemma 75.7 (strictly speaking this also uses Derived Categories of Schemes, Lemmas 3.5 and 10.7). Similarly, we may assume E' is represented by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h'_1} \to \mathcal{O}_S^{\oplus h'_2} \to \ldots$$

where $h'_i = \dim_{\kappa(s)} H^i(X'_s, \mathcal{O}_{X'_s})$. By functoriality of cohomology we have a map

$$E \longrightarrow E'$$

in $D(\mathcal{O}_S)$ whose formation commutes with change of base. Since the complex representing E is a finite complex of finite free modules and since S is affine, we can choose a map of complexes

$$\begin{tikzcd} \mathcal{O}_S \arrow[r, "d"'] \arrow[d, "a"'] & \mathcal{O}_S^{\oplus h_1} \arrow[r] \arrow[d, "b"'] & \mathcal{O}_S^{\oplus h_2} \arrow[r] \arrow[d, "c"'] & \ldots \\ \mathcal{O}_S \arrow[r, "d'"] & \mathcal{O}_S^{\oplus h'_1} \arrow[r] & \mathcal{O}_S^{\oplus h'_2} \arrow[r] & \ldots \end{tikzcd}$$

representing the given map $E \to E'$. Since $s \in Z$ we see that the trivializing section of $\mathcal{L}_s$ pulls back to a trivializing section of $\mathcal{L}'_s = \mathcal{O}_{X'_s}$. Thus $a \otimes \kappa(s)$ is an isomorphism, hence after shrinking S we see that a is an isomorphism. Finally, we use the hypothesis that $H^1(X_s, \mathcal{O}) \to H^1(X'_s, \mathcal{O})$ is injective, to see that there exists a $h_1 \times h_1$ minor of the matrix defining b which maps to a nonzero element in $\kappa(s)$. Hence after shrinking S we may assume that b is injective. However, since $\mathcal{L}' = \mathcal{O}_{X'}$ we see that $d' = 0$. It follows that $d = 0$. In this way we see that the trivializing section of $\mathcal{L}_s$ lifts to a section of $\mathcal{L}$ over X. A straightforward topological argument (omitted) shows that this means that $\mathcal{L}$ is trivial after possibly shrinking S a bit further. $\square$

0BF2 **Lemma 33.6.** *Consider n commutative diagrams of schemes*

$$\begin{tikzcd} X_i \arrow[rr] \arrow[dr, "f_i"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

with $f_i : X_i \to S$ and $f : X \to S$ satisfying the hypotheses of Lemma 33.1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and let $\mathcal{L}_i$ be the pullback to X_i. Let $Z \subset S$, resp. $Z_i \subset S$ be the locally closed subscheme constructed in Lemma 33.1 for $(f, \mathcal{L})$, resp. $(f_i, \mathcal{L}_i)$ so that $Z \subset \bigcap_{i=1,\ldots,n} Z_i$. If $s \in Z$ and

$$H^1(X_s, \mathcal{O}) \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} H^1(X_{i,s}, \mathcal{O})$$

is injective, then $Z \cap U = (\bigcap_{i=1,\ldots,n} Z_i) \cap U$ (scheme theoretic intersection) for some open neighbourhood U of s.

Proof. This lemma is a variant of Lemma 33.5 and we strongly urge the reader to read that proof first; this proof is basically a copy of that proof with minor modifications. It follows from the description of (scheme valued) points of Z and the Z_i that $Z \subset \bigcap_{i=1,\ldots,n} Z_i$ where we take the scheme theoretic intersection. Thus we may replace S by the scheme theoretic intersection $\bigcap_{i=1,\ldots,n} Z_i$. After shrinking S to an affine open neighbourhood of s we may assume that $\mathcal{L}_i = \mathcal{O}_{X_i}$ for $i = 1, \ldots, n$. Let $E = Rf_*\mathcal{L}$ and $E_i = Rf_{i,*}\mathcal{L}_i = Rf_{i,*}\mathcal{O}_{X_i}$. These are perfect complexes whose formation commutes with arbitrary change of base (Derived Categories of Schemes, Lemma 30.4). In particular we see that

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s) = R\Gamma(X_s, \mathcal{L}_s) = R\Gamma(X_s, \mathcal{O}_{X_s})$$

The second equality because $s \in Z$. Set $h_j = \dim_{\kappa(s)} H^j(X_s, \mathcal{O}_{X_s})$. After shrinking S we can represent E by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h_1} \to \mathcal{O}_S^{\oplus h_2} \to \ldots$$

see More on Algebra, Lemma 75.7 (strictly speaking this also uses Derived Categories of Schemes, Lemmas 3.5 and 10.7). Similarly, we may assume E_i is represented by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h_{i,1}} \to \mathcal{O}_S^{\oplus h_{i,2}} \to \ldots$$

where $h_{i,j} = \dim_{\kappa(s)} H^j(X_{i,s}, \mathcal{O}_{X_{i,s}})$. By functoriality of cohomology we have a map

$$E \longrightarrow E_i$$

in $D(\mathcal{O}_S)$ whose formation commutes with change of base. Since the complex representing E is a finite complex of finite free modules and since S is affine, we can choose a map of complexes

$$\begin{tikzcd} \mathcal{O}_S \arrow[r, "d"'] \arrow[d, "a_i"'] & \mathcal{O}_S^{\oplus h_1} \arrow[r] \arrow[d, "b_i"'] & \mathcal{O}_S^{\oplus h_2} \arrow[r] \arrow[d, "c_i"'] & \ldots \\ \mathcal{O}_S \arrow[r, "d_i"] & \mathcal{O}_S^{\oplus h_{i,1}} \arrow[r] & \mathcal{O}_S^{\oplus h_{i,2}} \arrow[r] & \ldots \end{tikzcd}$$

representing the given map $E \to E_i$. Since $s \in Z$ we see that the trivializing section of $\mathcal{L}_s$ pulls back to a trivializing section of $\mathcal{L}_{i,s} = \mathcal{O}_{X_{i,s}}$. Thus $a_i \otimes \kappa(s)$ is an isomorphism, hence after shrinking S we see that a_i is an isomorphism. Finally, we use the hypothesis that $H^1(X_s, \mathcal{O}) \to \bigoplus_{i=1,\ldots,n} H^1(X_{i,s}, \mathcal{O})$ is injective, to see that there exists a $h_1 \times h_1$ minor of the matrix defining $\oplus b_i$ which maps to a nonzero element in $\kappa(s)$. Hence after shrinking S we may assume that $(b_1, \ldots, b_n) : \mathcal{O}_S^{h_1} \to \bigoplus_{i=1,\ldots,n} \mathcal{O}_S^{h_{i,1}}$ is injective. However, since $\mathcal{L}_i = \mathcal{O}_{X_i}$ we see that $d_i = 0$ for $i = 1, \ldots n$. It follows that $d = 0$ because $(b_1, \ldots, b_n) \circ d = (\oplus d_i) \circ (a_1, \ldots, a_n)$. In this way we see that the trivializing section of $\mathcal{L}_s$ lifts to a section of $\mathcal{L}$ over X. A straightforward topological argument (omitted) shows that this means that $\mathcal{L}$ is trivial after possibly shrinking S a bit further. $\square$

0BF3 **Lemma 33.7.** *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes satisfying the hypotheses of Lemma 33.1. Let $\sigma : S \to X$ and $\tau : S \to Y$ be sections of f and g. Let $s \in S$. Let $\mathcal{L}$ be an invertible sheaf on $X \times_S Y$. If $(1 \times \tau)^*\mathcal{L}$ on X, $(\sigma \times 1)^*\mathcal{L}$ on Y, and $\mathcal{L}|_{(X \times_S Y)_s}$ are trivial, then there is an open neighbourhood U of s such that $\mathcal{L}$ is trivial over $(X \times_S Y)_U$.*

Proof. By Künneth (Varieties, Lemma 29.1) the map

$$H^1(X_s \times_{\mathrm{Spec}(\kappa(s))} Y_s, \mathcal{O}) \to H^1(X_s, \mathcal{O}) \oplus H^1(Y_s, \mathcal{O})$$

is injective. Thus we may apply Lemma 33.6 to the two morphisms

$$1 \times \tau : X \to X \times_S Y \quad \text{and} \quad \sigma \times 1 : Y \to X \times_S Y$$

to conclude. $\square$

0BF4 **Theorem 33.8** (Theorem of the cube)**.** *Let S be a scheme. Let X, Y, and Z be schemes over S. Let $x : S \to X$ and $y : S \to Y$ be sections of the structure morphisms. Let $\mathcal{L}$ be an invertible module on $X \times_S Y \times_S Z$. If*

1. *$X \to S$ and $Y \to S$ are flat, proper morphisms of finite presentation with geometrically integral fibres,*
2. *the pullbacks of $\mathcal{L}$ by $x \times id_Y \times id_Z$ and $id_X \times y \times id_Z$ are trivial over $Y \times_S Z$ and $X \times_S Z$,*
3. *there is a point $z \in Z$ such that $\mathcal{L}$ restricted to $X \times_S Y \times_S z$ is trivial, and*
4. *Z is connected,*

then $\mathcal{L}$ is trivial.

An often used special case is the following. Let k be a field. Let X, Y, Z be varieties with k-rational points x, y, z. Let $\mathcal{L}$ be an invertible module on $X \times Y \times Z$. If

1. $\mathcal{L}$ is trivial over $x \times Y \times Z$, $X \times y \times Z$, and $X \times Y \times z$, and
2. X and Y are geometrically integral and proper over k,

then $\mathcal{L}$ is trivial.

Proof. Observe that the morphism $X \times_S Y \to S$ is a flat, proper morphism of finite presentation whose geometrically integral fibres (see Varieties, Lemmas 9.2, 8.4, and 6.7 for the statement about the fibres). By Derived Categories of Schemes, Lemma 32.6 we see that the pushforward of the structure sheaf by $X \to S$, $Y \to S$, or $X \times_S Y \to S$ is the structure sheaf of S and the same remains true after any base change. Thus we may apply Lemma 33.1 to the morphism

$$p : X \times_S Y \times_S Z \longrightarrow Z$$

and the invertible module $\mathcal{L}$ to get a "universal" locally closed subscheme $Z' \subset Z$ such that $\mathcal{L}|_{X \times_S Y \times_S Z'}$ is the pullback of an invertible module $\mathcal{N}$ on Z'. The existence of z shows that Z' is nonempty. By Lemma 33.4 we see that $Z' \subset Z$ is a closed subscheme. Let $z' \in Z'$ be a point. Observe that we may write p as the product morphism

$$(X \times_S Z) \times_Z (Y \times_S Z) \longrightarrow Z$$

Hence we may apply Lemma 33.7 to the morphism p, the point z', and the sections $\sigma : Z \to X \times_S Z$ and $\tau : Z \to Y \times_S Z$ given by x and y. We conclude that Z' is open. Hence $Z' = Z$ and $\mathcal{L} = p^*\mathcal{N}$ for some invertible module $\mathcal{N}$ on Z. Pulling back via $x \times y \times \text{id}_Z : Z \to X \times_S Y \times_S Z$ we obtain on the one hand $\mathcal{N}$ and on the other hand we obtain the trivial invertible module by assumption (2). Thus $\mathcal{N} = \mathcal{O}_Z$ and the proof is complete. □

34. Limit arguments

05FA Some lemmas involving limits of schemes, and Noetherian approximation. We stick mostly to the affine case. Some of these lemmas are special cases of lemmas in the chapter on limits.

05FB **Lemma 34.1.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Then there exists a cartesian diagram*

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"'] \arrow[d, "f"] \\ S_0 & S \arrow[l] \end{tikzcd}$$

such that

1. *X_0, S_0 are affine schemes,*
2. *S_0 of finite type over $\mathbf{Z}$,*
3. *f_0 is of finite type.*

Proof. Write $S = \text{Spec}(A)$ and $X = \text{Spec}(B)$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 21.2. Thus the lemma follows from Algebra, Lemma 127.18. □

05FC **Lemma 34.2.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then there exists a diagram as in Lemma 34.1 such that there exists a coherent $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ with $g^*\mathcal{F}_0 = \mathcal{F}$.*

Proof. Write $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, and $\mathcal{F} = \widetilde{M}$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 21.2. As $\mathcal{F}$ is of finite presentation over $\mathcal{O}_X$ we see that M is of finite presentation as a B-module, see Properties, Lemma 16.2. Thus the lemma follows from Algebra, Lemma 127.18. □

05FD **Lemma 34.3.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation and flat over S. Then we may choose a diagram as in Lemma 34.2 and sheaf $\mathcal{F}_0$ such that in addition $\mathcal{F}_0$ is flat over S_0.*

Proof. Write $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, and $\mathcal{F} = \widetilde{M}$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 21.2. As $\mathcal{F}$ is of finite presentation over $\mathcal{O}_X$ we see that M is of finite presentation as a B-module, see Properties, Lemma 16.2. As $\mathcal{F}$ is flat over S we see that M is flat over A, see Morphisms, Lemma 25.2. Thus the lemma follows from Algebra, Lemma 168.1. $\square$

05FE **Lemma 34.4.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation and flat. Then there exists a diagram as in Lemma 34.1 such that in addition f_0 is flat.*

Proof. This is a special case of Lemma 34.3. $\square$

05FF **Lemma 34.5.** *Let $f : X \to S$ be a morphism of affine schemes, which is smooth. Then there exists a diagram as in Lemma 34.1 such that in addition f_0 is smooth.*

Proof. Write $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and as f is smooth we see that B is smooth as an A-algebra, see Morphisms, Lemma 34.2. Hence the lemma follows from Algebra, Lemma 138.14. $\square$

05FG **Lemma 34.6.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically reduced fibres. Then there exists a diagram as in Lemma 34.1 such that in addition f_0 has geometrically reduced fibres.*

Proof. Apply Lemma 34.1 to get a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"] \arrow[d, "f"] \\ S_0 & S \arrow[l, "h"'] \end{tikzcd}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 26.5 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically reduced is a constructible subset. By Lemma 26.2 we have $h(S) \subset E$. Write $S_0 = \mathrm{Spec}(A_0)$ and $S = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 4.10 we see that $\mathrm{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\mathrm{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \mathrm{Spec}(A_i)$ we see that all fibres of f_0 are geometrically reduced. $\square$

05FH **Lemma 34.7.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically irreducible fibres. Then there exists a diagram as in Lemma 34.1 such that in addition f_0 has geometrically irreducible fibres.*

Proof. Apply Lemma 34.1 to get a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"] \arrow[d, "f"] \\ S_0 & S \arrow[l, "h"'] \end{tikzcd}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 27.7 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically irreducible is a constructible subset. By Lemma 27.2 we have $h(S) \subset E$. Write $S_0 = \mathrm{Spec}(A_0)$ and $S = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 4.10 we see that $\mathrm{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\mathrm{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \mathrm{Spec}(A_i)$ we see that all fibres of f_0 are geometrically irreducible. $\square$

05FI **Lemma 34.8.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically connected fibres. Then there exists a diagram as in Lemma 34.1 such that in addition f_0 has geometrically connected fibres.*

Proof. Apply Lemma 34.1 to get a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"] \arrow[d, "f"] \\ S_0 & S \arrow[l, "h"] \end{tikzcd}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 28.6 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically connected is a constructible subset. By Lemma 28.2 we have $h(S) \subset E$. Write $S_0 = \text{Spec}(A_0)$ and $S = \text{Spec}(A)$. Write $A = \text{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 4.10 we see that $\text{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\text{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \text{Spec}(A_i)$ we see that all fibres of f_0 are geometrically connected. $\square$

05FJ **Lemma 34.9.** *Let $d \geq 0$ be an integer. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation all of whose fibres have dimension d. Then there exists a diagram as in Lemma 34.1 such that in addition all fibres of f_0 have dimension d.*

Proof. Apply Lemma 34.1 to get a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"] \arrow[d, "f"] \\ S_0 & S \arrow[l, "h"] \end{tikzcd}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 30.3 the set $E \subset S_0$ of points where the fibre of f_0 has dimension d is a constructible subset. By Lemma 30.2 we have $h(S) \subset E$. Write $S_0 = \text{Spec}(A_0)$ and $S = \text{Spec}(A)$. Write $A = \text{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 4.10 we see that $\text{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\text{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \text{Spec}(A_i)$ we see that all fibres of f_0 have dimension d. $\square$

05FK **Lemma 34.10.** *Let $f : X \to S$ be a morphism of affine schemes, which is standard syntomic (see Morphisms, Definition 30.1). Then there exists a diagram as in Lemma 34.1 such that in addition f_0 is standard syntomic.*

Proof. This lemma is a copy of Algebra, Lemma 136.11. $\square$

05FL **Lemma 34.11.** *(Noetherian approximation and combining properties.) Let P, Q be properties of morphisms of schemes which are stable under base change. Let $f : X \to S$ be a morphism of finite presentation of affine schemes. Assume we can find cartesian diagrams*

$$\begin{tikzcd} X_1 \arrow[d, "f_1"'] & X \arrow[l] \arrow[d, "f"] \\ S_1 & S \arrow[l] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_2 \arrow[d, "f_2"'] & X \arrow[l] \arrow[d, "f"] \\ S_2 & S \arrow[l] \end{tikzcd}$$

of affine schemes, with S_1, S_2 of finite type over $\mathbf{Z}$ and f_1, f_2 of finite type such that f_1 has property P and f_2 has property Q. Then we can find a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l] \arrow[d, "f"] \\ S_0 & S \arrow[l] \end{tikzcd}$$

of affine schemes with S_0 of finite type over $\mathbf{Z}$ and f_0 of finite type such that f_0 has both property P and property Q.

Proof. The given pair of diagrams correspond to cocartesian diagrams of rings

$$\begin{tikzcd} B_1 \arrow[r] & B \\ A_1 \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_2 \arrow[r] & B \\ A_2 \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

Let $A_0 \subset A$ be a finite type $\mathbf{Z}$-subalgebra of A containing the image of both $A_1 \to A$ and $A_2 \to A$. Such a subalgebra exists because by assumption both A_1 and A_2 are of finite type over $\mathbf{Z}$. Note that the rings $B_{0,1} = B_1 \otimes_{A_1} A_0$ and $B_{0,2} = B_2 \otimes_{A_2} A_0$ are finite type A_0-algebras with the property that $B_{0,1} \otimes_{A_0} A \cong B \cong B_{0,2} \otimes_{A_0} A$ as A-algebras. As A is the directed colimit of its finite type A_0-subalgebras, by Limits, Lemma 10.1 we may assume after enlarging A_0 that there exists an isomorphism $B_{0,1} \cong B_{0,2}$ as A_0-algebras. Since properties P and Q are assumed stable under base change we conclude that setting $S_0 = \mathrm{Spec}(A_0)$ and

$$X_0 = X_1 \times_{S_1} S_0 = \mathrm{Spec}(B_{0,1}) \cong \mathrm{Spec}(B_{0,2}) = X_2 \times_{S_2} S_0$$

works. $\square$

35. Étale neighbourhoods

02LD It turns out that some properties of morphisms are easier to study after doing an étale base change. It is convenient to introduce the following terminology.

02LE **Definition 35.1.** Let S be a scheme. Let $s \in S$ be a point.

(1) An *étale neighbourhood of* (S, s) is a pair (U, u) together with an étale morphism of schemes $\varphi : U \to S$ such that $\varphi(u) = s$.
(2) A *morphism of étale neighbourhoods* $f : (V, v) \to (U, u)$ of (S, s) is simply a morphism of S-schemes $f : V \to U$ such that $f(v) = u$.
(3) An *elementary étale neighbourhood* is an étale neighbourhood $\varphi : (U, u) \to (S, s)$ such that $\kappa(s) = \kappa(u)$.

The notion of an elementary étale neighbourhood has many different names in the literature, for example these are sometimes called "étale neighbourhoods" ([Mil80, Page 36] or "strongly étale" ([KPR75, Page 108]). Here we follow the convention of the paper [GR71] by calling them elementary étale neighbourhoods.

If $f : (V, v) \to (U, u)$ is a morphism of étale neighbourhoods, then f is automatically étale, see Morphisms, Lemma 36.18. Hence it turns (V, v) into an étale neighbourhood of (U, u). Of course, since the composition of étale morphisms is étale (Morphisms, Lemma 36.3) we see that conversely any étale neighbourhood (V, v) of (U, u) is an étale neighbourhood of (S, s) as well. We also remark that if $U \subset S$ is an open neighbourhood of s, then $(U, s) \to (S, s)$ is an étale neighbourhood. This follows from the fact that an open immersion is étale (Morphisms, Lemma 36.9). We will use these remarks without further mention throughout this section.

Note that $\kappa(u)/\kappa(s)$ is a finite separable extension if $(U, u) \to (S, s)$ is an étale neighbourhood, see Morphisms, Lemma 36.15.

02LF **Lemma 35.2.** *Let S be a scheme. Let $s \in S$. Let $k/\kappa(s)$ be a finite separable field extension. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ such that the field extension $\kappa(u)/\kappa(s)$ is isomorphic to $k/\kappa(s)$.*

Proof. We may assume S is affine. In this case the lemma follows from Algebra, Lemma 144.3. □

057A **Lemma 35.3.** *Let S be a scheme, and let s be a point of S. The category of étale neighborhoods has the following properties:*

1. *Let $(U_i, u_i)_{i=1,2}$ be two étale neighborhoods of s in S. Then there exists a third étale neighborhood (U, u) and morphisms $(U, u) \to (U_i, u_i)$, $i = 1, 2$.*
2. *Let $h_1, h_2 : (U, u) \to (U', u')$ be two morphisms between étale neighborhoods of s. Assume h_1, h_2 induce the same map $\kappa(u') \to \kappa(u)$ of residue fields. Then there exist an étale neighborhood (U'', u'') and a morphism $h : (U'', u'') \to (U, u)$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.*

Proof. For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change, see Morphisms, Lemma 36.4. There is a point of U mapping to both u_1 and u_2 for example by the description of points of a fibre product in Schemes, Lemma 17.5. For part (2), define U'' as the fibre product

$$\begin{tikzcd} U'' \arrow[r] \arrow[d] & U \arrow[d, "{(h_1,h_2)}"] \\ U' \arrow[r, "\Delta"] & U' \times_S U'. \end{tikzcd}$$

Since h_1 and h_2 induce the same map of residue fields $\kappa(u') \to \kappa(u)$ there exists a point $u'' \in U''$ lying over u' with $\kappa(u'') = \kappa(u')$. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over S, so is the fibre product $U' \times_S U'$ (see Morphisms, Lemmas 36.4 and 36.3). Hence the vertical arrow (h_1, h_2) is étale by Morphisms, Lemma 36.18. Therefore U'' is étale over U' by base change, and hence also étale over S (because compositions of étale morphisms are étale). Thus (U'', u'') is a solution to the problem. □

057B **Lemma 35.4.** *Let S be a scheme, and let s be a point of S. The category of elementary étale neighborhoods of (S, s) is cofiltered (see Categories, Definition 20.1).*

Proof. This is immediate from the definitions and Lemma 35.3. □

05KS **Lemma 35.5.** *Let S be a scheme. Let $s \in S$. Then we have*

$$\mathcal{O}^h_{S,s} = \mathrm{colim}_{(U,u)} \mathcal{O}(U)$$

where the colimit is over the filtered category which is opposite to the category of elementary étale neighbourhoods (U, u) of (S, s).

Proof. Let $\text{Spec}(A) \subset S$ be an affine neighbourhood of s. Let $\mathfrak p \subset A$ be the prime ideal corresponding to s. With these choices we have canonical isomorphisms $\mathcal{O}_{S,s} = A_\mathfrak{p}$ and $\kappa(s) = \kappa(\mathfrak p)$. A cofinal system of elementary étale neighbourhoods is given by those elementary étale neighbourhoods (U, u) such that U is affine and $U \to S$ factors through $\text{Spec}(A)$. In other words, we see that the right hand side is equal to $\text{colim}_{(B,\mathfrak q)} B$ where the colimit is over étale A-algebras B endowed with a prime $\mathfrak q$ lying over $\mathfrak p$ with $\kappa(\mathfrak p) = \kappa(\mathfrak q)$. Thus the lemma follows from Algebra, Lemma 155.7. $\square$

We can lift étale neighbourhoods of points on fibres to the total space.

0CAS **Lemma 35.6.** *Let $X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $(V, v) \to (X_s, x)$ be an étale neighbourhood. Then there exists an étale neighbourhood $(U, u) \to (X, x)$ such that there exists a morphism $(U_s, u) \to (V, v)$ of étale neighbourhoods of (X_s, x) which is an open immersion.*

Proof. We may assume X, V, and S affine. Say the morphism $X \to S$ is given by $A \to B$ the point x by a prime $\mathfrak q \subset B$, the point s by $\mathfrak p = A \cap \mathfrak q$, and the morphism $V \to X_s$ by $B \otimes_A \kappa(\mathfrak p) \to C$. Since $\kappa(\mathfrak p)$ is a localization of $A/\mathfrak p$ there exists an $f \in A$, $f \notin \mathfrak p$ and an étale ring map $B \otimes_A (A/\mathfrak p)_f \to D$ such that

$$C = (B \otimes_A \kappa(\mathfrak p)) \otimes_{B \otimes_A (A/\mathfrak p)_f} D$$

See Algebra, Lemma 143.3 part (9). After replacing A by A_f and B by B_f we may assume D is étale over $B \otimes_A A/\mathfrak p = B/\mathfrak p B$. Then we can apply Algebra, Lemma 143.10. This proves the lemma. $\square$

36. Étale neighbourhoods and branches

0CB2 The number of (geometric) branches of a scheme at a point was defined in Properties, Section 15. In Varieties, Section 40 we related this to fibres of the normalization morphism. In this section we discuss a characterization of this number in terms of étale neighbourhoods.

0CB3 **Lemma 36.1.** *Let $R = \text{colim}\, R_i$ be colimit of a directed system of rings whose transition maps are faithfully flat. Then the number of minimal primes of R taken as an element of $\{0, 1, 2, \ldots, \infty\}$ is the supremum of the numbers of minimal primes of the R_i.*

Proof. If $A \to B$ is a flat ring map, then $\text{Spec}(B) \to \text{Spec}(A)$ maps minimal primes to minimal primes by going down (Algebra, Lemma 39.19). If $A \to B$ is faithfully flat, then every minimal prime is the image of a minimal prime (by Algebra, Lemma 39.16 and 30.7). Hence the number of minimal primes of R_i is $\geq$ the number of minimal primes of $R_{i'}$ if $i \leq i'$. By Algebra, Lemma 39.20 each of the maps $R_i \to R$ is faithfully flat and we also see that the number of minimal primes of R is $\geq$ the number of minimal primes of R_i. Finally, suppose that $\mathfrak q_1, \ldots, \mathfrak q_n$ are pairwise distinct minimal primes of R. Then we can find an i such that $R_i \cap \mathfrak q_1, \ldots, R_i \cap \mathfrak q_n$ are pairwise distinct (as sets and hence as prime ideals). This implies the lemma. $\square$

0CB4 **Lemma 36.2.** *Let X be a scheme and $x \in X$ a point. Then*

(1) *the number of branches of X at x is equal to the supremum of the number of irreducible components of U passing through u taken over elementary étale neighbourhoods $(U, u) \to (X, x)$,*

(2) *the number of geometric branches of X at x is equal to the supremum of the number of irreducible components of U passing through u taken over étale neighbourhoods $(U, u) \to (X, x)$,*
(3) *X is unibranch at x if and only if for every elementary étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of U passing through u, and*
(4) *X is geometrically unibranch at x if and only if for every étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of U passing through u.*

Proof. Parts (3) and (4) follow from parts (1) and (2) via Properties, Lemma 15.6.

Proof of (1). Let $\mathrm{Spec}(A)$ be an affine open neighbourhood of x and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. We may replace X by $\mathrm{Spec}(A)$ and it suffices to consider affine elementary étale neighbourhoods (U, u) in the supremum as they form a cofinal subsystem. Recall that the henselization $A_\mathfrak{p}^h$ is the colimit of the rings $B_\mathfrak{q}$ over the category of pairs $(B, \mathfrak{q})$ where B is an étale A-algebra and $\mathfrak{q}$ is a prime lying over $\mathfrak{p}$ with $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$, see Algebra, Lemma 155.7. These pairs $(B, \mathfrak{q})$ correspond exactly to the affine elementary étale neighbourhoods (U, u) by the correspondence between rings and affine schemes. Observe that irreducible components of $\mathrm{Spec}(B)$ passing through $\mathfrak{q}$ are exactly the minimal prime ideals of $B_\mathfrak{q}$. The number of minimal primes of $A_\mathfrak{p}^h$ is the number of branches of X at x by Properties, Definition 15.4. Observe that the transition maps $B_\mathfrak{q} \to B'_{\mathfrak{q}'}$ in the system are all flat. Since a flat local ring map is faithfully flat (Algebra, Lemma 39.17) we see that the lemma follows from Lemma 36.1.

Proof of (2). The proof is the same as the proof of (1), except that we use Algebra, Lemma 155.11. There is a tiny difference: given a separable algebraic closure κ^{sep} of $\kappa(x)$ for every étale neighbourhood (U, u) we can choose a $\kappa(x)$-embedding $\phi : \kappa(u) \to \kappa^{sep}$ because $\kappa(u)/\kappa(x)$ is finite separable (Morphisms, Lemma 36.15). Hence we can look at the supremum over all triples (U, u, ϕ) where $(U, u) \to (X, x)$ is an affine étale neighbourhood and $\phi : \kappa(u) \to \kappa^{sep}$ is a $\kappa(x)$-embedding. These triples correspond exactly to the triples in Algebra, Lemma 155.11 and the rest of the proof is exactly the same. $\square$

We will need a relative variant of the lemma above.

0CB5 **Lemma 36.3.** *Let $X \to S$ be a morphism of schemes and $x \in X$ a point with image s. Then*

(1) *the number of branches of the fibre X_s at x is equal to the supremum of the number of irreducible components of the fibre U_s passing through u taken over elementary étale neighbourhoods $(U, u) \to (X, x)$,*
(2) *the number of geometric branches of the fibre X_s at x is equal to the supremum of the number of irreducible components of the fibre U_s passing through u taken over étale neighbourhoods $(U, u) \to (X, x)$,*
(3) *the fibre X_s is unibranch at x if and only if for every elementary étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of the fibre U_s passing through u, and*
(4) *X is geometrically unibranch at x if and only if for every étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of U_s passing through u.*

Proof. Combine Lemmas 36.2 and 35.6. ☐

0DQ2 **Lemma 36.4.** *Let $X \to S$ be a smooth morphism of schemes. Let $x \in X$ with image $s \in S$. Then*

1. *The number of geometric branches of X at x is equal to the number of geometric branches of S at s.*
2. *If $\kappa(x)/\kappa(s)$ is a purely inseparable[6] extension of fields, then number of branches of X at x is equal to the number of branches of S at s.*

Proof. Follows immediately from More on Algebra, Lemma 106.8 and the definitions. ☐

37. Unramified and étale morphisms

0GS7 Sometimes unramified morphisms are automatically étale.

0GS8 **Lemma 37.1.** *Let $f : X \to Y$ be a morphism of schemes. Let $x \in X$ with image $y \in Y$. Assume*

1. *Y is integral and geometrically unibranch at y,*
2. *f is locally of finite type,*
3. *there is a specialization $x' \rightsquigarrow x$ such that $f(x')$ is the generic point of Y,*
4. *f is unramified at x.*

Then f is étale at x.

Proof. We may replace X and Y by suitable affine open neighbourhoods of x and y. Then Y is the spectrum of a domain A and X is the spectrum of a finite type A-algebra B. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x and $\mathfrak{p} \subset A$ the prime ideal corresponding to y. The local ring $A_\mathfrak{p} = \mathcal{O}_{Y,y}$ is geometrically unibranch. The ring map $A \to B$ is unramified at $\mathfrak{q}$. Also, the point x' in (3) corresponds to a prime ideal $\mathfrak{q}' \subset \mathfrak{q}$ such that $A \cap \mathfrak{q}' = (0)$. It follows that $A_\mathfrak{p} \to B_\mathfrak{q}$ is injective. We conclude by More on Algebra, Lemma 107.2. ☐

0GS9 **Lemma 37.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume* [Gro71, Expose I, Corollary 9.11]

1. *Y is integral and geometrically unibranch,*
2. *at least one irreducible component of X dominates Y,*
3. *f is unramified, and*
4. *X is connected.*

Then f is étale and X is irreducible.

Proof. Let $X' \subset X$ be the irreducible component which dominates Y. This means that the generic point of X' maps to the generic point of Y (see for example Morphisms, Lemma 8.6). By Lemma 37.1 we see that f is étale at every point of X'. In particular, the open subscheme $U \subset X$ where f is étale contains X'. Note that every quasi-compact open of U has finitely many irreducible components, see Descent, Lemma 16.3. On the other hand since Y is geometrically unibranch and U is étale over Y, the scheme U is geometrically unibranch. In particular, through every point of U there passes at most one irreducible component. A simple topological argument now shows that $X' \subset U$ is both open and closed. Then of course X' is open and closed in X and by connectedness we find $X = U = X'$ as desired. ☐

[6] In fact, it would suffice if $\kappa(x)$ is geometrically irreducible over $\kappa(s)$. If we ever need this we will add a detailed proof.

0GSA **Lemma 37.3.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Let $x \in X$ with image $y \in Y$. Assume*

(1) *Y is integral and geometrically unibranch at y,*
(2) *$g \circ f$ is étale at x,*
(3) *there is a specialization $x' \rightsquigarrow x$ such that $f(x')$ is the generic point of Y.*

Then f is étale at x and g is étale at y.

Proof. After replacing X by an open neighbourhood of x we may assume $g \circ f$ is étale. Then we find f is unramified by Morphisms, Lemmas 35.16 and 36.5. Hence f is étale at x by Lemma 37.1. Then by étale descent we see that g is étale at y, see for example Descent, Lemma 14.4. $\square$

0GSB **Lemma 37.4.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Assume*

(1) *Y is integral and geometrically unibranch,*
(2) *$g \circ f$ is étale,*
(3) *every irreducible component of X dominates Y.*

Then f is étale and g is étale at every point in the image of f.

Proof. Immediate from the pointwise version Lemma 37.3. $\square$

38. Slicing smooth morphisms

055S In this section we explain a result that roughly states that smooth coverings of a scheme S can be refined by étale coverings. The technique to prove this relies on a slicing argument.

057C **Lemma 38.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Assume*

(1) *f is smooth at x, and*
(2) *the image $d\overline{h}$ of dh in*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is nonzero.

Then there exists an affine open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ smooth.

Proof. As f is smooth at x we may assume, after replacing X by an open neighbourhood of x that f is smooth. In particular we see that f is flat and locally of finite presentation. By Lemma 23.1 we already know there exists an open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ flat and of finite presentation. By Morphisms, Lemma 32.15 we have a short exact sequence

$$\mathcal{C}_{D/U} \to i^*\Omega_{U/S} \to \Omega_{D/S} \to 0$$

where $i : D \to U$ is the closed immersion and $\mathcal{C}_{D/U}$ is the conormal sheaf of D in U. As D is an effective Cartier divisor cut out by $h \in \Gamma(U, \mathcal{O}_U)$ we see that $\mathcal{C}_{D/U} = h \cdot \mathcal{O}_S$. Since $U \to S$ is smooth the sheaf $\Omega_{U/S}$ is finite locally free, hence its pullback $i^*\Omega_{U/S}$ is finite locally free also. The first arrow of the sequence maps the free generator h to the section $\mathrm{d}h|_D$ of $i^*\Omega_{U/S}$ which has nonzero value in the fibre $\Omega_{U/S,x} \otimes \kappa(x)$ by assumption. By right exactness of $\otimes\kappa(x)$ we conclude that

$$\dim_{\kappa(x)}\left(\Omega_{D/S,x} \otimes \kappa(x)\right) = \dim_{\kappa(x)}\left(\Omega_{U/S,x} \otimes \kappa(x)\right) - 1.$$

By Morphisms, Lemma 34.14 we see that $\Omega_{U/S,x}\otimes\kappa(x)$ can be generated by at most $\dim_x(U_s)$ elements. By the displayed formula we see that $\Omega_{D/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(U_s) - 1$ elements. Note that $\dim_x(D_s) = \dim_x(U_s) - 1$ for example because $\dim(\mathcal{O}_{D_s,x}) = \dim(\mathcal{O}_{U_s,x}) - 1$ by Algebra, Lemma 60.13 (also $D_s \subset U_s$ is effective Cartier, see Divisors, Lemma 18.1) and then using Morphisms, Lemma 28.1. Thus we conclude that $\Omega_{D/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(D_s)$ elements and we conclude that $D \to S$ is smooth at x by Morphisms, Lemma 34.14 again. After shrinking U we get that $D \to S$ is smooth and we win. $\square$

057D **Lemma 38.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

1. *f is smooth at x, and*
2. *the map*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) \longrightarrow \Omega_{\kappa(x)/\kappa(s)}$$
has a nonzero kernel.

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is smooth.

Proof. Write $k = \kappa(s)$ and $R = \mathcal{O}_{X_s,x}$. Denote $\mathfrak{m}$ the maximal ideal of R and $\kappa = R/\mathfrak{m}$ so that $\kappa = \kappa(x)$. As formation of modules of differentials commutes with localization (see Algebra, Lemma 131.8) we have $\Omega_{X_s/s,x} = \Omega_{R/k}$. By Algebra, Lemma 131.9 there is an exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\mathrm{d}} \Omega_{R/k} \otimes_R \kappa \to \Omega_{\kappa/k} \to 0.$$

Hence if (2) holds, there exists an element $\overline{h} \in \mathfrak{m}$ such that $\mathrm{d}\overline{h}$ is nonzero. Choose a lift $h \in \mathcal{O}_{X,x}$ of $\overline{h}$ and apply Lemma 38.1. $\square$

057E **Remark 38.3.** The second condition in Lemma 38.2 is necessary even if x is a closed point of a positive dimensional fibre. An example is the following: Let k be a field of characteristic $p > 0$ which is imperfect. Let $a \in k$ be an element which is not a pth power. Let $\mathfrak{m} = (x, y^p - a) \subset k[x, y]$. This corresponds to a closed point w of $X = \mathbf{A}^2_k$. Set $S = \mathbf{A}^1_k$ and let $f : X \to S$ be the morphism corresponding to $k[x] \to k[x, y]$. Then there does not exist any commutative diagram

$$\begin{tikzcd} S' \arrow[rr, "h"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

with g étale and w in the image of h. This is clear as the residue field extension $\kappa(w)/\kappa(f(w))$ is purely inseparable, but for any $s' \in S'$ with $g(s') = f(w)$ the extension $\kappa(s')/\kappa(f(w))$ would be separable.

If you assume the residue field extension is separable then the phenomenon of Remark 38.3 does not happen. Here is the precise result.

057F **Lemma 38.4.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is smooth at x,*
(2) *the residue field extension $\kappa(x)/\kappa(s)$ is separable, and*
(3) *x is not a generic point of X_s.*

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is smooth.

Proof. Write $k = \kappa(s)$ and $R = \mathcal{O}_{X_s,x}$. Denote $\mathfrak{m}$ the maximal ideal of R and $\kappa = R/\mathfrak{m}$ so that $\kappa = \kappa(x)$. As formation of modules of differentials commutes with localization (see Algebra, Lemma 131.8) we have $\Omega_{X_s/s,x} = \Omega_{R/k}$. By assumption (2) and Algebra, Lemma 140.4 the map

$$\mathrm{d} : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{R/k} \otimes_R \kappa(\mathfrak{m})$$

is injective. Assumption (3) implies that $\mathfrak{m}/\mathfrak{m}^2 \neq 0$. Thus there exists an element $\overline{h} \in \mathfrak{m}$ such that $\mathrm{d}\overline{h}$ is nonzero. Choose a lift $h \in \mathcal{O}_{X,x}$ of $\overline{h}$ and apply Lemma 38.1. □

The subscheme Z constructed in the following lemma is really a complete intersection in an affine open neighbourhood of x. If we ever need this we will explicitly formulate a separate lemma stating this fact.

057G **Lemma 38.5.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is smooth at x, and*
(2) *x is a closed point of X_s and $\kappa(s) \subset \kappa(x)$ is separable.*

Then there exists an immersion $Z \to X$ containing x such that

(1) *$Z \to S$ is étale, and*
(2) *$Z_s = \{x\}$ set theoretically.*

Proof. We may and do replace S by an affine open neighbourhood of s. We may and do replace X by an affine open neighbourhood of x such that $X \to S$ is smooth. We will prove the lemma for smooth morphisms of affines by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take Z to be an open neighbourhood of x. Namely, if $d = 0$, then $X \to S$ is quasi-finite at x, see Morphisms, Lemma 29.5. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma 56.2. Thus after replacing X by U we see that X is quasi-finite and smooth over S, hence smooth of relative dimension 0 over S, hence étale over S. Moreover, the fibre X_s is a finite discrete set. Hence after replacing X by a further affine open neighbourhood of X we see that $f^{-1}(\{s\}) = \{x\}$ (because the topology on X_s is induced from the topology on X, see Schemes, Lemma 18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because x is a closed point of its fibre the extension $\kappa(x)/\kappa(s)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 20.3). Thus we see $\Omega_{\kappa(x)/\kappa(s)} = 0$ as this holds for algebraic separable field extensions.

Thus we may apply Lemma 38.2 to find a diagram

$$\begin{tikzcd} D \arrow[r] \arrow[rrd] & U \arrow[r] \arrow[rd] & X \arrow[d] \\ & & S \end{tikzcd}$$

with $x \in D$. Note that $\dim_x(D_s) = \dim_x(X_s) - 1$ for example because $\dim(\mathcal{O}_{D_s,x}) = \dim(\mathcal{O}_{X_s,x}) - 1$ by Algebra, Lemma 60.13 (also $D_s \subset X_s$ is effective Cartier, see Divisors, Lemma 18.1) and then using Morphisms, Lemma 28.1. Thus the morphism $D \to S$ is smooth with $\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1$. By induction hypothesis we can find an immersion $Z \to D$ as desired, which finishes the proof. $\square$

055U **Lemma 38.6.** *Let $f : X \to S$ be a smooth morphism of schemes. Let $s \in S$ be a point in the image of f. Then there exists an étale neighbourhood $(S', s') \to (S, s)$ and a S-morphism $S' \to X$.*

First proof of Lemma 38.6. By assumption $X_s \neq \emptyset$. By Varieties, Lemma 25.6 there exists a closed point $x \in X_s$ such that $\kappa(x)$ is a finite separable field extension of $\kappa(s)$. Hence by Lemma 38.5 there exists an immersion $Z \to X$ such that $Z \to S$ is étale and such that $x \in Z$. Take $(S', s') = (Z, x)$. $\square$

Second proof of Lemma 38.6. Pick a point $x \in X$ with $f(x) = s$. Choose a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "\pi"'] \arrow[d] & \mathbf{A}^d_V \arrow[ld] \\ S & V \arrow[l] & \end{tikzcd}$$

with π étale, $x \in U$ and $V = \text{Spec}(R)$ affine, see Morphisms, Lemma 36.20. In particular $s \in V$. The morphism $\pi : U \to \mathbf{A}^d_V$ is open, see Morphisms, Lemma 36.13. Thus $W = \pi(U) \cap \mathbf{A}^d_s$ is a nonempty open subset of $\mathbf{A}^d_s$. Let $w \in W$ be a point with $\kappa(s) \subset \kappa(w)$ finite separable, see Varieties, Lemma 25.5. By Algebra, Lemma 114.1 there exist d elements $\overline{f}_1, \ldots, \overline{f}_d \in \kappa(s)[x_1, \ldots, x_d]$ which generate the maximal ideal corresponding to w in $\kappa(s)[x_1, \ldots, x_d]$. After replacing R by a principal localization we may assume there are $f_1, \ldots, f_d \in R[x_1, \ldots, x_d]$ which map to $\overline{f}_1, \ldots, \overline{f}_d \in \kappa(s)[x_1, \ldots, x_d]$. Consider the R-algebra

$$R' = R[x_1, \ldots, x_d]/(f_1, \ldots, f_d)$$

and set $S' = \text{Spec}(R')$. By construction we have a closed immersion $j : S' \to \mathbf{A}^d_V$ over V. By construction the fibre of $S' \to V$ over s is a single point s' whose residue field is finite separable over $\kappa(s)$. Let $\mathfrak{q}' \subset R'$ be the corresponding prime. By Algebra, Lemma 136.10 we see that $(R')_g$ is a relative global complete intersection over R for some $g \in R'$, $g \notin \mathfrak{q}$. Thus $S' \to V$ is flat and of finite presentation in a neighbourhood of s', see Algebra, Lemma 136.13. By construction the scheme theoretic fibre of $S' \to V$ over s is $\text{Spec}(\kappa(s'))$. Hence it follows from Morphisms, Lemma 36.15 that $S' \to S$ is étale at s'. Set

$$S'' = U \times_{\pi, \mathbf{A}^d_V, j} S'.$$

By construction there exists a point $s'' \in S''$ which maps to s' via the projection $p : S'' \to S'$. Note that p is étale as the base change of the étale morphism π, see Morphisms, Lemma 36.4. Choose a small affine neighbourhood $S''' \subset S''$ of s''

which maps into the nonempty open neighbourhood of $s' \in S'$ where the morphism $S' \to S$ is étale. Then the étale neighbourhood $(S''', s'') \to (S, s)$ is a solution to the problem posed by the lemma. $\square$

The following lemma shows that sheaves for the smooth topology are the same thing as sheaves for the étale topology.

055V **Lemma 38.7.** *Let S be a scheme. Let $\mathcal{U} = \{S_i \to S\}_{i \in I}$ be a smooth covering of S, see Topologies, Definition 5.1. Then there exists an étale covering $\mathcal{V} = \{T_j \to S\}_{j \in J}$ (see Topologies, Definition 4.1) which refines (see Sites, Definition 8.1) $\mathcal{U}$.*

Proof. For every $s \in S$ there exists an $i \in I$ such that s is in the image of $S_i \to S$. By Lemma 38.6 we can find an étale morphism $g_s : T_s \to S$ such that $s \in g_s(T_s)$ and such that g_s factors through $S_i \to S$. Hence $\{T_s \to S\}$ is an étale covering of S that refines $\mathcal{U}$. $\square$

0EY4 **Lemma 38.8.** *Let $f : X \to S$ be a smooth morphism of schemes. Then there exists an étale covering $\{U_i \to X\}_{i \in I}$ such that $U_i \to S$ factors as $U_i \to V_i \to S$ where $V_i \to S$ is étale and $U_i \to V_i$ is a smooth morphism of affine schemes, which has a section, and has geometrically connected fibres.*

Proof. Let $s \in S$. By Varieties, Lemma 25.6 the set of closed points $x \in X_s$ such that $\kappa(x)/\kappa(s)$ is separable is dense in X_s. Thus it suffices to construct an étale morphism $U \to X$ with x in the image such that $U \to S$ factors in the manner described in the lemma. To do this, choose an immersion $Z \to X$ passing through x such that $Z \to S$ is étale (Lemma 38.5). After replacing S by Z and X by $Z \times_S X$ we see that we may assume $X \to S$ has a section $\sigma : S \to X$ with $\sigma(s) = x$. Then we can first replace S by an affine open neighbourhood of s and next replace X by an affine open neighbourhood of x. Then finally, we consider the subset $X^0 \subset X$ of Section 29. By Lemmas 29.6 and 29.4 this is a retrocompact open subscheme containing σ such that the fibres $X^0 \to S$ are geometrically connected. If X^0 is not affine, then we choose an affine open $U \subset X^0$ containing x. Since $X^0 \to S$ is smooth, the image of U is open. Choose an affine open neighbourhood $V \subset S$ of s contained in $\sigma^{-1}(U)$ and in the image of $U \to S$. Finally, the reader sees that $U \cap f^{-1}(V) \to V$ has all the desired properties. For example $U \cap f^{-1}(V)$ is equal to $U \times_S V$ is affine as a fibre product of affine schemes. Also, the geometric fibres of $U \cap f^{-1}(V) \to V$ are nonempty open subschemes of the irreducible fibres of $X^0 \to S$ and hence connected. Some details omitted. $\square$

39. Étale neighbourhoods and Artin approximation

0CAT In this section we prove results of the form: if two pointed schemes have isomorphic complete local rings, then they have isomorphic étale neighbourhoods. We will rely on Popescu's theorem, see Smoothing Ring Maps, Theorem 12.1.

0CAU **Lemma 39.1.** *Let S be a locally Noetherian scheme. Let X, Y be schemes locally of finite type over S. Let $x \in X$ and $y \in Y$ be points lying over the same point $s \in S$. Assume $\mathcal{O}_{S,s}$ is a G-ring. Assume further we are given a local $\mathcal{O}_{S,s}$-algebra map*

$$\varphi : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}^\wedge$$

For every $N \geq 1$ there exists an elementary étale neighbourhood $(U, u) \to (X, x)$ and an S-morphism $f : U \to Y$ mapping u to y such that the diagram

$$\begin{tikzcd} \mathcal{O}_{X,x}^\wedge \arrow[r] & \mathcal{O}_{U,u}^\wedge \\ \mathcal{O}_{Y,y} \arrow[u, "\varphi"] \arrow[r, "f_u^\sharp"] & \mathcal{O}_{U,u} \arrow[u] \end{tikzcd}$$

commutes modulo $\mathfrak{m}_u^N$.

Proof. The question is local on X hence we may assume X, Y, S are affine. Say $S = \text{Spec}(R)$, $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$. Write $B = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. The local ring $\mathcal{O}_{X,x} = A_\mathfrak{p}$ is a G-ring by More on Algebra, Proposition 50.10. The map φ is a map

$$B_\mathfrak{q}^\wedge \longrightarrow A_\mathfrak{p}^\wedge$$

where $\mathfrak{q} \subset B$ is the prime corresponding to y. Let $a_1, \ldots, a_n \in A_\mathfrak{p}^\wedge$ be the images of $x_1, \ldots, x_n$ via $R[x_1, \ldots, x_n] \to B \to B_\mathfrak{q}^\wedge \to A_\mathfrak{p}^\wedge$. Then we can apply Smoothing Ring Maps, Lemma 13.4 to get an étale ring map $A \to A'$ and a prime ideal $\mathfrak{p}' \subset A'$ and $b_1, \ldots, b_n \in A'$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$, $a_i - b_i \in (\mathfrak{p}')^N (A'_{\mathfrak{p}'})^\wedge$, and $f_j(b_1, \ldots, b_n) = 0$ for $j = 1, \ldots, n$. This determines an R-algebra map $B \to A'$ by sending the class of x_i to $b_i \in A'$. This finishes the proof by taking $U = \text{Spec}(A') \to \text{Spec}(B)$ as the morphism f and $u = \mathfrak{p}'$. $\square$

0CAV **Lemma 39.2.** *Let S be a locally Noetherian scheme. Let X, Y be schemes locally of finite type over S. Let $x \in X$ and $y \in Y$ be points lying over the same point $s \in S$. Assume $\mathcal{O}_{S,s}$ is a G-ring. Assume we have an $\mathcal{O}_{S,s}$-algebra isomorphism*

$$\varphi : \mathcal{O}_{Y,y}^\wedge \longrightarrow \mathcal{O}_{X,x}^\wedge$$

between the complete local rings. Then for every $N \geq 1$ there exists morphisms

$$(X, x) \leftarrow (U, u) \to (Y, y)$$

of pointed schemes over S such that both arrows define elementary étale neighbourhoods and such that the diagram

$$\begin{tikzcd} & \mathcal{O}_{U,u}^\wedge & \\ \mathcal{O}_{Y,y}^\wedge \arrow[ur] \arrow[rr, "\varphi"] & & \mathcal{O}_{X,x}^\wedge \arrow[ul] \end{tikzcd}$$

commutes modulo $\mathfrak{m}_u^N$.

Proof. We may assume $N \geq 2$. Apply Lemma 39.1 to get $(U, u) \to (X, x)$ and $f : (U, u) \to (Y, y)$. We claim that f is étale at u which will finish the proof. In fact, we will show that the induced map $\mathcal{O}_{Y,y}^\wedge \to \mathcal{O}_{U,u}^\wedge$ is an isomorphism. Having proved this, Lemma 12.1 will show that f is smooth at u and of course f is unramified at u as well, so Morphisms, Lemma 36.5 tells us f is étale at u. For a local ring $(R, \mathfrak{m})$ we set $\text{Gr}_\mathfrak{m}(R) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. To prove the claim we look at the induced diagram of graded rings

$$\begin{tikzcd} & \mathrm{Gr}_{\mathfrak m_u}(\mathcal{O}_{U,u}) & \\ \mathrm{Gr}_{\mathfrak m_y}(\mathcal{O}_{Y,y}) \arrow[ur] \arrow[rr, "\varphi"] & & \mathrm{Gr}_{\mathfrak m_x}(\mathcal{O}_{X,x}) \arrow[ul] \end{tikzcd}$$

Since $N \geq 2$ this diagram is actually commutative as the displayed graded algebras are generated in degree 1! By assumption the lower arrow is an isomorphism. By More on Algebra, Lemma 43.9 (for example) the map $\mathcal{O}_{X,x}^\wedge \to \mathcal{O}_{U,u}^\wedge$ is an isomorphism and hence the north-west arrow in the diagram is an isomorphism. We conclude that f induces an isomorphism $\mathrm{Gr}_{\mathfrak m_x}(\mathcal{O}_{X,x}) \to \mathrm{Gr}_{\mathfrak m_y}(\mathcal{O}_{U,u})$. Using induction and the short exact sequences

$$0 \to \mathrm{Gr}^n_{\mathfrak m}(R) \to R/\mathfrak m^{n+1} \to R/\mathfrak m^n \to 0$$

for both local rings we conclude (from the snake lemma) that f induces isomorphisms $\mathcal{O}_{Y,y}/\mathfrak m_y^n \to \mathcal{O}_{U,u}/\mathfrak m_u^n$ for all n which is what we wanted to show. □

0GDX **Lemma 39.3.** *Let $X \to S$, $Y \to T$, x, s, y, t, σ, y_σ, and φ be given as follows: we have morphisms of schemes*

$$\begin{matrix} X & & Y & & & x & & y \\ \downarrow & & \downarrow & \text{with points} & & \downarrow & & \downarrow \\ S & & T & & & s & & t \end{matrix}$$

Here S is locally Noetherian and T is of finite type over $\mathbf{Z}$. The morphisms $X \to S$ and $Y \to T$ are locally of finite type. The local ring $\mathcal{O}_{S,s}$ is a G-ring. The map

$$\sigma : \mathcal{O}_{T,t} \longrightarrow \mathcal{O}_{S,s}^\wedge$$

is a local homomorphism. Set $Y_\sigma = Y \times_{T,\sigma} \mathrm{Spec}(\mathcal{O}_{S,s}^\wedge)$. Next, y_σ is a point of Y_σ mapping to y and the closed point of $\mathrm{Spec}(\mathcal{O}_{S,s}^\wedge)$. Finally

$$\varphi : \mathcal{O}_{X,x}^\wedge \longrightarrow \mathcal{O}_{Y_\sigma,y_\sigma}^\wedge$$

is an isomorphism of $\mathcal{O}_{S,s}^\wedge$-algebras. In this situation there exists a commutative diagram

$$\begin{tikzcd} X \arrow[d] & W \arrow[l] \arrow[r] \arrow[dr] & Y \times_{T,\tau} V \arrow[r] \arrow[d] & Y \arrow[d] \\ S & & V \arrow[ll] \arrow[r, "\tau"] & T \end{tikzcd}$$

of schemes and points $w \in W$, $v \in V$ such that

1. *$(V, v) \to (S, s)$ is an elementary étale neighbourhood,*
2. *$(W, w) \to (X, x)$ is an elementary étale neighbourhood, and*
3. *$\tau(v) = t$.*

Let $y_\tau \in Y \times_T V$ correspond to y_σ via the identification $(Y_\sigma)_s = (Y \times_T V)_v$. Then

4. *$(W, w) \to (Y \times_{T,\tau} V, y_\tau)$ is an elementary étale neighbourhood.*

Proof. Denote $X_\sigma = X \times_S \mathrm{Spec}(\mathcal{O}_{S,s}^\wedge)$ and $x_\sigma \in X_\sigma$ the unique point lying over x. Observe that $\mathcal{O}_{S,s}^\wedge$ is a G-ring by More on Algebra, Proposition 50.6. By Lemma 39.2 we can choose

$$(X_\sigma, x_\sigma) \leftarrow (U, u) \rightarrow (Y_\sigma, y_\sigma)$$

where both arrows are elementary étale neighbourhoods.

After replacing S by an open neighbourhood of s, we may assume $S = \mathrm{Spec}(R)$ is affine. Since $\mathcal{O}_{S,s}$ is a G-ring by Smoothing Ring Maps, Theorem 12.1 the ring $\mathcal{O}_{S,s}^\wedge$ is a filtered colimit of smooth R-algebras. Thus we can write

$$\mathrm{Spec}(\mathcal{O}_{S,s}^\wedge) = \lim S_i$$

as a directed limit of affine schemes S_i smooth over S. Denote $s_i \in S_i$ the image of the closed point of $\mathrm{Spec}(\mathcal{O}_{S,s}^\wedge)$. Observe that $\kappa(s) = \kappa(s_i)$. Set $X_i = X \times_S S_i$ and denote $x_i \in X_i$ the unique point mapping to x. Note that $\kappa(x) = \kappa(x_i)$. Since T is of finite type over $\mathbf{Z}$ by Limits, Proposition 6.1 we can choose an i and a morphism $\sigma_i : (S_i, s_i) \to (T, t)$ of pointed schemes whose composition with $\mathrm{Spec}(\mathcal{O}_{S,s}^\wedge) \to S_i$ is equal to σ. Set $Y_i = Y \times_T S_i$ and denote y_i the image of y_σ. Note that $\kappa(y_i) = \kappa(y_\sigma)$. By Limits, Lemma 10.1 we can choose an i and a diagram

$$\begin{tikzcd} X_i \arrow[dr] & U_i \arrow[l] \arrow[r] \arrow[d] & Y_i \arrow[dl] \\ & S_i & \end{tikzcd}$$

whose base change to $\mathrm{Spec}(\mathcal{O}_{S,s}^\wedge)$ recovers $X_\sigma \leftarrow U \to Y_\sigma$. By Limits, Lemma 8.10 after increasing i we may assume the morphisms $X_i \leftarrow U_i \to Y_i$ are étale. Let $u_i \in U_i$ be the image of u. Then $u_i \mapsto x_i$ hence $\kappa(x) = \kappa(x_\sigma) = \kappa(u) \supset \kappa(u_i) \supset \kappa(x_i) = \kappa(x)$ and we see that $\kappa(u_i) = \kappa(x_i)$. Hence $(X_i, x_i) \leftarrow (U_i, u_i)$ is an elementary étale neighbourhood. Since also $\kappa(y_i) = \kappa(y_\sigma) = \kappa(u)$ we see that also $(U_i, u_i) \to (Y_i, y_i)$ is an elementary étale neighbourhood.

At this point we have constructed a diagram

$$\begin{tikzcd} X \arrow[d] & X \times_S S_i \arrow[l] \arrow[dr] & U_i \arrow[l] \arrow[r] \arrow[d] & Y \times_T S_i \arrow[r] \arrow[dl] & Y \arrow[d] \\ S & & S_i \arrow[ll] \arrow[rr] & & T \end{tikzcd}$$

as in the statement of the lemma, except that $S_i \to S$ is smooth. By Lemma 38.5 and after shrinking S_i we can assume there exists a closed subscheme $V \subset S_i$ passing through s_i such that $V \to S$ is étale. Setting W equal to the scheme theoretic inverse image of V in U_i we conclude. $\square$

We strongly encourage the reader to skip the rest of this section.

0CAW **Lemma 39.4.** *Consider a diagram*

$$\begin{tikzcd} X \arrow[d] & Y \arrow[d] \\ S & T \arrow[l] \end{tikzcd} \quad \text{with points} \quad \begin{tikzcd} x \arrow[d] & y \arrow[d] \\ s & t \arrow[l] \end{tikzcd}$$

where S be a locally Noetherian scheme and the morphisms are locally of finite type. Assume $\mathcal{O}_{S,s}$ is a G-ring. Assume further we are given a local $\mathcal{O}_{S,s}$-algebra map

$$\sigma : \mathcal{O}_{T,t} \longrightarrow \mathcal{O}_{S,s}^\wedge$$

and a local $\mathcal{O}_{S,s}$-algebra map

$$\varphi : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y_\sigma, y_\sigma}^\wedge$$

where $Y_\sigma = Y \times_{T,\sigma} \mathrm{Spec}(\mathcal{O}_{S,s}^\wedge)$ and y_σ is the unique point of Y_σ lying over y. For every $N \geq 1$ there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[d] & X \times_S V \arrow[l] \arrow[dr] & W \arrow[l, "f"'] \arrow[d] \arrow[r] & Y \times_{T,\tau} V \arrow[dl] \arrow[r] & Y \arrow[d] \\
S & & V \arrow[ll] \arrow[rr, "\tau"] & & T
\end{tikzcd}$$

of schemes over S and points $w \in W$, $v \in V$ such that

(1) *$v \mapsto s$, $\tau(v) = t$, $f(w) = (x, v)$, and $w \mapsto (y, v)$,*
(2) *$(V, v) \to (S, s)$ is an elementary étale neighbourhood,*
(3) *the diagram*

$$\begin{tikzcd}
\mathcal{O}_{S,s}^\wedge \arrow[r] & \mathcal{O}_{V,v}^\wedge \\
\mathcal{O}_{T,t} \arrow[u, "\sigma"] \arrow[r, "\tau_v^\sharp"] & \mathcal{O}_{V,v} \arrow[u]
\end{tikzcd}$$

commutes module $\mathfrak{m}_v^N$,

(4) *$(W, w) \to (Y \times_{T,\tau} V, (y, v))$ is an elementary étale neighbourhood,*
(5) *the diagram*

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[r, "\varphi"'] \arrow[d, equal] & \mathcal{O}_{Y_\sigma,y_\sigma}^\wedge \arrow[r] & \mathcal{O}_{Y_\sigma,y_\sigma}/\mathfrak{m}_{y_\sigma}^N \arrow[r, equal] & \mathcal{O}_{Y \times_{T,\tau} V,(y,v)}/\mathfrak{m}_{(y,v)}^N \arrow[d, "\cong"'] \\
\mathcal{O}_{X,x} \arrow[r] & \mathcal{O}_{X \times_S V,(x,v)} \arrow[r, "f_w^\sharp"] & \mathcal{O}_{W,w} \arrow[r] & \mathcal{O}_{W,w}/\mathfrak{m}_w^N
\end{tikzcd}$$

commutes. The equality comes from the fact that Y_σ and $Y \times_{T,\tau} V$ are canonically isomorphic over $\mathcal{O}_{V,v}/\mathfrak{m}_v^N = \mathcal{O}_{S,s}/\mathfrak{m}_s^N$ by parts (2) and (3).

Proof. After replacing X, S, T, Y by affine open subschemes we may assume the diagram in the statement of the lemma comes from applying Spec to a diagram

$$\begin{tikzcd}
A & B \\
R \arrow[u] \arrow[r] & C \arrow[u]
\end{tikzcd}
\quad \text{with primes} \quad
\begin{tikzcd}
\mathfrak{p}_A \arrow[d, dash] & \mathfrak{p}_B \arrow[d, dash] \\
\mathfrak{p}_R \arrow[r, dash] & \mathfrak{p}_C
\end{tikzcd}$$

of Noetherian rings and finite type ring maps. In this proof every ring E will be a Noetherian R-algebra endowed with a prime ideal $\mathfrak{p}_E$ lying over $\mathfrak{p}_R$ and all ring maps will be R-algebra maps compatible with the given primes. Moreover, if we write $E^\wedge$ we mean the completion of the localization of E at $\mathfrak{p}_E$. We will also use without further mention that an étale ring map $E_1 \to E_2$ such that $\kappa(\mathfrak{p}_{E_1}) = \kappa(\mathfrak{p}_{E_2})$ induces an isomorphism $E_1^\wedge = E_2^\wedge$ by More on Algebra, Lemma 43.9.

With this notation σ and φ correspond to ring maps

$$\sigma : C \to R^\wedge \quad \text{and} \quad \varphi : A \longrightarrow (B \otimes_{C,\sigma} R^\wedge)^\wedge$$

Here is a picture

$$
\begin{tikzcd}
A \arrow[rrr, bend left, "\varphi"] & B \arrow[r] & B \otimes_{C,\sigma} R^\wedge \arrow[r] & (B \otimes_{C,\sigma} R^\wedge)^\wedge \\
R \arrow[u] \arrow[r] & C \arrow[u] \arrow[r, "\sigma"] & R^\wedge \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

Observe that $R^\wedge$ is a G-ring by More on Algebra, Proposition 50.6. Thus $B \otimes_{C,\sigma} R^\wedge$ is a G-ring by More on Algebra, Proposition 50.10. By Lemma 39.1 (translated into algebra) there exists an étale ring map $B \otimes_{C,\sigma} R^\wedge \to B'$ inducing an isomorphism $\kappa(\mathfrak p_{B \otimes_{C,\sigma} R^\wedge}) \to \kappa(\mathfrak p_{B'})$ and an R-algebra map $A \to B'$ such that the composition

$$A \to B' \to (B')^\wedge = (B \otimes_{C,\sigma} R^\wedge)^\wedge$$

is the same as φ modulo $(\mathfrak p_{(B \otimes_{C,\sigma} R^\wedge)^\wedge})^N$. Thus we may replace φ by this composition because the only way φ enters the conclusion is via the commutativity requirement in part (5) of the statement of the lemma. Picture:

$$
\begin{tikzcd}
 & & B' \arrow[r] & (B')^\wedge \arrow[d, equal] \\
A \arrow[urr] & B \arrow[r] & B \otimes_{C,\sigma} R^\wedge \arrow[u] \arrow[r] & (B \otimes_{C,\sigma} R^\wedge)^\wedge \\
R \arrow[u] \arrow[r] & C \arrow[u] \arrow[r, "\sigma"] & R^\wedge \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

Next, we use that $R^\wedge$ is a filtered colimit of smooth R-algebras (Smoothing Ring Maps, Theorem 12.1) because $R_{\mathfrak p_R}$ is a G-ring by assumption. Since C is of finite presentation over R we get a factorization

$$C \to R' \to R^\wedge$$

for some $R \to R'$ smooth, see Algebra, Lemma 127.3. After increasing R' we may assume there exists an étale $B \otimes_C R'$-algebra B'' whose base change to $B \otimes_{C,\sigma} R^\wedge$ is B', see Algebra, Lemma 143.3. Then B' is the filtered colimit of these B'' and we conclude that after increasing R' we may assume there is an R-algebra map $A \to B''$ such that $A \to B'' \to B'$ is the previously constructed map (same reference as above). Picture

$$
\begin{tikzcd}
 & & B'' \arrow[r] & B' \arrow[r] & (B')^\wedge \arrow[d, equal] \\
A \arrow[urr] & B \arrow[r] & B \otimes_C R' \arrow[u] \arrow[r] & B \otimes_{C,\sigma} R^\wedge \arrow[u] \arrow[r] & (B \otimes_{C,\sigma} R^\wedge)^\wedge \\
R \arrow[u] \arrow[r] & C \arrow[u] \arrow[r] & R' \arrow[u] \arrow[r] & R^\wedge \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

and

$$B' = B'' \otimes_{(B \otimes_C R')} (B \otimes_{C,\sigma} R^\wedge)$$

This means that we may replace C by R', $\sigma : C \to R^\wedge$ by $R' \to R^\wedge$, and B by B'' so that we simplify to the diagram

$$\begin{tikzcd} A \arrow[r] & B \arrow[r] & B \otimes_{C,\sigma} R^\wedge \\ R \arrow[r] \arrow[u] & C \arrow[r, "\sigma"] \arrow[u] & R^\wedge \arrow[u] \end{tikzcd}$$

with φ equal to the composition of the horizontal arrows followed by the canonical map from $B \otimes_{C,\sigma} R^\wedge$ to its completion. The final step in the proof is to apply Lemma 39.1 (or its proof) one more time to $\text{Spec}(C)$ and $\text{Spec}(R)$ over $\text{Spec}(R)$ and the map $C \to R^\wedge$. The lemma produces a ring map $C \to D$ such that $R \to D$ is étale, such that $\kappa(\mathfrak p_R) = \kappa(\mathfrak p_D)$, and such that

$$C \to D \to D^\wedge = R^\wedge$$

is equal to $\sigma : C \to R^\wedge$ modulo $(\mathfrak p_{R^\wedge})^N$. Then we can take

$$V = \text{Spec}(D) \quad \text{and} \quad W = \text{Spec}(B \otimes_C D)$$

as our solution to the problem posed by the lemma. Namely the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & B \otimes_{C,\sigma} R^\wedge \arrow[r] & B \otimes_{C,\sigma} R^\wedge/(\mathfrak p_{R^\wedge})^N \arrow[r, equal] & B \otimes_C D/(\mathfrak p_D)^N \arrow[d, equal] \\ A \arrow[r] & A \otimes_R D \arrow[r] & B \otimes_R D \arrow[r] & B \otimes_C D/(\mathfrak p_D)^N \end{tikzcd}$$

commutes because $C \to D \to D^\wedge = R^\wedge$ is equal to σ modulo $(\mathfrak p_{R^\wedge})^N$. This proves part (5) and the other properties are immediate from the construction. $\square$

0CAX **Lemma 39.5.** *Let $T \to S$ be finite type morphisms of Noetherian schemes. Let $t \in T$ map to $s \in S$ and let $\sigma : \mathcal O_{T,t} \to \mathcal O_{S,s}^\wedge$ be a local $\mathcal O_{S,s}$-algebra map. For every $N \geq 1$ there exists a finite type morphism $(T', t') \to (T, t)$ such that σ factors through $\mathcal O_{T,t} \to \mathcal O_{T',t'}$ and such that for every local $\mathcal O_{S,s}$-algebra map $\sigma' : \mathcal O_{T,t} \to \mathcal O_{S,s}^\wedge$ which factors through $\mathcal O_{T,t} \to \mathcal O_{T',t'}$ the maps σ and σ' agree modulo $\mathfrak m_s^N$.*

Proof. We may assume S and T are affine. Say $S = \text{Spec}(R)$ and $T = \text{Spec}(C)$. Let $c_1, \ldots, c_n \in C$ be generators of C as an R-algebra. Let $\mathfrak p \subset R$ be the prime ideal corresponding to s. Say $\mathfrak p = (f_1, \ldots, f_m)$. After replacing R by a principal localization (to clear denominators in $R_\mathfrak p$) we may assume there exist $r_1, \ldots, r_n \in R$ and $a_{i,I} \in \mathcal O_{S,s}^\wedge$ where $I = (i_1, \ldots, i_m)$ with $\sum i_j = N$ such that

$$\sigma(c_i) = r_i + \sum\nolimits_I a_{i,I} f_1^{i_1} \ldots f_m^{i_m}$$

in $\mathcal O_{S,s}^\wedge$. Then we consider

$$C' = C[t_{i,I}]/\left(c_i - r_i - \sum\nolimits_I t_{i,I} f_1^{i_1} \ldots f_m^{i_m}\right)$$

with $\mathfrak p' = \mathfrak p C' + (t_{i,I})$ and factorization of $\sigma : C \to \mathcal O_{S,s}^\wedge$ through C' given by sending $t_{i,I}$ to $a_{i,I}$. Taking $T' = \text{Spec}(C')$ works because any σ' as in the statement of the lemma will send c_i to r_i modulo the maximal ideal to the power N. $\square$

0CAY **Lemma 39.6.** *Let $Y \to T \to S$ be finite type morphisms of Noetherian schemes. Let $t \in T$ map to $s \in S$ and let $\sigma : \mathcal O_{T,t} \to \mathcal O_{S,s}^\wedge$ be a local $\mathcal O_{S,s}$-algebra map. There exists a finite type morphism $(T', t') \to (T, t)$ such that σ factors through*

$\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ and such that for every local $\mathcal{O}_{S,s}$-algebra map $\sigma' : \mathcal{O}_{T,t} \to \mathcal{O}_{S,s}^\wedge$ which factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ the closed immersions

$$Y \times_{T,\sigma} \mathrm{Spec}(\mathcal{O}_{S,s}^\wedge) = Y_\sigma \longleftarrow Y_t \longrightarrow Y_{\sigma'} = Y \times_{T,\sigma'} \mathrm{Spec}(\mathcal{O}_{S,s}^\wedge)$$

have isomorphic conormal algebras.

Proof. A useful observation is that $\kappa(s) = \kappa(t)$ by the existence of σ. Observe that the statement makes sense as the fibres of Y_σ and $Y_{\sigma'}$ over $s \in \mathrm{Spec}(\mathcal{O}_{S,s}^\wedge)$ are both canonically isomorphic to Y_t. We will think of the property "σ' factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$" as a constraint on σ'. If we have several such constraints, say given by $(T'_i, t'_i) \to (T, t)$, $i = 1, \ldots, n$ then we can combined them by considering $(T'_1 \times_T \ldots \times_T T'_n, (t'_1, \ldots, t'_n)) \to (T, t)$. We will use this without further mention in the following.

By Lemma 39.5 we can assume that any σ' as in the statement of the lemma is the same as σ modulo $\mathfrak{m}_s^2$. Note that the conormal algebra of Y_t in Y_σ is just the quasi-coherent graded $\mathcal{O}_{Y_t}$-algebra

$$\bigoplus\nolimits_{n \geq 0} \mathfrak{m}_s^n \mathcal{O}_{Y_\sigma} / \mathfrak{m}_s^{n+1} \mathcal{O}_{Y_\sigma}$$

and similarly for $Y_{\sigma'}$. Since σ and σ' agree modulo $\mathfrak{m}_s^2$ we see that these two algebras are the same in degrees 0 and 1. On the other hand, these conormal algebras are generated in degree 1 over degree 0. Hence if there is an isomorphism extending the isomorphism just constructed in degrees 0 and 1, then it is unique.

We may assume S and T are affine. Let $Y = Y_1 \cup \ldots \cup Y_n$ be an affine open covering. If we can construct $(T'_i, t'_i) \to (T, t)$ as in the lemma such that the desired isomorphism (see previous paragraph) exists for $Y_i \to T \to S$ and σ, then these glue by uniqueness to prove the result for $Y \to T$. Thus we may assume Y is affine.

Write $S = \mathrm{Spec}(R)$, $T = \mathrm{Spec}(C)$, and $Y = \mathrm{Spec}(B)$. Choose a presentation $B = C[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Denote $R^\wedge = \mathcal{O}_{S,s}^\wedge$. Let $a_{kj} \in R^\wedge[x_1, \ldots, x_n]$ be polynomials such that

$$\sum\nolimits_{j=1,\ldots,m} a_{kj}\sigma(f_j) = 0, \quad \text{for } k = 1, \ldots, K$$

is a set of generators for the module of relations among the $\sigma(f_j) \in R^\wedge[x_1, \ldots, x_n]$. Thus we have an exact sequence

(39.6.1)

0CAZ
$$R^\wedge[x_1, \ldots, x_n]^{\oplus K} \to R^\wedge[x_1, \ldots, x_n]^{\oplus m} \to R^\wedge[x_1, \ldots, x_n] \to B \otimes_{C,\sigma} R^\wedge \to 0$$

Let c be an integer which works in the Artin-Rees lemma for both the first and the second map in this sequence and the ideal $\mathfrak{m}_{R^\wedge} R^\wedge[x_1, \ldots, x_n]$ as defined in More on Algebra, Section 4. Write

$$a_{kj} = \sum\nolimits_{I \in \Omega} a_{kj,I} x^I \quad \text{and} \quad f_j = \sum\nolimits_{I \in \Omega} f_{j,I} x^I$$

in multiindex notation where $a_{kj,I} \in R^\wedge$, $f_{j,I} \in C$, and Ω a finite set of multiindices. Then we see that

$$\sum\nolimits_{j=1,\ldots,m,\ I,I' \in \Omega,\ I+I'=I''} a_{kj,I}\sigma(f_{j,I'}) = 0, \quad I'' \text{ a multiindex}$$

in $R^\wedge$. Thus we take

$$C' = C[t_{jk,I}]/\left(\sum\nolimits_{j=1,\ldots,m,\ I,I' \in \Omega,\ I+I'=I''} t_{kj,I} f_{j,I'},\ I'' \text{ a multiindex}\right)$$

Then σ factors through a map $\tilde{\sigma} : C' \to R^\wedge$ sending $t_{kj,I}$ to $a_{jk,I}$. Thus $T' = \text{Spec}(C')$ comes with a point $t' \in T'$ such that σ factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$. Let $t_{kj} = \sum t_{kj,I} x^I$ in $C'[x_1, \ldots, x_n]$. Then we see that we have a complex

0CB0 (39.6.2) $$C'[x_1, \ldots, x_n]^{\oplus K} \to C'[x_1, \ldots, x_n]^{\oplus m} \to C'[x_1, \ldots, x_n] \to B \otimes_C C' \to 0$$

which is exact at $C'[x_1, \ldots, x_n]$ and whose base change by $\tilde{\sigma}$ gives (39.6.1).

By Lemma 39.5 we can find a further morphism $(T'', t'') \to (T', t')$ such that $\tilde{\sigma}$ factors through $\mathcal{O}_{T',t'} \to \mathcal{O}_{T'',t''}$ and such that if $\sigma' : C \to R^\wedge$ factors through $\mathcal{O}_{T'',t''}$, then the induced map $\tilde{\sigma}' : C' \to R^\wedge$ agrees modulo $\mathfrak{m}_s^{c+1}$ with $\tilde{\sigma}$. Thus if σ' is such a map, then we obtain a complex

$$R^\wedge[x_1, \ldots, x_n]^{\oplus K} \to R^\wedge[x_1, \ldots, x_n]^{\oplus m} \to R^\wedge[x_1, \ldots, x_n] \to B \otimes_{C,\sigma'} R^\wedge \to 0$$

over $R^\wedge[x_1, \ldots, x_n]$ by applying $\tilde{\sigma}'$ to the polynomials t_{kj} and f_j. In other words, this is the base change of the complex (39.6.2) by $\tilde{\sigma}'$. The matrices defining this complex are congruent modulo $\mathfrak{m}_s^{c+1}$ to the matrices defining the complex (39.6.1) because $\tilde{\sigma}$ and $\tilde{\sigma}'$ are congruent modulo $\mathfrak{m}_s^{c+1}$. Since (39.6.1) is exact, we can apply More on Algebra, Lemma 4.2 to conclude that

$$\text{Gr}_{\mathfrak{m}_s}(B \otimes_{C,\sigma'} R^\wedge) \cong \text{Gr}_{\mathfrak{m}_s}(B \otimes_{C,\sigma} R^\wedge)$$

as desired. □

0CB1 **Lemma 39.7.** *With notation an assumptions as in Lemma 39.4 assume that φ induces an isomorphism on completions. Then we can choose our diagram such that f is étale.*

Proof. We may assume $N \geq 2$ and we may replace (T, t) with (T', t') as in Lemma 39.6. Since $(V, v) \to (S, s)$ is an elementary étale neighbourhood, so is $(X \times_S V, (x, v)) \to (X, x)$. Thus $\mathcal{O}_{X,x} \to \mathcal{O}_{X \times_S V,(x,v)}$ induces an isomorphism on completions by More on Algebra, Lemma 43.9. We claim $\mathcal{O}_{X,x} \to \mathcal{O}_{W,w}$ induces an isomorphism on completions. Having proved this, Lemma 12.1 will show that f is smooth at w and of course f is unramified at u as well, so Morphisms, Lemma 36.5 tells us f is étale at w.

First we use the commutativity in part (5) of Lemma 39.4 to see that for $i \leq N$ there is a commutative diagram

$$\begin{tikzcd}
\text{Gr}^i_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) \arrow[r, "\varphi"'] \arrow[d, equal] & \text{Gr}^i_{\mathfrak{m}_{y_\sigma}}(\mathcal{O}^\wedge_{Y_\sigma,y_\sigma}) \arrow[r, equal] & \text{Gr}^i_{\mathfrak{m}_{(y,v)}}(\mathcal{O}_{Y \times_{T,\tau} V,(y,v)}) \arrow[d, "\cong"'] \\
\text{Gr}^i_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) \arrow[r, "\cong"] & \text{Gr}^i_{\mathfrak{m}_{(x,v)}}(\mathcal{O}_{X \times_S V,(x,v)}) \arrow[r, "f^\sharp_w"] & \text{Gr}^i_{\mathfrak{m}_w}(\mathcal{O}_{W,w})
\end{tikzcd}$$

This implies that $f^\sharp_w$ defines an isomorphism $\kappa(x) \to \kappa(w)$ on residue fields and an isomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathfrak{m}_w/\mathfrak{m}_w^2$ on cotangent spaces. Hence $f^\sharp_w$ defines a surjection $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{W,w}$ on complete local rings.

By Lemma 39.6 there is an isomorphism of $\text{Gr}_{\mathfrak{m}_s}(\mathcal{O}_{(Y \times_{T,\tau} V,(y,v)})$ with $\text{Gr}_{\mathfrak{m}_s}(\mathcal{O}_{Y_\sigma,y_\sigma})$. This follows by taking stalks of the isomorphism of conormal sheaves at the point y. Since our local rings are Noetherian taking associated graded with respect to $\mathfrak{m}_s$ commutes with completion because completion with respect to an ideal is an exact functor on finite modules over Noetherian rings. This produces the right vertical isomorphism in the diagram of graded rings

$$\begin{tikzcd}
\mathrm{Gr}_{\mathfrak m_s}(\mathcal{O}^\wedge_{W,w}) & \mathrm{Gr}_{\mathfrak m_s}(\mathcal{O}^\wedge_{(Y \times_{T,\tau} V,(y,v))}) \arrow[l] \\
\mathrm{Gr}_{\mathfrak m_s}(\mathcal{O}^\wedge_{X,x}) \arrow[u] \arrow[r, "\varphi"] & \mathrm{Gr}_{\mathfrak m_s}(\mathcal{O}^\wedge_{Y_\sigma,y_\sigma}) \arrow[u, "\cong"']
\end{tikzcd}$$

We do not claim the diagram commutes. By the result of the previous paragraph the left arrow is surjective. The other three arrows are isomorphisms. It follows that the left arrow is a surjective map between isomorphic Noetherian rings. Hence it is an isomorphism by Algebra, Lemma 31.10 (you can argue this directly using Hilbert functions as well). In particular $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{W,w}$ must be injective as well as surjective which finishes the proof. □

40. Finite free locally dominates étale

04HE In this section we explain a result that roughly states that étale coverings of a scheme S can be refined by Zariski coverings of finite locally free covers of S.

02LG **Lemma 40.1.** *Let S be a scheme. Let $s \in S$. Let $f : (U,u) \to (S,s)$ be an étale neighbourhood. There exists an affine open neighbourhood $s \in V \subset S$ and a surjective, finite locally free morphism $\pi : T \to V$ such that for every $t \in \pi^{-1}(s)$ there exists an open neighbourhood $t \in W_t \subset T$ and a commutative diagram*

$$\begin{tikzcd}
T \arrow[d, "\pi"] & W_t \arrow[l] \arrow[rr, "h_t"] \arrow[dr] & & U \arrow[dl] \\
V \arrow[rr] & & S &
\end{tikzcd}$$

with $h_t(t) = u$.

Proof. The problem is local on S hence we may replace S by any open neighbourhood of s. We may also replace U by an open neighbourhood of u. Hence, by Morphisms, Lemma 36.14 we may assume that $U \to S$ is a standard étale morphism of affine schemes. In this case the lemma (with $V = S$) follows from Algebra, Lemma 144.5. □

02LH **Lemma 40.2.** *Let $f : U \to S$ be a surjective étale morphism of affine schemes. There exists a surjective, finite locally free morphism $\pi : T \to S$ and a finite open covering $T = T_1 \cup \ldots \cup T_n$ such that each $T_i \to S$ factors through $U \to S$. Diagram:*

$$\begin{tikzcd}
& \coprod T_i \arrow[dl] \arrow[dr] & \\
T \arrow[dr, "\pi"] & & U \arrow[dl, "f"'] \\
& S &
\end{tikzcd}$$

where the south-west arrow is a Zariski-covering.

Proof. This is a restatement of Algebra, Lemma 144.6. □

02LI **Remark 40.3.** In terms of topologies Lemmas 40.1 and 40.2 mean the following. Let S be any scheme. Let $\{f_i : U_i \to S\}$ be an étale covering of S. There exists a Zariski open covering $S = \bigcup V_j$, for each j a finite locally free, surjective morphism $W_j \to V_j$, and for each j a Zariski open covering $\{W_{j,k} \to W_j\}$ such that the family $\{W_{j,k} \to S\}$ refines the given étale covering $\{f_i : U_i \to S\}$. What does this mean in practice? Well, for example, suppose we have a descent problem which we know how to solve for Zariski coverings and for fppf coverings of the form $\{\pi : T \to S\}$ with π finite locally free and surjective. Then this descent problem has an affirmative answer for étale coverings as well. This trick was used by Gabber in his proof that $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for an affine scheme X, see [Hoo82].

41. Étale localization of quasi-finite morphisms

04HF Now we come to a series of lemmas around the theme "quasi-finite morphisms become finite after étale localization". The general idea is the following. Suppose given a morphism of schemes $f : X \to S$ and a point $s \in S$. Let $\varphi : (U, u) \to (S, s)$ be an étale neighbourhood of s in S. Consider the fibre product $X_U = U \times_S X$ and the basic diagram

02LJ (41.0.1)

$$\begin{tikzcd} V \arrow[r] \arrow[dr] & X_U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & U \arrow[r, "\varphi"] & S \end{tikzcd}$$

where $V \subset X_U$ is open. Is there some standard model for the morphism $f_U : X_U \to U$, or for the morphism $V \to U$ for suitable opens V? Of course the answer is no in general. But for quasi-finite morphisms we can say something.

02LK **Lemma 41.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume that*

1. *f is locally of finite type, and*
2. *$x \in X_s$ is isolated*[7].

Then there exist

(a) *an elementary étale neighbourhood $(U, u) \to (S, s)$,*
(b) *an open subscheme $V \subset X_U$ (see 41.0.1)*

such that

(i) *$V \to U$ is a finite morphism,*
(ii) *there is a unique point v of V mapping to u in U, and*
(iii) *the point v maps to x under the morphism $X_U \to X$, inducing $\kappa(x) = \kappa(v)$.*

Moreover, for any elementary étale neighbourhood $(U', u') \to (U, u)$ setting $V' = U' \times_U V \subset X_{U'}$ the triple (U', u', V') satisfies the properties (i), (ii), and (iii) as well.

Proof. Let $Y \subset X$, $W \subset S$ be affine opens such that $f(Y) \subset W$ and such that $x \in Y$. Note that x is also an isolated point of the fibre of the morphism $f|_Y : Y \to W$. If we can prove the theorem for $f|_Y : Y \to W$, then the theorem follows for f. Hence we reduce to the case where f is a morphism of affine schemes. This case is Algebra, Lemma 145.2. $\square$

[7]In the presence of (1) this means that f is quasi-finite at x, see Morphisms, Lemma 20.6.

In the preceding and following lemma we do not assume that the morphism f is separated. This means that the opens V, V_i created in them are not necessarily closed in X_U. Moreover, if we choose the neighbourhood U to be affine, then each V_i is affine, but the intersections $V_i \cap V_j$ need not be affine (in the nonseparated case).

02LL **Lemma 41.2.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume that*

(1) *f is locally of finite type, and*
(2) *$x_i \in X_s$ is isolated for $i = 1, \ldots, n$.*

Then there exist

(a) *an elementary étale neighbourhood $(U, u) \to (S, s)$,*
(b) *for each i an open subscheme $V_i \subset X_U$,*

such that for each i we have

(i) *$V_i \to U$ is a finite morphism,*
(ii) *there is a unique point v_i of V_i mapping to u in U, and*
(iii) *the point v_i maps to x_i in X and $\kappa(x_i) = \kappa(v_i)$.*

Proof. We will use induction on n. Namely, suppose $(U, u) \to (S, s)$ and $V_i \subset X_U$, $i = 1, \ldots, n-1$ work for $x_1, \ldots, x_{n-1}$. Since $\kappa(s) = \kappa(u)$ the fibre $(X_U)_u = X_s$. Hence there exists a unique point $x'_n \in X_u \subset X_U$ corresponding to $x_n \in X_s$. Also x'_n is isolated in X_u. Hence by Lemma 41.1 there exists an elementary étale neighbourhood $(U', u') \to (U, u)$ and an open $V_n \subset X_{U'}$ which works for x'_n and hence for x_n. By the final assertion of Lemma 41.1 the open subschemes $V'_i = U' \times_U V_i$ for $i = 1, \ldots, n-1$ still work with respect to $x_1, \ldots, x_{n-1}$. Hence we win. $\square$

If we allow a nontrivial field extension $\kappa(u)/\kappa(s)$, i.e., general étale neighbourhoods, then we can split the points as follows.

02LM **Lemma 41.3.** *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume that*

(1) *f is locally of finite type, and*
(2) *$x_i \in X_s$ is isolated for $i = 1, \ldots, n$.*

Then there exist

(a) *an étale neighbourhood $(U, u) \to (S, s)$,*
(b) *for each i an integer m_i and open subschemes $V_{i,j} \subset X_U$, $j = 1, \ldots, m_i$*

such that we have

(i) *each $V_{i,j} \to U$ is a finite morphism,*
(ii) *there is a unique point $v_{i,j}$ of $V_{i,j}$ mapping to u in U with $\kappa(u) \subset \kappa(v_{i,j})$ finite purely inseparable,*
(iv) *if $v_{i,j} = v_{i',j'}$, then $i = i'$ and $j = j'$, and*
(iii) *the points $v_{i,j}$ map to x_i in X and no other points of $(X_U)_u$ map to x_i.*

Proof. This proof is a variant of the proof of Algebra, Lemma 145.4 in the language of schemes. By Morphisms, Lemma 20.6 the morphism f is quasi-finite at each of the points x_i. Hence $\kappa(s) \subset \kappa(x_i)$ is finite for each i (Morphisms, Lemma 20.5). For each i, let $\kappa(s) \subset L_i \subset \kappa(x_i)$ be the subfield such that $L_i/\kappa(s)$ is separable, and $\kappa(x_i)/L_i$ is purely inseparable. Choose a finite Galois extension

$L/\kappa(s)$ such that there exist $\kappa(s)$-embeddings $L_i \to L$ for $i = 1, \ldots, n$. Choose an étale neighbourhood $(U, u) \to (S, s)$ such that $L \cong \kappa(u)$ as $\kappa(s)$-extensions (Lemma 35.2).

Let $y_{i,j}$, $j = 1, \ldots, m_i$ be the points of X_U lying over $x_i \in X$ and $u \in U$. By Schemes, Lemma 17.5 these points $y_{i,j}$ correspond exactly to the primes in the rings $\kappa(u) \otimes_{\kappa(s)} \kappa(x_i)$. This also explains why there are finitely many; in fact $m_i = [L_i : \kappa(s)]$ but we do not need this. By our choice of L (and elementary field theory) we see that $\kappa(u) \subset \kappa(y_{i,j})$ is finite purely inseparable for each pair i, j. Also, by Morphisms, Lemma 20.13 for example, the morphism $X_U \to U$ is quasi-finite at the points $y_{i,j}$ for all i, j.

Apply Lemma 41.2 to the morphism $X_U \to U$, the point $u \in U$ and the points $y_{i,j} \in (X_U)_u$. This gives an étale neighbourhood $(U', u') \to (U, u)$ with $\kappa(u) = \kappa(u')$ and opens $V_{i,j} \subset X_{U'}$ with the properties (i), (ii), and (iii) of that lemma. We claim that the étale neighbourhood $(U', u') \to (S, s)$ and the opens $V_{i,j} \subset X_{U'}$ are a solution to the problem posed by the lemma. We omit the verifications. $\square$

02LN **Lemma 41.4.** *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that*

1. *f is locally of finite type,*
2. *f is separated, and*
3. *$x_1, \ldots, x_n$ are pairwise distinct isolated points of X_s.*

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg V_1 \amalg \ldots \amalg V_n$$

into open and closed subschemes such that the morphisms $V_i \to U$ are finite, the fibres of $V_i \to U$ over u are singletons $\{v_i\}$, each v_i maps to x_i with $\kappa(x_i) = \kappa(v_i)$, and the fibre of $W \to U$ over u contains no points mapping to any of the x_i.

Proof. Choose $(U, u) \to (S, s)$ and $V_i \subset X_U$ as in Lemma 41.2. Since $X_U \to U$ is separated (Schemes, Lemma 21.12) and $V_i \to U$ is finite hence proper (Morphisms, Lemma 44.11) we see that $V_i \subset X_U$ is closed by Morphisms, Lemma 41.7. Hence $V_i \cap V_j$ is a closed subset of V_i which does not contain v_i. Hence the image of $V_i \cap V_j$ in U is a closed set (because $V_i \to U$ proper) not containing u. After shrinking U we may therefore assume that $V_i \cap V_j = \emptyset$ for all i, j. This gives the decomposition as in the lemma. $\square$

Here is the variant where we reduce to purely inseparable field extensions.

02LO **Lemma 41.5.** *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that*

1. *f is locally of finite type,*
2. *f is separated, and*
3. *$x_1, \ldots, x_n$ are pairwise distinct isolated points of X_s.*

Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

into open and closed subschemes such that the morphisms $V_{i,j} \to U$ are finite, the fibres of $V_{i,j} \to U$ over u are singletons $\{v_{i,j}\}$, each $v_{i,j}$ maps to x_i, $\kappa(u) \subset \kappa(v_{i,j})$

is purely inseparable, and the fibre of $W \to U$ over u contains no points mapping to any of the x_i.

Proof. This is proved in exactly the same way as the proof of Lemma 41.4 except that it uses Lemma 41.3 instead of Lemma 41.2. $\square$

The following version may be a little easier to parse.

02LP **Lemma 41.6.** *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

1. *f is locally of finite type,*
2. *f is separated, and*
3. *X_s has at most finitely many isolated points.*

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg V$$

into open and closed subschemes such that the morphism $V \to U$ is finite, and the fibre W_u of the morphism $W \to U$ contains no isolated points. In particular, if $f^{-1}(s)$ is a finite set, then $W_u = \emptyset$.

Proof. This is clear from Lemma 41.4 by choosing $x_1, \ldots, x_n$ the complete set of isolated points of X_s and setting $V = \bigcup V_i$. $\square$

42. Étale localization of integral morphisms

0BUH Some variants of the results of Section 41 for the case of integral morphisms.

0BSR **Lemma 42.1.** *Let $R \to S$ be an integral ring map. Let $\mathfrak p \subset R$ be a prime ideal. Assume*

1. *there are finitely many primes $\mathfrak q_1, \ldots, \mathfrak q_n$ lying over $\mathfrak p$, and*
2. *for each i the maximal separable subextension $\kappa(\mathfrak q)/\kappa(\mathfrak q_i)_{sep}/\kappa(\mathfrak p)$ (Fields, Lemma 14.6) is finite over $\kappa(\mathfrak p)$.*

Then there exists an étale ring map $R \to R'$ and a prime $\mathfrak p'$ lying over $\mathfrak p$ such that

$$S \otimes_R R' = A_1 \times \ldots \times A_m$$

with $R' \to A_j$ integral having a unique prime $\mathfrak r_j$ over $\mathfrak p'$ such that $\kappa(\mathfrak r_j)/\kappa(\mathfrak p')$ is purely inseparable.

First proof. This proof uses Algebra, Lemma 145.4. Namely, choose a generator $\theta_i \in \kappa(\mathfrak q_i)_{sep}$ of this field over $\kappa(\mathfrak p)$ (Fields, Lemma 19.1). The spectrum of the fibre ring $S \otimes_R \kappa(\mathfrak p)$ is finite discrete with points corresponding to $\mathfrak q_1, \ldots, \mathfrak q_n$. By the Chinese remainder theorem (Algebra, Lemma 15.4) we see that $S \otimes_R \kappa(\mathfrak p) \to \prod \kappa(\mathfrak q_i)$ is surjective. Hence after replacing R by R_g for some $g \in R$, $g \notin \mathfrak p$ we may assume that $(0, \ldots, 0, \theta_i, 0, \ldots, 0) \in \prod \kappa(\mathfrak q_i)$ is the image of some $x_i \in S$. Let $S' \subset S$ be the R-subalgebra generated by our x_i. Since $\text{Spec}(S) \to \text{Spec}(S')$ is surjective (Algebra, Lemma 36.17) we conclude that $\mathfrak q'_i = S' \cap \mathfrak q_i$ are the primes of S' over $\mathfrak p$. By our choice of x_i we conclude these primes are distinct that and $\kappa(\mathfrak q'_i)_{sep} = \kappa(\mathfrak q_i)_{sep}$. In particular the field extensions $\kappa(\mathfrak q_i)/\kappa(\mathfrak q'_i)$ are purely inseparable. Since $R \to S'$ is finite we may apply Algebra, Lemma 145.4. and we get $R \to R'$ and $\mathfrak p'$ and a decomposition

$$S' \otimes_R R' = A'_1 \times \ldots \times A'_m \times B'$$

with $R' \to A'_j$ integral having a unique prime $\mathfrak r'_j$ over $\mathfrak p'$ such that $\kappa(\mathfrak r'_j)/\kappa(\mathfrak p')$ is purely inseparable and such that B' does not have a prime lying over $\mathfrak p'$. Since

$R' \to B'$ is finite (as $R \to S'$ is finite) we can after localizing R' at some $g' \in R'$, $g' \notin \mathfrak{p}'$ assume that $B' = 0$. Via the map $S' \otimes_R R' \to S \otimes_R R'$ we get the corresponding decomposition for S. □

Second proof. This proof uses strict henselization. First, assume R is strictly henselization with maximal ideal $\mathfrak{p}$. Then $S/\mathfrak{p}S$ has finitely many primes corresponding to $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$, each maximal, each with purely inseparable residue field over $\kappa(\mathfrak{p})$. Hence $S/\mathfrak{p}S$ is equal to $\prod (S/\mathfrak{p}S)_{\mathfrak{p}_i}$. By More on Algebra, Lemma 11.6 we can lift this product decomposition to a product composition of S as in the statement.

In the general case, let R^{sh} be the strict henselization of $R_\mathfrak{p}$. Then we can apply the result of the first paragraph to $R^{sh} \to S \otimes_R R^{sh}$. Consider the m mutually orthogonal idempotents in $S \otimes_R R^{sh}$ corresponding to the product decomposition. Since R^{sh} is a filtered colimit of étale ring maps $(R, \mathfrak{p}) \to (R', \mathfrak{p}')$ by Algebra, Lemma 155.11 we see that these idempotents descend to some R' as desired. □

43. Zariski's Main Theorem

02LQ In this section we prove Zariski's main theorem as reformulated by Grothendieck. Often when we say "Zariski's main theorem" in this content we mean either of Lemma 43.1, Lemma 43.2, or Lemma 43.3. In most texts people refer to the last of these as Zariski's main theorem.

We have already proved the algebraic version in Algebra, Theorem 123.12 and we have already restated this algebraic version in the language of schemes, see Morphisms, Theorem 56.1. The version in this section is more subtle; to get the full result we use the étale localization techniques of Section 41 to reduce to the algebraic case.

03GW **Lemma 43.1.** *Let $f : X \to S$ be a morphism of schemes. Assume f is of finite type and separated. Let S' be the normalization of S in X, see Morphisms, Definition 53.3. Picture:*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & S' \arrow[dl, "\nu"] \\ & S & \end{tikzcd}$$

Then there exists an open subscheme $U' \subset S'$ such that

1. *$(f')^{-1}(U') \to U'$ is an isomorphism, and*
2. *$(f')^{-1}(U') \subset X$ is the set of points at which f is quasi-finite.*

Proof. By Morphisms, Lemma 56.2 the subset $U \subset X$ of points where f is quasi-finite is open. The lemma is equivalent to

(a) $U' = f'(U) \subset S'$ is open,
(b) $U = (f')^{-1}(U')$, and
(c) $U \to U'$ is an isomorphism.

Let $x \in U$ be arbitrary. We claim there exists an open neighbourhood $f'(x) \in V \subset S'$ such that $(f')^{-1}V \to V$ is an isomorphism. We first prove the claim implies the lemma. Namely, then $(f')^{-1}V \cong V$ is both locally of finite type over S (as an open subscheme of X) and for $v \in V$ the residue field extension $\kappa(v)/\kappa(\nu(v))$ is algebraic (as $V \subset S'$ and S' is integral over S). Hence the fibres of $V \to S$ are discrete

(Morphisms, Lemma 20.2) and $(f')^{-1}V \to S$ is locally quasi-finite (Morphisms, Lemma 20.8). This implies $(f')^{-1}V \subset U$ and $V \subset U'$. Since x was arbitrary we see that (a), (b), and (c) are true.

Let $s = f(x)$. Let $(T, t) \to (S, s)$ be an elementary étale neighbourhood. Denote by a subscript $_T$ the base change to T. Let $y = (x, t) \in X_T$ be the unique point in the fibre X_t lying over x. Note that $U_T \subset X_T$ is the set of points where f_T is quasi-finite, see Morphisms, Lemma 20.13. Note that

$$X_T \xrightarrow{f'_T} S'_T \xrightarrow{\nu_T} T$$

is the normalization of T in X_T, see Lemma 19.2. Suppose that the claim holds for $y \in U_T \subset X_T \to S'_T \to T$, i.e., suppose that we can find an open neighbourhood $f'_T(y) \in V' \subset S'_T$ such that $(f'_T)^{-1}V' \to V'$ is an isomorphism. The morphism $S'_T \to S'$ is étale hence the image $V \subset S'$ of V' is open. Observe that $f'(x) \in V$ as $f'_T(y) \in V'$. Observe that

$$\begin{tikzcd} (f'_T)^{-1}V' \arrow[r] \arrow[d] & (f')^{-1}(V) \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

is a fibre square (as $S'_T \times_{S'} X = X_T$). Since the left vertical arrow is an isomorphism and $\{V' \to V\}$ is a étale covering, we conclude that the right vertical arrow is an isomorphism by Descent, Lemma 23.17. In other words, the claim holds for $x \in U \subset X \to S' \to S$.

By the result of the previous paragraph we may replace S by an elementary étale neighbourhood of $s = f(x)$ in order to prove the claim. Thus we may assume there is a decomposition

$$X = V \amalg W$$

into open and closed subschemes where $V \to S$ is finite and $x \in V$, see Lemma 41.4. Since X is a disjoint union of V and W over S and since $V \to S$ is finite we see that the normalization of S in X is the morphism

$$X = V \amalg W \longrightarrow V \amalg W' \longrightarrow S$$

where W' is the normalization of S in W, see Morphisms, Lemmas 53.10, 44.4, and 53.12. The claim follows and we win. □

02LR **Lemma 43.2.** *Let $f : X \to S$ be a morphism of schemes. Assume f is quasi-finite and separated. Let S' be the normalization of S in X, see Morphisms, Definition 53.3. Picture:*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & S' \arrow[dl, "\nu"] \\ & S & \end{tikzcd}$$

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. This follows from Lemma 43.1. Namely, by that lemma there exists an open subscheme $U' \subset S'$ such that $(f')^{-1}(U') = X$ and $X \to U'$ is an isomorphism. In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : S' \to S$ is separated (Schemes, Lemma 21.14). It follows that f is quasi-affine by Morphisms, Lemma 13.3. $\square$

05K0 **Lemma 43.3** (Zariski's Main Theorem)**.** *Let $f : X \to S$ be a morphism of schemes. Assume f is quasi-finite and separated and assume that S is quasi-compact and quasi-separated. Then there exists a factorization* [DG67, IV Corollary 18.12.13]

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & T \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

where j is a quasi-compact open immersion and π is finite.

Proof. Let $X \to S' \to S$ be as in the conclusion of Lemma 43.2. By Properties, Lemma 22.13 we can write $\nu_*\mathcal{O}_{S'} = \text{colim}_{i \in I} \mathcal{A}_i$ as a directed colimit of finite quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_i \subset \nu_*\mathcal{O}_{S'}$. Then $\pi_i : T_i = \underline{\text{Spec}}_S(\mathcal{A}_i) \to S$ is a finite morphism for each i. Note that the transition morphisms $T_{i'} \to T_i$ are affine and that $S' = \lim T_i$.

By Limits, Lemma 4.11 there exists an i and a quasi-compact open $U_i \subset T_i$ whose inverse image in S' equals $f'(X)$. For $i' \geq i$ let $U_{i'}$ be the inverse image of U_i in $T_{i'}$. Then $X \cong f'(X) = \lim_{i' \geq i} U_{i'}$, see Limits, Lemma 2.2. By Limits, Lemma 4.16 we see that $X \to U_{i'}$ is a closed immersion for some $i' \geq i$. (In fact $X \cong U_{i'}$ for sufficiently large i' but we don't need this.) Hence $X \to T_{i'}$ is an immersion. By Morphisms, Lemma 3.2 we can factor this as $X \to T \to T_{i'}$ where the first arrow is an open immersion and the second a closed immersion. Thus we win. $\square$

0F2N **Lemma 43.4.** *With notation and hypotheses as in Lemma 43.3. Assume moreover that f is locally of finite presentation. Then we can choose the factorization such that T is finite and of finite presentation over S.*

Proof. By Limits, Lemma 9.8 we can write $T = \lim T_i$ where all T_i are finite and of finite presentation over Y and the transition morphisms $T_{i'} \to T_i$ are closed immersions. By Limits, Lemma 4.11 there exists an i and an open subscheme $U_i \subset T_i$ whose inverse image in T is X. By Limits, Lemma 4.16 we see that $X \cong U_i$ for large enough i. Replacing T by T_i finishes the proof. $\square$

44. Applications of Zariski's Main Theorem, I

0F2P A first application is the characterization of finite morphisms as proper morphisms with finite fibres.

02LS **Lemma 44.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is finite,*
(2) *f is proper with finite fibres,*
(3) *f is proper and locally quasi-finite,*
(4) *f is universally closed, separated, locally of finite type and has finite fibres.*

Proof. We have (1) implies (2) by Morphisms, Lemmas 44.11, 20.10, and 44.10. We have (2) implies (3) by Morphisms, Lemma 20.7. We have (3) implies (4) by the definition of proper morphisms and Morphisms, Lemmas 20.9 and 20.10.

Assume (4). Pick $s \in S$. By Morphisms, Lemma 20.7 we see that all the finitely many points of X_s are isolated in X_s. Choose an elementary étale neighbourhood $(U, u) \to (S, s)$ and decomposition $X_U = V \amalg W$ as in Lemma 41.6. Note that $W_u = \emptyset$ because all points of X_s are isolated. Since f is universally closed we see that the image of W in U is a closed set not containing u. After shrinking U we may assume that $W = \emptyset$. In other words we see that $X_U = V$ is finite over U. Since $s \in S$ was arbitrary this means there exists a family $\{U_i \to S\}$ of étale morphisms whose images cover S such that the base changes $X_{U_i} \to U_i$ are finite. Note that $\{U_i \to S\}$ is an étale covering, see Topologies, Definition 4.1. Hence it is an fpqc covering, see Topologies, Lemma 9.7. Hence we conclude f is finite by Descent, Lemma 23.23. □

As a consequence we have the following useful results.

02UP **Lemma 44.2.** *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that f is proper and $f^{-1}(\{s\})$ is a finite set. Then there exists an open neighbourhood $V \subset S$ of s such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.*

Proof. The morphism f is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 20.7. By Morphisms, Lemma 56.2 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since f is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ of s with $f(Z) \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 20.9), hence has finite fibres (Morphisms, Lemma 20.10), hence is finite by Lemma 44.1. □

0AH8 **Lemma 44.3.** *Consider a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Let $s \in S$. Assume

1. *$X \to S$ is a proper morphism,*
2. *$Y \to S$ is separated and locally of finite type, and*
3. *the image of $X_s \to Y_s$ is finite.*

Then there is an open subspace $U \subset S$ containing s such that $X_U \to Y_U$ factors through a closed subscheme $Z \subset Y_U$ finite over U.

Proof. Let $Z \subset Y$ be the scheme theoretic image of h, see Morphisms, Section 6. By Morphisms, Lemma 41.10 the morphism $X \to Z$ is surjective and $Z \to S$ is proper. Thus $X_s \to Z_s$ is surjective. We see that either (3) implies Z_s is finite. Hence $Z \to S$ is finite in an open neighbourhood of s by Lemma 44.2. □

45. Applications of Zariski's Main Theorem, II

0F2Q In this section we give a few more consequences of Zariski's main theorem to the structure of quasi-finite morphisms.

07S0 **Lemma 45.1.** *Let $f : X \to Y$ be a separated, locally quasi-finite morphism with Y affine. Then every finite set of points of X is contained in an open affine of X.*

Proof. Let $x_1, \ldots, x_n \in X$. Choose a quasi-compact open $U \subset X$ with $x_i \in U$. Then $U \to Y$ is quasi-affine by Lemma 43.2. Hence there exists an affine open $V \subset U$ containing $x_1, \ldots, x_n$ by Properties, Lemma 29.5. $\square$

03I1 **Lemma 45.2.** *Let $f : Y \to X$ be a quasi-finite morphism. There exists a dense open $U \subset X$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite.*

Proof. If $U_i \subset X$, $i \in I$ is a collection of opens such that the restrictions $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ are finite, then with $U = \bigcup U_i$ the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite, see Morphisms, Lemma 44.3. Thus the problem is local on X and we may assume that X is affine.

Assume X is affine. Write $Y = \bigcup_{j=1,\ldots,m} V_j$ with V_j affine. This is possible since f is quasi-finite and hence in particular quasi-compact. Each $V_j \to X$ is quasi-finite and separated. Let $\eta \in X$ be a generic point of an irreducible component of X. We see from Morphisms, Lemmas 20.10 and 51.1 that there exists an open neighbourhood $\eta \in U_\eta$ such that $f^{-1}(U_\eta) \cap V_j \to U_\eta$ is finite. We may choose U_η such that it works for each $j = 1, \ldots, m$. Note that the collection of generic points of X is dense in X. Thus we see there exists a dense open $W = \bigcup_\eta U_\eta$ such that each $f^{-1}(W) \cap V_j \to W$ is finite. It suffices to show that there exists a dense open $U \subset W$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. Thus we may replace X by an affine open subscheme of W and assume that each $V_j \to X$ is finite.

Assume X is affine, $Y = \bigcup_{j=1,\ldots,m} V_j$ with V_j affine, and the restrictions $f|_{V_j} : V_j \to X$ are finite. Set

$$\Delta_{ij} = \left(\overline{V_i \cap V_j} \setminus V_i \cap V_j\right) \cap V_j.$$

This is a nowhere dense closed subset of V_j because it is the boundary of the open subset $V_i \cap V_j$ in V_j. By Morphisms, Lemma 48.7 the image $f(\Delta_{ij})$ is a nowhere dense closed subset of X. By Topology, Lemma 21.2 the union $T = \bigcup f(\Delta_{ij})$ is a nowhere dense closed subset of X. Thus $U = X \setminus T$ is a dense open subset of X. We claim that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. To see this let $U' \subset U$ be an affine open. Set $Y' = f^{-1}(U') = U' \times_X Y$, $V'_j = Y' \cap V_j = U' \times_X V_j$. Consider the restriction

$$f' = f|_{Y'} : Y' \longrightarrow U'$$

of f. This morphism now has the property that $Y' = \bigcup_{j=1,\ldots,m} V'_j$ is an affine open covering, each $V'_j \to U'$ is finite, and $V'_i \cap V'_j$ is (open and) closed both in V'_i and V'_j. Hence $V'_i \cap V'_j$ is affine, and the map

$$\mathcal{O}(V'_i) \otimes_{\mathbf{Z}} \mathcal{O}(V'_j) \longrightarrow \mathcal{O}(V'_i \cap V'_j)$$

is surjective. This implies that Y' is separated, see Schemes, Lemma 21.7. Finally, consider the commutative diagram

$$\begin{tikzcd} \coprod_{j=1,\ldots,m} V'_j \arrow[rr] \arrow[rd] & & Y' \arrow[ld] \\ & U' & \end{tikzcd}$$

The south-east arrow is finite, hence proper, the horizontal arrow is surjective, and the south-west arrow is separated. Hence by Morphisms, Lemma 41.9 we conclude that $Y' \to U'$ is proper. Since it is also quasi-finite, we see that it is finite by Lemma 44.1, and we win. $\square$

07RY **Lemma 45.3.** *Let $f : X \to S$ be flat, locally of finite presentation, separated, locally quasi-finite with universally bounded fibres. Then there exist closed subsets*

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S$$

such that with $S_r = Z_r \setminus Z_{r-1}$ the stratification $S = \coprod_{r=0,\ldots,n} S_r$ is characterized by the following universal property: Given $g : T \to S$ the projection $X \times_S T \to T$ is finite locally free of degree r if and only if $g(T) \subset S_r$ (set theoretically).

Proof. Let n be an integer bounding the degree of the fibres of $X \to S$. By Morphisms, Lemma 57.5 we see that any base change has degrees of fibres bounded by n also. In particular, all the integers r that occur in the statement of the lemma will be $\leq n$. We will prove the lemma by induction on n. The base case is $n = 0$ which is obvious.

We claim the set of points $s \in S$ with $\deg_{\kappa(s)}(X_s) = n$ is an open subset $S_n \subset S$ and that $X \times_S S_n \to S_n$ is finite locally free of degree n. Namely, suppose that $s \in S$ is such a point. Choose an elementary étale morphism $(U, u) \to (S, s)$ and a decomposition $U \times_S X = W \amalg V$ as in Lemma 41.6. Since $V \to U$ is finite, flat, and locally of finite presentation, we see that $V \to U$ is finite locally free, see Morphisms, Lemma 48.2. After shrinking U to a smaller neighbourhood of u we may assume $V \to U$ is finite locally free of some degree d, see Morphisms, Lemma 48.5. As $u \mapsto s$ and $W_u = \emptyset$ we see that $d = n$. Since n is the maximum degree of a fibre we see that $W = \emptyset$! Thus $U \times_S X \to U$ is finite locally free of degree n. By Descent, Lemma 23.30 we conclude that $X \to S$ is finite locally free of degree n over $\text{Im}(U \to S)$ which is an open neighbourhood of s (Morphisms, Lemma 36.13). This proves the claim.

Let $S' = S \setminus S_n$ endowed with the reduced induced scheme structure and set $X' = X \times_S S'$. Note that the degrees of fibres of $X' \to S'$ are universally bounded by $n - 1$. By induction we find a stratification $S' = S_0 \amalg \ldots \amalg S_{n-1}$ adapted to the morphism $X' \to S'$. We claim that $S = \coprod_{r=0,\ldots,n} S_r$ works for the morphism $X \to S$. Let $g : T \to S$ be a morphism of schemes and assume that $X \times_S T \to T$ is finite locally free of degree r. As remarked above this implies that $r \leq n$. If $r = n$, then it is clear that $T \to S$ factors through S_n. If $r < n$, then $g(T) \subset S' = S \setminus S_d$ (set theoretically) hence $T_{red} \to S$ factors through S', see Schemes, Lemma 12.7. Note that $X \times_S T_{red} \to T_{red}$ is also finite locally free of degree r as a base change. By the universal property of the stratification $S' = \coprod_{r=0,\ldots,n-1} S_r$ we see that $g(T) = g(T_{red})$ is contained in S_r. Conversely, suppose that we have $g : T \to S$ such that $g(T) \subset S_r$ (set theoretically). If $r = n$, then g factors through S_n and it is clear that $X \times_S T \to T$ is finite locally free of degree n as a base change. If $r < n$, then $X \times_S T \to T$ is a morphism which is separated, flat, and locally of finite presentation, such that the restriction to T_{red} is finite locally free of degree r. Since $T_{red} \to T$ is a universal homeomorphism, we conclude that $X \times_S T_{red} \to X \times_S T$ is a universal homeomorphism too and hence $X \times_S T \to T$ is universally closed (as this is true for the finite morphism $X \times_S T_{red} \to T_{red}$). It follows that $X \times_S T \to T$ is finite, for example by Lemma 44.1. Then we can use Morphisms, Lemma 48.2 to see that $X \times_S T \to T$ is finite locally free. Finally, the degree is r as all the fibres have degree r. $\square$

07RZ **Lemma 45.4.** *Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, separated, and quasi-finite. Then there exist closed subsets*

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset S$$

such that with $S_r = Z_r \setminus Z_{r-1}$ the stratification $S = \coprod S_r$ is characterized by the following universal property: Given a morphism $g : T \to S$ the projection $X \times_S T \to T$ is finite locally free of degree r if and only if $g(T) \subset S_r$ (set theoretically). Moreover, the inclusion maps $S_r \to S$ are quasi-compact.

Proof. The question is local on S, hence we may assume that S is affine. By Morphisms, Lemma 57.9 the fibres of f are universally bounded in this case. Hence the existence of the stratification follows from Lemma 45.3.

We will show that $U_r = S \setminus Z_r \to S$ is quasi-compact for each $r \geq 0$. This will prove the final statement by elementary topology. Since a composition of quasi-compact maps is quasi-compact it suffices to prove that $U_r \to U_{r-1}$ is quasi-compact. Choose an affine open $W \subset U_{r-1}$. Write $W = \mathrm{Spec}(A)$. Then $Z_r \cap W = V(I)$ for some ideal $I \subset A$ and $X \times_S \mathrm{Spec}(A/I) \to \mathrm{Spec}(A/I)$ is finite locally free of degree r. Note that $A/I = \mathrm{colim}\, A/I_i$ where $I_i \subset I$ runs through the finitely generated ideals. By Limits, Lemma 8.8 we see that $X \times_S \mathrm{Spec}(A/I_i) \to \mathrm{Spec}(A/I_i)$ is finite locally free of degree r for some i. (This uses that $X \to S$ is of finite presentation, as it is locally of finite presentation, separated, and quasi-compact.) Hence $\mathrm{Spec}(A/I_i) \to \mathrm{Spec}(A) = W$ factors (set theoretically) through $Z_r \cap W$. It follows that $Z_r \cap W = V(I_i)$ is the zero set of a finite subset of elements of A. This means that $W \setminus Z_r$ is a finite union of standard opens, hence quasi-compact, as desired. □

086R **Lemma 45.5.** *Let $f : X \to S$ be a flat, locally of finite presentation, separated, and locally quasi-finite morphism of schemes. Then there exist open subschemes*

$$S = U_0 \supset U_1 \supset U_2 \supset \ldots$$

such that a morphism $\mathrm{Spec}(k) \to S$ where k is a field factors through U_d if and only if $X \times_S \mathrm{Spec}(k)$ has degree $\geq d$ over k.

Proof. The statement simply means that the collection of points where the degree of the fibre is $\geq d$ is open. Thus we can work locally on S and assume S is affine. In this case, for every $W \subset X$ quasi-compact open, the set of points $U_d(W)$ where the fibres of $W \to S$ have degree $\geq d$ is open by Lemma 45.4. Since $U_d = \bigcup_W U_d(W)$ the result follows. □

082V **Lemma 45.6.** *Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, and locally quasi-finite. Let $g \in \Gamma(X, \mathcal{O}_X)$ nonzero. Then there exist an open $V \subset X$ such that $g|_V \neq 0$, an open $U \subset S$ fitting into a commutative diagram*

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\pi"'] & X \arrow[d, "f"] \\ U \arrow[r] & S, \end{tikzcd}$$

a quasi-coherent subsheaf $\mathcal{F} \subset \mathcal{O}_U$, an integer $r > 0$, and an injective $\mathcal{O}_U$-module map $\mathcal{F}^{\oplus r} \to \pi_*\mathcal{O}_V$ whose image contains $g|_V$.

Proof. We may assume X and S affine. We obtain a filtration $\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S$ as in Lemmas 45.3 and 45.4. Let $T \subset X$ be the scheme theoretic support of the finite $\mathcal{O}_X$-module $\mathrm{Im}(g : \mathcal{O}_X \to \mathcal{O}_X)$. Note that T is the support of g as a section of $\mathcal{O}_X$ (Modules, Definition 5.1) and for any open $V \subset X$ we have $g|_V \neq 0$ if and only if $V \cap T \neq \emptyset$. Let r be the smallest integer such that $f(T) \subset Z_r$ set theoretically. Let $\xi \in T$ be a generic point of an irreducible component of T such that $f(\xi) \notin Z_{r-1}$ (and hence $f(\xi) \in Z_r$). We may replace S by an affine neighbourhood of $f(\xi)$ contained in $S \setminus Z_{r-1}$. Write $S = \mathrm{Spec}(A)$ and let $I = (a_1, \ldots, a_m) \subset A$ be a finitely generated ideal such that $V(I) = Z_r$ (set theoretically, see Algebra, Lemma 29.1). Since the support of g is contained in $f^{-1}V(I)$ by our choice of r we see that there exists an integer N such that $a_j^N g = 0$ for $j = 1, \ldots, m$. Replacing a_j by a_j^r we may assume that $Ig = 0$. For any A-module M write $M[I]$ for the I-torsion of M, i.e., $M[I] = \{m \in M \mid Im = 0\}$. Write $X = \mathrm{Spec}(B)$, so $g \in B[I]$. Since $A \to B$ is flat we see that

$$B[I] = A[I] \otimes_A B \cong A[I] \otimes_{A/I} B/IB$$

By our choice of Z_r, the A/I-module B/IB is finite locally free of rank r. Hence after replacing S by a smaller affine open neighbourhood of $f(\xi)$ we may assume that $B/IB \cong (A/IA)^{\oplus r}$ as A/I-modules. Choose a map $\psi : A^{\oplus r} \to B$ which reduces modulo I to the isomorphism of the previous sentence. Then we see that the induced map

$$A[I]^{\oplus r} \longrightarrow B[I]$$

is an isomorphism. The lemma follows by taking $\mathcal{F}$ the quasi-coherent sheaf associated to the A-module $A[I]$ and the map $\mathcal{F}^{\oplus r} \to \pi_*\mathcal{O}_V$ the one corresponding to $A[I]^{\oplus r} \subset A^{\oplus r} \to B$. $\square$

09Z0 **Lemma 45.7.** *Let $U \to X$ be a surjective étale morphism of schemes. Assume X is quasi-compact and quasi-separated. Then there exists a surjective integral morphism $Y \to X$, such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through U. In fact, we may assume $Y \to X$ is finite and of finite presentation.*

Proof. Since X is quasi-compact, there exist finitely many affine opens $U_i \subset U$ such that $U' = \coprod U_i \to X$ is surjective. After replacing U by U', we see that we may assume U is affine. In particular $U \to X$ is separated (Schemes, Lemma 21.15). Then there exists an integer d bounding the degree of the geometric fibres of $U \to X$ (see Morphisms, Lemma 57.9). We will prove the lemma by induction on d for all quasi-compact and separated schemes U mapping surjective and étale onto X. If $d = 1$, then $U = X$ and the result holds with $Y = U$. Assume $d > 1$.

We apply Lemma 43.2 and we obtain a factorization

$$\begin{tikzcd} U \arrow[rr, "j"] \arrow[dr] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

with π integral and j a quasi-compact open immersion. We may and do assume that $j(U)$ is scheme theoretically dense in Y. Note that

$$U \times_X Y = U \amalg W$$

where the first summand is the image of $U \to U \times_X Y$ (which is closed by Schemes, Lemma 21.10 and open because it is étale as a morphism between schemes étale over Y) and the second summand is the (open and closed) complement. The image $V \subset Y$ of W is an open subscheme containing $Y \setminus U$.

The étale morphism $W \to Y$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $U \subset Y$ by inspection. Since $U \subset Y$ is dense, it holds for all geometric points of Y for example by Lemma 45.3 (the degree of the fibres of a quasi-compact separated étale morphism does not go up under specialization). Thus we may apply the induction hypothesis to $W \to V$ and find a surjective integral morphism $Z \to V$ with Z a scheme, which Zariski locally factors through W. Choose a factorization $Z \to Z' \to Y$ with $Z' \to Y$ integral and $Z \to Z'$ open immersion (Lemma 43.2). After replacing Z' by the scheme theoretic closure of Z in Z' we may assume that Z is scheme theoretically dense in Z'. After doing this we have $Z' \times_Y V = Z$. Finally, let $T \subset Y$ be the induced reduced closed subscheme structure on $Y \setminus V$. Consider the morphism

$$Z' \amalg T \longrightarrow X$$

This is a surjective integral morphism by construction. Since $T \subset U$ it is clear that the morphism $T \to X$ factors through U. On the other hand, let $z \in Z'$ be a point. If $z \notin Z$, then z maps to a point of $Y \setminus V \subset U$ and we find a neighbourhood of z on which the morphism factors through U. If $z \in Z$, then we have a neighbourhood $\Omega \subset Z$ which factors through $W \subset U \times_X Y$ and hence through U. This proves existence.

Assume we have found $Y \to X$ integral and surjective which Zariski locally factors through U. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through U. We can write $Y = \lim Y_i$ with $Y_i \to X$ finite and of finite presentation, see Limits, Lemma 7.3. For large enough i we can find affine opens $V_{i,j} \subset Y_i$ whose inverse image in Y recovers V_j, see Limits, Lemma 4.11. For even larger i the morphisms $V_j \to U$ over X come from morphisms $V_{i,j} \to U$ over X, see Limits, Proposition 6.1. This finishes the proof. $\square$

46. Application to morphisms with connected fibres

057H In this section we prove some lemmas that produce morphisms all of whose fibres are geometrically connected or geometrically integral. This will be useful in our study of the local structure of morphisms of finite type later.

057I **Lemma 46.1.** *Consider a diagram of morphisms of schemes*

$$\begin{tikzcd} Z \arrow[r, "\sigma"'] \arrow[dr] & X \arrow[d] \\ & Y \end{tikzcd}$$

an a point $y \in Y$. Assume

1. *$X \to Y$ is of finite presentation and flat,*
2. *$Z \to Y$ is finite locally free,*
3. *$Z_y \neq \emptyset$,*
4. *all fibres of $X \to Y$ are geometrically reduced, and*
5. *X_y is geometrically connected over $\kappa(y)$.*

Then there exists a quasi-compact open $X^0 \subset X$ such that $X^0_y = X_y$ and such that all nonempty fibres of $X^0 \to Y$ are geometrically connected.

Proof. In this proof we will use that flat, finite presentation, finite locally free are properties that are preserved under base change and composition. We will also use that a finite locally free morphism is both open and closed. You can find these facts as Morphisms, Lemmas 25.8, 21.4, 48.4, 25.6, 21.3, 48.3, 25.10, and 44.11.

Note that $X_Z \to Z$ is flat morphism of finite presentation which has a section s coming from σ. Let X_Z^0 denote the subset of X_Z defined in Situation 29.1. By Lemma 29.6 it is an open subset of X_Z.

The pullback $X_{Z \times_Y Z}$ of X to $Z \times_Y Z$ comes equipped with two sections s_0, s_1, namely the base changes of s by $\text{pr}_0, \text{pr}_1 : Z \times_Y Z \to Z$. The construction of Situation 29.1 gives two subsets $(X_{Z\times_Y Z})^0_{s_0}$ and $(X_{Z\times_Y Z})^0_{s_1}$. By Lemma 29.2 these are the inverse images of X_Z^0 under the morphisms $1_X \times \text{pr}_0, 1_X \times \text{pr}_1 : X_{Z\times_Y Z} \to X_Z$. In particular these subsets are open.

Let $(Z \times_Y Z)_y = \{z_1, \ldots, z_n\}$. As X_y is geometrically connected, we see that the fibres of $(X_{Z\times_Y Z})^0_{s_0}$ and $(X_{Z\times_Y Z})^0_{s_1}$ over each z_i agree (being equal to the whole fibre). Another way to say this is that

$$s_0(z_i) \in (X_{Z\times_Y Z})^0_{s_1} \quad \text{and} \quad s_1(z_i) \in (X_{Z\times_Y Z})^0_{s_0}.$$

Since the sets $(X_{Z\times_Y Z})^0_{s_0}$ and $(X_{Z\times_Y Z})^0_{s_1}$ are open in $X_{Z\times_Y Z}$ there exists an open neighbourhood $W \subset Z \times_Y Z$ of $(Z \times_Y Z)_y$ such that

$$s_0(W) \subset (X_{Z\times_Y Z})^0_{s_1} \quad \text{and} \quad s_1(W) \subset (X_{Z\times_Y Z})^0_{s_0}.$$

Then it follows directly from the construction in Situation 29.1 that

$$p^{-1}(W) \cap (X_{Z\times_Y Z})^0_{s_0} = p^{-1}(W) \cap (X_{Z\times_Y Z})^0_{s_1}$$

where $p : X_{Z\times_Y Z} \to Z \times_W Z$ is the projection. Because $Z \times_Y Z \to Y$ is finite locally free, hence open and closed, there exists an affine open neighbourhood $V \subset Y$ of y such that $q^{-1}(V) \subset W$, where $q : Z \times_Y Z \to Y$ is the structure morphism. To prove the lemma we may replace Y by V. After we do this we see that $X_Z^0 \subset Y_Z$ is an open such that

$$(1_X \times \text{pr}_0)^{-1}(X_Z^0) = (1_X \times \text{pr}_1)^{-1}(X_Z^0).$$

This means that the image $X^0 \subset X$ of X_Z^0 is an open such that $(X_Z \to X)^{-1}(X^0) = X_Z^0$, see Descent, Lemma 13.6. Finally, X^0 is quasi-compact because X_Z^0 is quasi-compact by Lemma 29.4 (use that at this point Y is affine, hence X is quasi-compact and quasi-separated, hence locally constructible is the same as constructible and in particular quasi-compact; details omitted). In this way we see that X^0 has all the desired properties. □

055W **Lemma 46.2.** *Let $h : Y \to S$ be a morphism of schemes. Let $s \in S$ be a point. Let $T \subset Y_s$ be an open subscheme. Assume*

1. *h is flat and of finite presentation,*
2. *all fibres of h are geometrically reduced, and*
3. *T is geometrically connected over $\kappa(s)$.*

Then we can find an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $V \subset Y_{S'}$ such that

(a) *all fibres of $V \to S'$ are geometrically connected,*

(b) $V_{s'} = T \times_s s'$.

Proof. The problem is clearly local on S, hence we may replace S by an affine open neighbourhood of s. The topology on Y_s is induced from the topology on Y, see Schemes, Lemma 18.5. Hence we can find a quasi-compact open $V \subset Y$ such that $V_s = T$. The restriction of h to V is quasi-compact (as S affine and V quasi-compact), quasi-separated, locally of finite presentation, and flat hence flat of finite presentation. Thus after replacing Y by V we may assume, in addition to (1) and (2) that $Y_s = T$ and S affine.

Pick a closed point $y \in Y_s$ such that h is Cohen-Macaulay at y, see Lemma 22.7. By Lemma 23.4 there exists a diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$

such that $Z \to S$ is flat, locally of finite presentation, locally quasi-finite with $Z_s = \{y\}$. Apply Lemma 41.1 to find an elementary neighbourhood $(S', s') \to (S, s)$ and an open $Z' \subset Z_{S'} = S' \times_S Z$ with $Z' \to S'$ finite with a unique point $z' \in Z'$ lying over s. Note that $Z' \to S'$ is also locally of finite presentation and flat (as an open of the base change of $Z \to S$), hence $Z' \to S'$ is finite locally free, see Morphisms, Lemma 48.2. Note that $Y_{S'} \to S'$ is flat and of finite presentation with geometrically reduced fibres as a base change of h. Also $Y_{s'} = Y_s$ is geometrically connected. Apply Lemma 46.1 to $Z' \to Y_{S'}$ over S' to get $V \subset Y_{S'}$ quasi-compact open satisfying (2) whose fibres over S' are either empty or geometrically connected. As $V \to S'$ is open (Morphisms, Lemma 25.10), after replacing S' by an affine open neighbourhood of s' we may assume $V \to S'$ is surjective, whence (1) holds. $\square$

0EY5 **Lemma 46.3.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite presentation and flat with geometrically reduced fibres. Then there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that $X_i \to S$ factors as $X_i \to S_i \to S$ where $S_i \to S$ is étale and $X_i \to S_i$ is flat of finite presentation with geometrically connected and geometrically reduced fibres.*

Proof. Pick a point $x \in X$ with image $s \in S$. We will produce a diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[dr] & S' \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ & S' \arrow[r] & S \end{tikzcd}$$

and points $s' \in S'$, $x' \in X'$, $y \in S' \times_S X$ such that x' maps to x, $(S', s') \to (S, s)$ is an étale neighbourhood, $(X', x') \to (S' \times_S X, y)$ is an étale neighbourhood[8], and $X' \to S'$ has geometrically connected fibres. If we can do this for every $x \in X$, then the lemma follows (with members of the covering given by the collection of étale morphisms $X' \to X$ so produced). The first step is the replace X and S by affine open neighbourhoods of x and s which reduces us to the case that X and S are affine (and hence f of finite presentation).

[8]The proof actually gives an open $X' \subset S' \times_S X$.

Choose a separable algebraic extension $\overline{k}$ of $\kappa(s)$. Denote $X_{\overline{k}}$ the base change of X_s. Choose a point $\overline{x}$ in $X_{\overline{k}}$ mapping to $x \in X_s$. Choose a connected quasi-compact open neighbourhood $\overline{V} \subset X_{\overline{k}}$ of $\overline{x}$. (This is possible because any scheme locally of finite type over a field is locally connected as a locally Noetherian topological space.) By Varieties, Lemma 7.9 we can find a finite separable extension $k'/\kappa(s)$ and a quasi-compact open $V' \subset X_{k'}$ whose base change is $\overline{V}$. In particular V' is geometrically connected over k', see Varieties, Lemma 7.7. By Lemma 35.2 we can find an étale neighbourhood $(S', s') \to (S, s)$ such that $\kappa(s')$ is isomorphic to k' as an extension of $\kappa(s)$. Denote $x' \in (S' \times_S X)_{s'} = X_{k'}$ the image of $\overline{x}$. Thus after replacing (S, s) by (S', s') and (X, x) by $(S' \times_S X, x')$ we reduce to the case handled in the next paragraph.

Assume there is a quasi-compact open $V \subset X_s$ which contains x and is geometrically connected. Then we can apply Lemma 46.2 to find an affine étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $X' \subset S' \times_S X$ such that $X' \to S'$ has geometrically connected fibres and such that X' contains a point mapping to x. This finishes the proof. □

057J **Lemma 46.4.** *Let $h : Y \to S$ be a morphism of schemes. Let $s \in S$ be a point. Let $T \subset Y_s$ be an open subscheme. Assume*

(1) *h is of finite presentation,*
(2) *h is normal, and*
(3) *T is geometrically irreducible over $\kappa(s)$.*

Then we can find an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $V \subset Y_{S'}$ such that

(a) *all fibres of $V \to S'$ are geometrically integral,*
(b) *$V_{s'} = T \times_s s'$.*

Proof. Apply Lemma 46.2 to find an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $V \subset Y_{S'}$ such that all fibres of $V \to S'$ are geometrically connected and $V_{s'} = T \times_s s'$. As V is an open of the base change of h all fibres of $V \to S'$ are geometrically normal, see Lemma 20.2. In particular, they are geometrically reduced. To finish the proof we have to show they are geometrically irreducible. But, if $t \in S'$ then V_t is of finite type over $\kappa(t)$ and hence $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is of finite type over $\overline{\kappa(t)}$ hence Noetherian. By choice of $S' \to S$ the scheme $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is connected. Hence $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is irreducible by Properties, Lemma 7.6 and we win. □

47. Application to the structure of finite type morphisms

052D The result in this section can be found in [GR71]. Loosely stated it says that a finite type morphism is étale locally on the source and target the composition of a finite morphism by a smooth morphism with geometrically connected fibres of relative dimension equal to the fibre dimension of the original morphism.

052E **Lemma 47.1.** *Let $f : X \to S$ be a morphism. Let $x \in X$ and set $s = f(x)$. Assume that f is locally of finite type and that $n = \dim_x(X_s)$. Then there exists a* *commutative diagram*

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] & & x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] \\ & Y \arrow[d, "h"] & & & y \arrow[d, mapsto] \\ S \arrow[r, equal] & S & & s \arrow[r, equal] & s \end{tikzcd}$$

and a point $x' \in X'$ with $g(x') = x$ such that with $y = \pi(x')$ we have

(1) *$h : Y \to S$ is smooth of relative dimension n,*
(2) *$g : (X', x') \to (X, x)$ is an elementary étale neighbourhood,*
(3) *π is finite, and $\pi^{-1}(\{y\}) = \{x'\}$, and*
(4) *$\kappa(y)$ is a purely transcendental extension of $\kappa(s)$.*

Moreover, if f is locally of finite presentation then π is of finite presentation.

Proof. The problem is local on X and S, hence we may assume that X and S are affine. By Algebra, Lemma 125.3 after replacing X by a standard open neighbourhood of x in X we may assume there is a factorization

$$X \xrightarrow{\pi} \mathbf{A}^n_S \longrightarrow S$$

such that π is quasi-finite and such that $\kappa(\pi(x))$ is purely transcendental over $\kappa(s)$. By Lemma 41.1 there exists an elementary étale neighbourhood

$$(Y, y) \to (\mathbf{A}^n_S, \pi(x))$$

and an open $X' \subset X \times_{\mathbf{A}^n_S} Y$ which contains a unique point x' lying over y such that $X' \to Y$ is finite. This proves (1) – (4) hold. For the final assertion, use Morphisms, Lemma 21.11. □

057K **Lemma 47.2.** *Let $f : X \to S$ be a morphism. Let $x \in X$ and set $s = f(x)$. Assume that f is locally of finite type and that $n = \dim_x(X_s)$. Then there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] & & x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] \\ & Y' \arrow[d, "h"] & & & y' \arrow[d, mapsto] \\ S & S' \arrow[l, "e"'] & & s & s' \arrow[l, mapsto] \end{tikzcd}$$

and a point $x' \in X'$ with $g(x') = x$ such that with $y' = \pi(x')$, $s' = h(y')$ we have

(1) *$h : Y' \to S'$ is smooth of relative dimension n,*
(2) *all fibres of $Y' \to S'$ are geometrically integral,*
(3) *$g : (X', x') \to (X, x)$ is an elementary étale neighbourhood,*
(4) *π is finite, and $\pi^{-1}(\{y'\}) = \{x'\}$,*
(5) *$\kappa(y')$ is a purely transcendental extension of $\kappa(s')$, and*
(6) *$e : (S', s') \to (S, s)$ is an elementary étale neighbourhood.*

Moreover, if f is locally of finite presentation, then π is of finite presentation.

Proof. The question is local on S, hence we may replace S by an affine open neighbourhood of s. Next, we apply Lemma 47.1 to get a commutative diagram

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] \\ & Y \arrow[d, "h"] \\ S \arrow[r, equal] & S \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] \\ & y \arrow[d, mapsto] \\ s \arrow[r, equal] & s \end{tikzcd}$$

where h is smooth of relative dimension n and $\kappa(y)$ is a purely transcendental extension of $\kappa(s)$. Since the question is local on X also, we may replace Y by an affine neighbourhood of y (and X' by the inverse image of this under π). As S is affine this guarantees that $Y \to S$ is quasi-compact, separated and smooth, in particular of finite presentation. Let T be the connected component of Y_s containing y. As Y_s is Noetherian we see that T is open. We also see that T is geometrically connected over $\kappa(s)$ by Varieties, Lemma 7.14. Since T is also smooth over $\kappa(s)$ it is geometrically normal, see Varieties, Lemma 25.4. We conclude that T is geometrically irreducible over $\kappa(s)$ (as a connected Noetherian normal scheme is irreducible, see Properties, Lemma 7.6). Finally, note that the smooth morphism h is normal by Lemma 20.3. At this point we have verified all assumption of Lemma 46.4 hold for the morphism $h : Y \to S$ and open $T \subset Y_s$. As a result of applying Lemma 46.4 we obtain $e : S' \to S$, $s' \in S'$, Y' as in the commutative diagram

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] & X' \times_Y Y' \arrow[l] \arrow[d] \\ & Y \arrow[d, "h"] & Y' \arrow[l] \arrow[d] \\ S \arrow[r, equal] & S & S' \arrow[l, "e"'] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] & (x', s') \arrow[l, mapsto] \arrow[d, mapsto] \\ & y \arrow[d, mapsto] & (y, s') \arrow[l, mapsto] \arrow[d, mapsto] \\ s \arrow[r, equal] & s & s' \arrow[l, mapsto] \end{tikzcd}$$

where $e : (S', s') \to (S, s)$ is an elementary étale neighbourhood, and where $Y' \subset Y_{S'}$ is an open neighbourhood all of whose fibres over S' are geometrically irreducible, such that $Y'_{s'} = T$ via the identification $Y_s = Y_{S', s'}$. Let $(y, s') \in Y'$ be the point corresponding to $y \in T$; this is also the unique point of $Y \times_S S'$ lying over y with residue field equal to $\kappa(y)$ which maps to s' in S'. Similarly, let $(x', s') \in X' \times_Y Y' \subset X' \times_S S'$ be the unique point over x' with residue field equal to $\kappa(x')$ lying over s'. Then the outer part of this diagram is a solution to the problem posed in the lemma. Some minor details omitted. $\square$

057L **Lemma 47.3.** *Assumption and notation as in Lemma 47.2. In addition to properties (1) – (6) we may also arrange it so that*

(7) *S', Y', X' are affine.*

Proof. Note that if Y' is affine, then X' is affine as π is finite. Choose an affine open neighbourhood $U' \subset S'$ of s'. Choose an affine open neighbourhood $V' \subset h^{-1}(U')$ of y'. Let $W' = h(V')$. This is an open neighbourhood of s' in S', see Morphisms, Lemma 34.10, contained in U'. Choose an affine open neighbourhood $U'' \subset W'$ of s'. Then $h^{-1}(U'') \cap V'$ is affine because it is equal to $U'' \times_{U'} V'$. By construction $h^{-1}(U'') \cap V' \to U''$ is a surjective smooth morphism whose fibres are (nonempty) open subschemes of geometrically integral fibres of $Y' \to S'$, and hence geometrically integral. Thus we may replace S' by U'' and Y' by $h^{-1}(U'') \cap V'$. ☐

The significance of the property $\pi^{-1}(\{y'\}) = \{x'\}$ is partially explained by the following lemma.

05B8 **Lemma 47.4.** *Let $\pi : X \to Y$ be a finite morphism. Let $x \in X$ with $y = \pi(x)$ such that $\pi^{-1}(\{y\}) = \{x\}$. Then*

(1) *For every neighbourhood $U \subset X$ of x in X, there exists a neighbourhood $V \subset Y$ of y such that $\pi^{-1}(V) \subset U$.*
(2) *The ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is finite.*
(3) *If π is of finite presentation, then $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is of finite presentation.*
(4) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $\mathcal{F}_x = \pi_*\mathcal{F}_y$ as $\mathcal{O}_{Y,y}$-modules.*

Proof. The first assertion is purely topological; use that π is a continuous and closed map such that $\pi^{-1}(\{y\}) = \{x\}$. To prove the second and third parts we may assume $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$. Then $A \to B$ is a finite ring map and y corresponds to a prime $\mathfrak{p}$ of A such that there exists a unique prime $\mathfrak{q}$ of B lying over $\mathfrak{p}$. Then $B_{\mathfrak{q}} = B_{\mathfrak{p}}$, see Algebra, Lemma 41.11. In other words, the map $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is equal to the map $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ you get from localizing $A \to B$ at $\mathfrak{p}$. Thus (2) and (3) follow from simple properties of localization (some details omitted). For the final statement, suppose that $\mathcal{F} = \widetilde{M}$ for some B-module M. Then $\mathcal{F} = M_{\mathfrak{q}}$ and $\pi_*\mathcal{F}_y = M_{\mathfrak{p}}$. By the above these localizations agree. Alternatively you can use part (1) and the definition of stalks to see that $\mathcal{F}_x = \pi_*\mathcal{F}_y$ directly. ☐

48. Application to the fppf topology

05WM We can use the above étale localization techniques to prove the following result describing the fppf topology as being equal to the topology "generated by" Zariski coverings and by coverings of the form $\{f : T \to S\}$ where f is surjective finite locally free.

0DET **Lemma 48.1.** *Let S be a scheme. Let $\{S_i \to S\}_{i \in I}$ be an fppf covering. Then there exist*

(1) *an étale covering $\{S'_a \to S\}$,*
(2) *surjective finite locally free morphisms $V_a \to S'_a$,*

such that the fppf covering $\{V_a \to S\}$ refines the given covering $\{S_i \to S\}$.

Proof. We may assume that each $S_i \to S$ is locally quasi-finite, see Lemma 23.6.

Fix a point $s \in S$. Pick an $i \in I$ and a point $s_i \in S_i$ mapping to s. Choose an elementary étale neighbourhood $(S', s) \to (S, s)$ such that there exists an open

$$S_i \times_S S' \supset V$$

which contains a unique point $v \in V$ mapping to $s \in S'$ and such that $V \to S'$ is finite, see Lemma 41.1. Then $V \to S'$ is finite locally free, because it is finite and because $S_i \times_S S' \to S'$ is flat and locally of finite presentation as a base change of the morphism $S_i \to S$, see Morphisms, Lemmas 21.4, 25.8, and 48.2. Hence $V \to S'$ is open, and after shrinking S' we may assume that $V \to S'$ is surjective finite locally free. Since we can do this for every point of S we conclude that $\{S_i \to S\}$ can be refined by a covering of the form $\{V_a \to S\}_{a \in A}$ where each $V_a \to S$ factors as $V_a \to S'_a \to S$ with $S'_a \to S$ étale and $V_a \to S'_a$ surjective finite locally free. ☐

05WN **Lemma 48.2.** *Let S be a scheme. Let $\{S_i \to S\}_{i \in I}$ be an fppf covering. Then there exist*

1. *a Zariski open covering $S = \bigcup U_j$,*
2. *surjective finite locally free morphisms $W_j \to U_j$,*
3. *Zariski open coverings $W_j = \bigcup_k W_{j,k}$,*
4. *surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$*

such that the fppf covering $\{T_{j,k} \to S\}$ refines the given covering $\{S_i \to S\}$.

Proof. Let $\{V_a \to S\}_{a \in A}$ be the fppf covering found in Lemma 48.1. In other words, this covering refines $\{S_i \to S\}$ and each $V_a \to S$ factors as $V_a \to S'_a \to S$ with $S'_a \to S$ étale and $V_a \to S'_a$ surjective finite locally free.

By Remark 40.3 there exists a Zariski open covering $S = \bigcup U_j$, for each j a finite locally free, surjective morphism $W_j \to U_j$, and for each j a Zariski open covering $\{W_{j,k} \to W_j\}$ such that the family $\{W_{j,k} \to S\}$ refines the étale covering $\{S'_a \to S\}$, i.e., for each pair j, k there exists an $a(j,k)$ and a factorization $W_{j,k} \to S'_a \to S$ of the morphism $W_{j,k} \to S$. Set $T_{j,k} = W_{j,k} \times_{S'_a} V_a$ and everything is clear. $\square$

0CNX **Lemma 48.3.** *Let S be a scheme. If $U \subset S$ is open and $V \to U$ is a surjective integral morphism, then there exists a surjective integral morphism $\overline{V} \to S$ with $\overline{V} \times_S U$ isomorphic to V as schemes over U.*

Proof. Let $V' \to S$ be the normalization of S in U, see Morphisms, Section 53. By construction $V' \to S$ is integral. By Morphisms, Lemmas 53.6 and 53.12 we see that the inverse image of U in V' is V. Let Z be the reduced induced scheme structure on $S \setminus U$. Then $\overline{V} = V' \amalg Z$ works. $\square$

0CNY **Lemma 48.4.** *Let S be a quasi-compact and quasi-separated scheme. If $U \subset S$ is a quasi-compact open and $V \to U$ is a surjective finite morphism, then there exists a surjective finite morphism $\overline{V} \to S$ with $\overline{V} \times_S U$ isomorphic to V as schemes over U.*

Proof. By Zariski's Main Theorem (Lemma 43.3) we can assume V is a quasi-compact open in a scheme V' finite over S. After replacing V' by the scheme theoretic image of V we may assume that V is dense in V'. It follows that $V' \times_S U = V$ because $V \to V' \times_S U$ is closed as V is finite over U. Let Z be the reduced induced scheme structure on $S \setminus U$. Then $\overline{V} = V' \amalg Z$ works. $\square$

0CNZ **Lemma 48.5.** *Let S be a scheme. Let $\{S_i \to S\}_{i \in I}$ be an fppf covering. Then there exists a surjective integral morphism $S' \to S$ and an open covering $S' = \bigcup U'_\alpha$ such that for each α the morphism $U'_\alpha \to S$ factors through $S_i \to S$ for some i.*

Proof. Choose $S = \bigcup U_j$, $W_j \to U_j$, $W_j = \bigcup W_{j,k}$, and $T_{j,k} \to W_{j,k}$ as in Lemma 48.2. By Lemma 48.3 we can extend $W_j \to U_j$ to a surjective integral morphism $\overline{W}_j \to S$. After this we can extend $T_{j,k} \to W_{j,k}$ to a surjective integral morphism $\overline{T}_{j,k} \to \overline{W}_j$. We set $\overline{T}_j$ equal to the product of all the schemes $\overline{T}_{j,k}$ over $\overline{W}_j$ (Limits, Lemma 3.1). Then we set S' equal to the product of all the schemes $\overline{T}_j$ over S. If $x \in S'$, then there is a j such that the image of x in S lies in U_j. Hence there is a k such that the image of x under the projection $S' \to \overline{W}_j$ lies in $W_{j,k}$. Hence under the projection $S' \to \overline{T}_j \to \overline{T}_{j,k}$ the point x ends up in $T_{j,k}$. And $T_{j,k} \to S$ factors through S_i for some i. Finally, the morphism $S' \to S$ is integral and surjective by Limits, Lemmas 3.3 and 3.2. $\square$

0CP0 **Lemma 48.6.** *Let S be a quasi-compact and quasi-separated scheme. Let $\{S_i \to S\}_{i\in I}$ be an fppf covering. Then there exists a surjective finite morphism $S' \to S$ of finite presentation and an open covering $S' = \bigcup U'_\alpha$ such that for each α the morphism $U'_\alpha \to S$ factors through $S_i \to S$ for some i.*

Proof. Let $Y \to X$ be the integral surjective morphism found in Lemma 48.5. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through $S_{i(j)}$. We can write $Y = \lim Y_\lambda$ with $Y_\lambda \to X$ finite and of finite presentation, see Limits, Lemma 7.3. For large enough λ we can find affine opens $V_{\lambda,j} \subset Y_\lambda$ whose inverse image in Y recovers V_j, see Limits, Lemma 4.11. For even larger λ the morphisms $V_j \to S_{i(j)}$ over X come from morphisms $V_{\lambda,j} \to S_{i(j)}$ over X, see Limits, Proposition 6.1. Setting $S' = Y_\lambda$ for this λ finishes the proof. □

0DBT **Lemma 48.7.** *An fppf covering of schemes is a ph covering.*

Proof. Let $\{T_i \to T\}$ be an fppf covering of schemes, see Topologies, Definition 7.1. Observe that $T_i \to T$ is locally of finite type. Let $U \subset T$ be an affine open. It suffices to show that $\{T_i \times_T U \to U\}$ can be refined by a standard ph covering, see Topologies, Definition 8.4. This follows immediately from Lemma 48.6 and the fact that a finite morphism is proper (Morphisms, Lemma 44.11). □

0DBU **Remark 48.8.** As a consequence of Lemma 48.7 we obtain a comparison morphism

$$\epsilon : (Sch/S)_{ph} \longrightarrow (Sch/S)_{fppf}$$

This is the morphism of sites given by the identity functor on underlying categories (with suitable choices of sites as in Topologies, Remark 11.1). The functor ϵ_* is the identity on underlying presheaves and the functor ϵ^{-1} associated to an fppf sheaf its ph sheafification. By composition we can in addition compare the ph topology with the syntomic, smooth, étale, and Zariski topologies.

49. Quasi-projective schemes

0B41 The term “quasi-projective scheme” has not yet been defined. A possible definition could be a scheme which has an ample invertible sheaf. However, if X is a scheme over a base scheme S, then we say that X *is quasi-projective over* S if the morphism $X \to S$ is quasi-projective (Morphisms, Definition 40.1). Since the identity morphism of any scheme is quasi-projective, we see that a scheme quasi-projective over S doesn’t necessarily have an ample invertible sheaf. For this reason it seems better to leave the term “quasi-projective scheme” undefined.

0B42 **Lemma 49.1.** *Let S be a scheme which has an ample invertible sheaf. Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *$X \to S$ is quasi-projective,*
2. *$X \to S$ is H-quasi-projective,*
3. *there exists a quasi-compact open immersion $X \to X'$ of schemes over S with $X' \to S$ projective,*
4. *$X \to S$ is of finite type and X has an ample invertible sheaf, and*
5. *$X \to S$ is of finite type and there exists an f-very ample invertible sheaf.*

Proof. The implication (2) ⇒ (1) is Morphisms, Lemma 40.5. The implication (1) ⇒ (2) is Morphisms, Lemma 43.16. The implication (2) ⇒ (3) is Morphisms, Lemma 43.11

Assume $X \subset X'$ is as in (3). In particular $X \to S$ is of finite type. By Morphisms, Lemma 43.11 the morphism $X \to S$ is H-projective. Thus there exists a quasi-compact immersion $i : X \to \mathbf{P}^n_S$. Hence $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$ is f-very ample. As $X \to S$ is quasi-compact we conclude from Morphisms, Lemma 38.2 that $\mathcal{L}$ is f-ample. Thus $X \to S$ is quasi-projective by definition.

The implication (4) $\Rightarrow$ (2) is Morphisms, Lemma 39.3.

Assume the equivalent conditions (1), (2), (3) hold. Choose an immersion $i : X \to \mathbf{P}^n_S$ over S. Let $\mathcal{L}$ be an ample invertible sheaf on S. To finish the proof we will show that $\mathcal{N} = f^*\mathcal{L} \otimes_{\mathcal{O}_X} i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$ is ample on X. By Properties, Lemma 26.14 we reduce to the case $X = \mathbf{P}^n_S$. Let $s \in \Gamma(S, \mathcal{L}^{\otimes d})$ be a section such that the corresponding open S_s is affine. Say $S_s = \text{Spec}(A)$. Recall that $\mathbf{P}^n_S$ is the projective bundle associated to $\mathcal{O}_S T_0 \oplus \ldots \oplus \mathcal{O}_S T_n$, see Constructions, Lemma 21.5 and its proof. Let $s_i \in \Gamma(\mathbf{P}^n_S, \mathcal{O}(1))$ be the global section corresponding to the section T_i of $\mathcal{O}_S T_0 \oplus \ldots \oplus \mathcal{O}_S T_n$. Then we see that $X_{f^*s \otimes s_i^{\otimes n}}$ is affine because it is equal to $\text{Spec}(A[T_0/T_i, \ldots, T_n/T_i])$. This proves that $\mathcal{N}$ is ample by definition.

The equivalence of (1) and (5) follows from Morphisms, Lemmas 38.2 and 39.5. $\square$

0B43 **Lemma 49.2.** *Let S be a scheme which has an ample invertible sheaf. Let QP_S be the full subcategory of the category of schemes over S satisfying the equivalent conditions of Lemma 49.1.*

1. *if $S' \to S$ is a morphism of schemes and S' has an ample invertible sheaf, then base change determines a functor $QP_S \to QP_{S'}$,*
2. *if $X \in QP_S$ and $Y \in QP_X$, then $Y \in QP_S$,*
3. *the category QP_S is closed under fibre products,*
4. *the category QP_S is closed under finite disjoint unions,*
5. *if $X \to S$ is projective, then $X \in QP_S$,*
6. *if $X \to S$ is quasi-affine of finite type, then X is in QP_S,*
7. *if $X \to S$ is quasi-finite and separated, then $X \in QP_S$,*
8. *if $X \to S$ is a quasi-compact immersion, then $X \in QP_S$,*
9. *add more here.*

Proof. Part (1) follows from Morphisms, Lemma 40.2.

Part (2) follows from the fourth characterization of Lemma 49.1.

If $X \to S$ and $Y \to S$ are quasi-projective, then $X \times_S Y \to Y$ is quasi-projective by Morphisms, Lemma 40.2. Hence (3) follows from (2).

If $X = Y \amalg Z$ is a disjoint union of schemes and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module such that $\mathcal{L}|_Y$ and $\mathcal{L}|_Z$ are ample, then $\mathcal{L}$ is ample (details omitted). Thus part (4) follows from the fourth characterization of Lemma 49.1.

Part (5) follows from Morphisms, Lemma 43.10.

Part (6) follows from Morphisms, Lemma 40.7.

Part (7) follows from part (6) and Lemma 43.2.

Part (8) follows from part (7) and Morphisms, Lemma 20.16. $\square$

The following lemma doesn't really belong in this section, but there does not seem to be a good spot for it anywhere else.

0EJY **Lemma 49.3.** *Let X be a quasi-affine scheme. Let $f : U \to X$ be an integral morphism. Then U is quasi-affine and the diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathrm{Spec}(\Gamma(U, \mathcal{O}_U)) \arrow[d] \\ X \arrow[r] & \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)) \end{tikzcd}$$

is cartesian.

Proof. The scheme U is quasi-affine because integral morphisms are affine, affine morphisms are quasi-affine, a scheme is quasi-affine if and only if the structure morphism to $\mathrm{Spec}(\mathbf{Z})$ is quasi-affine, and compositions of quasi-affine morphisms are quasi-affine. The first two statements follow immediately from the definition and the third is Morphisms, Lemma 13.4. Set $U' = X \times_{\mathrm{Spec}(\Gamma(X, \mathcal{O}_X))} \mathrm{Spec}(\Gamma(U, \mathcal{O}_U))$ and consider the extended diagram

$$\begin{tikzcd} U \arrow[r, "j"'] \arrow[dr] & U' \arrow[r] \arrow[d] & \mathrm{Spec}(\Gamma(U, \mathcal{O}_U)) \arrow[d] \\ & X \arrow[r] & \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)) \end{tikzcd}$$

The morphism j is closed by Morphisms, Lemma 41.7 combined with the fact that an integral morphism is universally closed (Morphisms, Lemma 44.7) and the fact that the vertical arrows are in the diagram are separated. On the other hand, j is open because the horizontal arrows in the diagram of the lemma are open by Properties, Lemma 18.4. Thus j identifies U with an open and closed subscheme of U'. If $U \neq U'$ then U isn't dense in U' and a fortiori not dense in the spectrum of $\Gamma(U, \mathcal{O}_U)$. However, the scheme theoretic image of U in $\mathrm{Spec}(\Gamma(U, \mathcal{O}_U))$ is $\mathrm{Spec}(\Gamma(U, \mathcal{O}_U))$ because any ideal in $\Gamma(U, \mathcal{O}_U)$ cutting out a closed subscheme through which U factors would have to be zero. Hence U is dense in $\mathrm{Spec}(\Gamma(U, \mathcal{O}_U))$ for example by Morphisms, Lemma 6.3. Thus $U = U'$ and we win. $\square$

50. Projective schemes

0B44 This section is the analogue of Section 49 for projective morphisms.

0B45 **Lemma 50.1.** *Let S be a scheme which has an ample invertible sheaf. Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *$X \to S$ is projective,*
2. *$X \to S$ is H-projective,*
3. *$X \to S$ is quasi-projective and proper,*
4. *$X \to S$ is H-quasi-projective and proper,*
5. *$X \to S$ is proper and X has an ample invertible sheaf,*
6. *$X \to S$ is proper and there exists an f-ample invertible sheaf,*
7. *$X \to S$ is proper and there exists an f-very ample invertible sheaf,*
8. *there is a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ generated by $\mathcal{A}_1$ over $\mathcal{A}_0$ with $\mathcal{A}_1$ a finite type $\mathcal{O}_S$-module such that $X = \underline{Proj}_S(\mathcal{A})$.*

Proof. Observe first that in each case the morphism f is proper, see Morphisms, Lemmas 43.3 and 43.5. Hence it suffices to prove the equivalence of the notions in case f is a proper morphism. We will use this without further mention in the following.

The equivalences (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4) are Morphisms, Lemma 43.13.

The implication (2) $\Rightarrow$ (1) is Morphisms, Lemma 43.3.

The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are Morphisms, Lemma 43.16.

The implication (1) $\Rightarrow$ (7) is immediate from Morphisms, Definitions 43.1 and 38.1.

The conditions (3) and (6) are equivalent by Morphisms, Definition 40.1.

Thus (1) – (4), (6) are equivalent and imply (7). By Lemma 49.1 conditions (3), (5), and (7) are equivalent. Thus we see that (1) – (7) are equivalent.

By Divisors, Lemma 30.5 we see that (8) implies (1). Conversely, if (2) holds, then we can choose a closed immersion

$$i : X \longrightarrow \mathbf{P}^n_S = \underline{\text{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n]).$$

See Constructions, Lemma 21.5 for the equality. By Divisors, Lemma 31.1 we see that X is the relative Proj of a quasi-coherent graded quotient algebra $\mathcal{A}$ of $\mathcal{O}_S[T_0, \ldots, T_n]$. Then $\mathcal{A}$ satisfies the conditions of (8). $\square$

0B46 **Lemma 50.2.** *Let S be a scheme which has an ample invertible sheaf. Let P_S be the full subcategory of the category of schemes over S satisfying the equivalent conditions of Lemma 50.1.*

1. *if $S' \to S$ is a morphism of schemes and S' has an ample invertible sheaf, then base change determines a functor $P_S \to P_{S'}$,*
2. *if $X \in P_S$ and $Y \in P_X$, then $Y \in P_S$,*
3. *the category P_S is closed under fibre products,*
4. *the category P_S is closed under finite disjoint unions,*
5. *if $X \to S$ is finite, then X is in P_S,*
6. *add more here.*

Proof. Part (1) follows from Morphisms, Lemma 43.9.

Part (2) follows from the fifth characterization of Lemma 50.1 and the fact that compositions of proper morphisms are proper (Morphisms, Lemma 41.4).

If $X \to S$ and $Y \to S$ are projective, then $X \times_S Y \to Y$ is projective by Morphisms, Lemma 43.9. Hence (3) follows from (2).

If $X = Y \amalg Z$ is a disjoint union of schemes and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module such that $\mathcal{L}|_Y$ and $\mathcal{L}|_Z$ are ample, then $\mathcal{L}$ is ample (details omitted). Thus part (4) follows from the fifth characterization of Lemma 50.1.

Part (5) follows from Morphisms, Lemma 44.16. $\square$

Here is a slightly different type of result.

0D2S **Lemma 50.3.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $y \in Y$ be a point such that $\mathcal{L}_y$ is ample on X_y. Then there is an open neighbourhood $V \subset Y$ of y such that $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)/V$.* [DG67, IV Corollary 9.6.4]

Proof. We may assume Y is affine. Then we find a directed set I and an inverse system of morphisms $X_i \to Y_i$ of schemes with Y_i of finite type over $\mathbf{Z}$, with affine transition morphisms $X_i \to X_{i'}$ and $Y_i \to Y_{i'}$, with $X_i \to Y_i$ proper, such that $X \to Y = \lim(X_i \to Y_i)$. See Limits, Lemma 13.3. After shrinking I we can assume we have a compatible system of invertible $\mathcal{O}_{X_i}$-modules $\mathcal{L}_i$ pulling back to $\mathcal{L}$, see Limits, Lemma 10.3. Let $y_i \in Y_i$ be the image of y. Then $\kappa(y) = \operatorname{colim} \kappa(y_i)$. Hence for some i we have $\mathcal{L}_{i,y_i}$ is ample on X_{i,y_i} by Limits, Lemma 4.15. By Cohomology of Schemes, Lemma 21.4 we find an open neighbourhood $V_i \subset Y_i$ of y_i such that $\mathcal{L}_i$ restricted to $f_i^{-1}(V_i)$ is ample relative to V_i. Letting $V \subset Y$ be the inverse image of V_i finishes the proof (hints: use Morphisms, Lemma 37.9 and the fact that $X \to Y \times_{Y_i} X_i$ is affine and the fact that the pullback of an ample invertible sheaf by an affine morphism is ample by Morphisms, Lemma 37.7). $\square$

51. Proj and Spec

0EKF In this section we clarify the relationship between the Proj and the spectrum of a graded ring.

Let R be a ring. Let A be a graded R-algebra, see Algebra, Section 56. For $m \geq 0$ we denote $A_{\geq m} = \bigoplus_{d \geq m} A_d$. Consider the graded ring

$$B = \bigoplus\nolimits_{d \geq 0} A_{\geq d}$$

For $d' \geq d$ and $a \in A_{d'}$ let us denote $a^{(d)} \in B$ the element in B_d corresponding to a. Let us denote $\sigma : A \to B$ and $\psi : A \to B$ the two obvious ring maps: if $a \in A_d$, then $\sigma(a) = a^{(0)}$ and $\psi(a) = a^{(d)}$. Then ψ is a graded ring map and σ turns B into a graded algebra over A. There is also a surjective graded ring map $\tau : B \to A$ which for $d' \geq d$ and $a \in A_{d'}$ sends $a^{(d)}$ to 0 if $d' > d$ and to a if $d' = d$.

Affine schemes and spectra. We set $X = \operatorname{Spec}(A)$. The irrelevant ideal A_+ cuts out a closed subscheme $Z = V(A_+) = \operatorname{Spec}(A/A_+) = \operatorname{Spec}(A_0)$. Set $U = X \setminus Z$.

$$U \longrightarrow X \longrightarrow Z$$

Projective schemes and Proj. Set $P = \operatorname{Proj}(A)$. We may and do view P as a scheme over $\operatorname{Spec}(A_0) = Z$. Set $L = \operatorname{Proj}(B)$. We may and do view L as a scheme over $\operatorname{Spec}(B_0) = \operatorname{Spec}(A) = X$; observe that the identification of B_0 with A is given by σ. The surjection τ defines a closed immersion $0 : P \to L$. Since $A \xrightarrow{\sigma} B \to A$ is equal to the map $A \to A_0 \to A$ we conclude that

$$\begin{tikzcd} P \arrow[r, "0"'] \arrow[d] & L \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

is commutative.

We claim that ψ defines a morphism $L \to P$. To see this, by Constructions, Lemma 11.1, it suffices to check $\psi(A_+) \not\subset \mathfrak{p}$ for every homogeneous prime ideal $\mathfrak{p} \subset B$ with $B_+ \not\subset \mathfrak{p}$. First, pick $g \in B_+$ homogeneous $g \notin \mathfrak{p}$. Then we can write g as a finite sum $g = \sum a_i^{(d)}$ with $a_i \in A_{d_i}$ for some $d_i \geq d$. We conclude that there exist $d' \geq d$ and $a \in A_{d'}$ such that $a^{(d)} \notin \mathfrak{p}$. Then

$$(a^{(d)})^{d'} = (a^{d'})^{(d'd)} = a^{(d)}(a^{d'-1})^{(d(d'-1))} = \psi(a)(a^{d'-1})^{(d(d'-1))}$$

(the notation leaves something to be desired) is not in $\mathfrak{p}$. Hence $\psi(a) \notin \mathfrak{p}$, proving the claim. Thus we can extend our diagram above to a commutative diagram

$$\begin{tikzcd} P \arrow[r, "0"'] \arrow[d] & L \arrow[r, "\pi"'] \arrow[d] & P \arrow[d] \\ Z \arrow[r] & X \arrow[r] & Z \end{tikzcd}$$

where $X \to Z$ is given by $A_0 \to A$. Since $\tau \circ \psi = \mathrm{id}_A$ we see $\pi \circ 0 = \mathrm{id}_P$.

Observe that π is an affine morphism. This is clear from the construction in Constructions, Lemma 11.1. In fact, if $f \in A_d$ for some $d > 0$, then setting $g = \psi(f)$ we have $\pi^{-1}(D_+(f)) = D_+(g)$. In this case we have the following equality of homogeneous parts

$$(B[1/g])_{m'} = \bigoplus\nolimits_{m \geq m'} (A[1/f])_m$$

This isomorphism is compatible with further localization. Taking $m' = 0$ we see that $\pi_*\mathcal{O}_L$ is the direct sum of $\mathcal{O}_P(m)$ for $m \geq 0$[9]. We conclude L is identified with the relative spectrum:

$$L = \underline{\mathrm{Spec}}_P\left(\bigoplus\nolimits_{m\geq 0} \mathcal{O}_P(m)\right)$$

In particular $L \to P$ is a cone[10], see Constructions, Section 7. Moreover, it is clear that $0 : P \to L$ is the vertex of the cone.

Let $f \in A_d$ for some $d > 0$ and $g = \psi(f) \in B_d$ as in the previous paragraph. Looking at the structure of the ring maps

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\sigma"] & A_0 \arrow[d] \\ (A[1/f])_0 \arrow[r, "\psi"] & (B[1/g])_0 = \bigoplus_{m\geq 0}(A[1/f])_m \arrow[r, "\tau"] & (A[1/f])_0 \end{tikzcd}$$

some compuations[11] in graded rings will show that

(1) $\sigma(A_+)(B[1/g])_0 \subset \mathrm{Ker}(\tau : (B[1/g])_0 \to (A[1/f])_0)$,
(2) $\sigma(f) \in (B[1/g])_0$ is a nonzerodivisor,
(3) $\sigma(f)(B[1/g])_0 = \sigma(A_d)(B[1/g])_0$ as ideals,
(4) $\sigma(f)(B[1/g])_0$ and $\mathrm{Ker}(\tau : (B[1/g])_0 \to (A[1/f])_0)$ have the same radical,
(5) if $d = 1$, then $\sigma(f)(B[1/g])_0 = \mathrm{Ker}(\tau : (B[1/g])_0 \to (A[1/f])_0)$.

We see in particular that

$$0(D_+(f)) = V(\sigma(f)) \subset D_+(g) = \mathrm{Spec}((B[1/g])_0)$$

[9] It similarly follows that $\pi_*\mathcal{O}_L(i) = \bigoplus_{m\geq -i} \mathcal{O}_P(m)$.

[10] Often L is a line bundle over P, see below.

[11] Parts (1) and (2) are clear. To see (3), note that if $a \in A_d$, then $\sigma(a) = \sigma(f)\psi(a/f)$. For (4) note that b/g^m is in the kernel of τ if and only if $b \in A_{\geq md}$ maps to zero in A_{md}. Thus it suffices to show if $m' > md$ and $a \in A_{m'}$, then some power of $a^{(md)}/g^m$ is in the ideal generated by $\sigma(f)$. Take e such that $em' - emd \geq d$. Then

$$(a^{(md)}/g^m)^e = (a^e)^{(emd)}/g^{em} = (fa^e)^{(emd+d)}/g^{em+1} = \sigma(f) \cdot (a^e)^{(emd+d)}/g^{em+1}$$

as desired (apologies for the terrible notation). To see (5) argue as before and note that $a^{(md)}/g^m = \sigma(f) \cdot a^{(md+1)}/g^{m+1}$ if $d = 1$.

set theoretically. In other words, the ideal generated by $\sigma(A_d)$ cuts out an effective Cartier divisor on $D_+(g)$ which is set theoretically equal to the image of the closed immersion $0 : P \to L$.

We claim that $L \to X$ is an isomorphism over U. Namely, if $f \in A_d$ for some $d > 0$, then

$$\mathrm{Spec}(A_f) \times_X L = \mathrm{Proj}(A_f \otimes_A B) = \mathrm{Proj}(B_{\sigma(f)})$$

For each e we have $(B_{\sigma(f)})_e = A_f \otimes_B B_e = A_f \otimes_A A_{\geq e} = A_f$, the final equality induced by the injection $A_{\geq e} \subset A$. Hence $B_{\sigma(f)} \cong A_f[T]$ with T in degree 1. This proves the claim as $\mathrm{Proj}(A_f[T]) \to \mathrm{Spec}(A_f)$ is an isomorphism. From now on we identify U with the corresponding open of L.

The identification made in the previous paragraph lets us consider the restriction $\pi|_U : U \to P$. Pick $f \in A_d$ for some $d > 0$ and $g = \psi(f) \in B_d$ as we have done above several times. Then

$$U \cap \pi^{-1}(D_+(f)) = U \cap D_+(g)$$

is the complement of the zero locus of $\sigma(f) \in (B[1/g])_0$ via the identification of $D_+(g)$ with the spectrum of $(B[1/g])_0$. This is assertion (4) above. Therefore $U \cap D_+(g)$ is affine and

$$\mathcal{O}_L(U \cap D_+(g)) = (B[1/g])_0[1/\sigma(f)] = \bigoplus\nolimits_{m \in \mathbf{Z}} (A[1/f])_m$$

where the last equal sign is the natural extension of the identification $(B[1/g])_0 = \bigoplus_{m \geq 0} (A[1/f])_m$ made above. Exactly as we did before with $\pi : L \to P$ we conclude that $\pi|_U : U \to P$ is affine and

$$U = \underline{\mathrm{Spec}}_P\left(\bigoplus\nolimits_{m \in \mathbf{Z}} \mathcal{O}_P(m)\right)$$

as schemes over P.

Summarising the above, our constructions produce a commutative diagram

0EKG (51.0.1)
$$\begin{tikzcd} \underline{\mathrm{Spec}}_P\left(\bigoplus_{m \in \mathbf{Z}} \mathcal{O}_P(m)\right) \arrow[r] \arrow[d, equal] & L = \underline{\mathrm{Spec}}_P\left(\bigoplus_{m \geq 0} \mathcal{O}_P(m)\right) \arrow[r, "\pi"'] \arrow[d, "\sigma"] & P \arrow[d] \\ U \arrow[r] & X \arrow[r] & Z \end{tikzcd}$$

of schemes where π is a cone whose zero section $0 : P \to L$ maps set theoretically onto the inverse image of Z in L.

Let $W \subset P$ be the largest open such that $\mathcal{O}_P(1)|_W$ is invertible and the natural maps induce isomorphisms $\mathcal{O}_P(m)|_W \cong \mathcal{O}_P(1)^{\otimes m}|_W$ for all $m \in \mathbf{Z}$, i.e., the open of Constructions, Lemma 10.4 for $d = 1$. Then we see that $L|_W = \pi^{-1}(W) \to W$ is a vector bundle (Constructions, Section 6) of rank 1, namely,

$$L|_W = \mathbf{V}(\mathcal{O}_P(1)|_W)$$

in Grothendieckian notation. This is immediate from the above showing that $L|_W$ is equal to the relative spectrum of the symmetric algebra over $\mathcal{O}_W$ on $\mathcal{O}_P(1)|_W$. Then clearly the morphism $0|_W : W \to L|_W$ is the zero section of this vector bundle. In particular $0(W)$ is an effective Cartier divisor on $L|_W$. Moreover, the open $U|_W = (\pi|_U)^{-1}(W)$ is the complement of the zero section.

If A is generated by $f_1, \ldots, f_r \in A_1$ over A_0, then $(f_1, \ldots, f_r)^m = A_{\geq m}$ for all $m \geq 0$ and hence our B above is the Rees algebra for $A_+ = (f_1, \ldots, f_r)$. Thus in this case $L \to X$ is the blowup of Z and $W = P$ where W is as in the preceding paragraph.

If P is quasi-compact, then for d sufficiently divisible, the closed subscheme $D \subset L$ cut out by $\sigma(A_d)\mathcal{O}_L$ is an effective Cartier divisor, $0 : P \to L$ factors through D, and $0(P) = D$ set theoretically. This follows from Constructions, Lemma 8.9 and (1), (2), (3), and (4) proved above. (Take any d divisible by the lcm of the degrees of the elements found in the lemma.)

We continue to assume P is quasi-compact. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_P$-module. Let us set $\mathcal{F}_U = \pi^*\mathcal{F}|_U$. Then we have

0EKH (51.0.2)
$$R\Gamma(U, \mathcal{F}_U) = \bigoplus\nolimits_{m \in \mathbf{Z}} R\Gamma(P, \mathcal{F} \otimes_{\mathcal{O}_P} \mathcal{O}_P(m))$$

Moreover, this direct sum decomposition is functorial in $\mathcal{F}$ and the induced A-module structure on the right is the same as the A-module structure on the left coming from $U \subset X$. To prove the formula, since $\pi|_U$ is affine and $(\pi|_U)_*\mathcal{O}_U = \bigoplus_{m \in \mathbf{Z}} \mathcal{O}_P(m)$ we get

$$\begin{aligned} R(\pi|_U)_*\mathcal{F}_U &= (\pi|_U)_*\mathcal{F}_U \\ &= (\pi|_U)_*(\pi|_U)^*\mathcal{F} \\ &= \mathcal{F} \otimes_{\mathcal{O}_P} \bigoplus\nolimits_{m \in \mathbf{Z}} \mathcal{O}_P(m) \\ &= \bigoplus\nolimits_{m \in \mathbf{Z}} \mathcal{F} \otimes_{\mathcal{O}_P} \mathcal{O}_P(m) \end{aligned}$$

By Leray we find that $R\Gamma(U, \mathcal{F}_U) = R\Gamma(P, R(\pi|_U)_*\mathcal{F}_U)$, see Cohomology, Lemma 13.6. The proof is finished because taking cohomology commutes with direct sums in this case, see Derived Categories of Schemes, Lemma 4.5. This is where we use that P is quasi-compact; P is separated by Constructions, Lemma 8.8.

0EKI **Lemma 51.1.** *Let R be a ring. Let P be a proper scheme over R and let $\mathcal{L}$ be an ample invertible $\mathcal{O}_P$-module. Set $A = \bigoplus_{m \geq 0} \Gamma(P, \mathcal{L}^{\otimes m})$. Then $P = Proj(A)$ and diagram (51.0.1) becomes the diagram*

$$\begin{tikzcd} \underline{\mathrm{Spec}}_P\left(\bigoplus_{m \in \mathbf{Z}} \mathcal{L}^{\otimes m}\right) \arrow[r] \arrow[d, equal] & L = \underline{\mathrm{Spec}}_P\left(\bigoplus_{m \geq 0} \mathcal{L}^{\otimes m}\right) \arrow[r, "\pi"] \arrow[d, "\sigma"] & P \arrow[d] \\ U \arrow[r] & X \arrow[r] & Z \end{tikzcd}$$

having the properties explained above.

Proof. We have $P = \mathrm{Proj}(A)$ by Morphisms, Lemma 43.17. Moreover, by Properties, Lemma 28.2 via this identification we have $\mathcal{O}_P(m) = \mathcal{L}^{\otimes m}$ for all $m \in \mathbf{Z}$. $\square$

52. Closed points in fibres

053Q Some of the material in this section is taken from the preprint [OP10].

053R **Lemma 52.1.** *Let $f : X \to S$ be a morphism of schemes. Let $Z \subset X$ be a closed subscheme. Let $s \in S$. Assume*

(1) *S is irreducible with generic point η,*
(2) *X is irreducible,*

(3) *f is dominant,*
(4) *f is locally of finite type,*
(5) $\dim(X_s) \leq \dim(X_\eta)$,
(6) *Z is locally principal in X, and*
(7) $Z_\eta = \emptyset$.

Then the fibre Z_s is (set theoretically) a union of irreducible components of X_s.

Proof. Let X_{red} denote the reduction of X. Then $Z \cap X_{red}$ is a locally principal closed subscheme of X_{red}, see Divisors, Lemma 13.11. Hence we may assume that X is reduced. In other words X is integral, see Properties, Lemma 3.4. In this case the morphism $X \to S$ factors through S_{red}, see Schemes, Lemma 12.7. Thus we may replace S by S_{red} and assume that S is integral too.

The assertion that f is dominant signifies that the generic point of X is mapped to η, see Morphisms, Lemma 8.6. Moreover, the scheme X_η is an integral scheme which is locally of finite type over the field $\kappa(\eta)$. Hence $d = \dim(X_\eta) \geq 0$ is equal to $\dim_\xi(X_\eta)$ for every point ξ of X_η, see Algebra, Lemmas 114.4 and 114.5. In view of Morphisms, Lemma 28.4 and condition (5) we conclude that $\dim_x(X_s) = d$ for every $x \in X_s$.

In the Noetherian case the assertion can be proved as follows. If the lemma does not holds there exists $x \in Z_s$ which is a generic point of an irreducible component of Z_s but not a generic point of any irreducible component of X_s. Then we see that $\dim_x(Z_s) \leq d - 1$, because $\dim_x(X_s) = d$ and in a neighbourhood of x in X_s the closed subscheme Z_s does not contain any of the irreducible components of X_s. Hence after replacing X by an open neighbourhood of x we may assume that $\dim_z(Z_{f(z)}) \leq d - 1$ for all $z \in Z$, see Morphisms, Lemma 28.4. Let $\xi' \in Z$ be a generic point of an irreducible component of Z and set $s' = f(\xi)$. As $Z \neq X$ is locally principal we see that $\dim(\mathcal{O}_{X,\xi}) = 1$, see Algebra, Lemma 60.11 (this is where we use X is Noetherian). Let $\xi \in X$ be the generic point of X and let ξ_1 be a generic point of any irreducible component of $X_{s'}$ which contains ξ'. Then we see that we have the specializations

$$\xi \rightsquigarrow \xi_1 \rightsquigarrow \xi'.$$

As $\dim(\mathcal{O}_{X,\xi}) = 1$ one of the two specializations has to be an equality. By assumption $s' \neq \eta$, hence the first specialization is not an equality. Hence $\xi' = \xi_1$ is a generic point of an irreducible component of $X_{s'}$. Applying Morphisms, Lemma 28.4 one more time this implies $\dim_{\xi'}(Z_{s'}) = \dim_{\xi'}(X_{s'}) \geq \dim(X_\eta) = d$ which gives the desired contradiction.

In the general case we reduce to the Noetherian case as follows. If the lemma is false then there exists a point $x \in X$ lying over s such that x is a generic point of an irreducible component of Z_s, but not a generic point of any of the irreducible components of X_s. Let $U \subset S$ be an affine neighbourhood of s and let $V \subset X$ be an affine neighbourhood of x with $f(V) \subset U$. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ so that $f|_V$ is given by a ring map $A \to B$. Let $\mathfrak{q} \subset B$, resp. $\mathfrak{p} \subset A$ be the prime corresponding to x, resp. s. After possibly shrinking V we may assume $Z \cap V$ is cut out by some element $g \in B$. Denote K the fraction field of A. What we know at this point is the following:

(1) $A \subset B$ is a finitely generated extension of domains,
(2) the element $g \otimes 1$ is invertible in $B \otimes_A K$,

(3) $d = \dim(B \otimes_A K) = \dim(B \otimes_A \kappa(\mathfrak{p}))$,
(4) $g \otimes 1$ is not a unit of $B \otimes_A \kappa(\mathfrak{p})$, and
(5) $g \otimes 1$ is not in any of the minimal primes of $B \otimes_A \kappa(\mathfrak{p})$.

We are seeking a contradiction.

Pick elements $x_1, \ldots, x_n \in B$ which generate B over A. For a finitely generated **Z**-algebra $A_0 \subset A$ let $B_0 \subset B$ be the A_0-subalgebra generated by $x_1, \ldots, x_n$, denote K_0 the fraction field of A_0, and set $\mathfrak{p}_0 = A_0 \cap \mathfrak{p}$. We claim that when A_0 is large enough then (1) – (5) also hold for the system $(A_0 \subset B_0, g, \mathfrak{p}_0)$.

We prove each of the conditions in turn. Part (1) holds by construction. For part (2) write $(g \otimes 1)h = 1$ for some $h \otimes 1/a \in B \otimes_A K$. Write $g = \sum a_I x^I$, $h = \sum a'_I x^I$ (multi-index notation) for some coefficients $a_I, a'_I \in A$. As soon as A_0 contains a and the a_I, a'_I then (2) holds because $B_0 \otimes_{A_0} K_0 \subset B \otimes_A K$ (as localizations of the injective map $B_0 \to B$). To achieve (3) consider the exact sequence

$$0 \to I \to A[X_1, \ldots, X_n] \to B \to 0$$

which defines I where the second map sends X_i to x_i. Since $\otimes$ is right exact we see that $I \otimes_A K$, respectively $I \otimes_A \kappa(\mathfrak{p})$ is the kernel of the surjection $K[X_1, \ldots, X_n] \to B \otimes_A K$, respectively $\kappa(\mathfrak{p})[X_1, \ldots, X_n] \to B \otimes_A \kappa(\mathfrak{p})$. As a polynomial ring over a field is Noetherian there exist finitely many elements $h_j \in I$, $j = 1, \ldots, m$ which generate $I \otimes_A K$ and $I \otimes_A \kappa(\mathfrak{p})$. Write $h_j = \sum a_{j,I} X^I$. As soon as A_0 contains all $a_{j,I}$ we get to the situation where

$$B_0 \otimes_{A_0} K_0 \otimes_{K_0} K = B \otimes_A K \quad \text{and} \quad B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0) \otimes_{\kappa(\mathfrak{p}_0)} \kappa(\mathfrak{p}) = B \otimes_A \kappa(\mathfrak{p}).$$

By either Morphisms, Lemma 28.3 or Algebra, Lemma 116.5 we see that the dimension equalities of (3) are satisfied. Part (4) is immediate. As $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0) \subset B \otimes_A \kappa(\mathfrak{p})$ each minimal prime of $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)$ lies under a minimal prime of $B \otimes_A \kappa(\mathfrak{p})$ by Algebra, Lemma 30.6. This implies that (5) holds. In this way we reduce the problem to the Noetherian case which we have dealt with above. $\square$

Here is an algebraic application of the lemma above. The fourth assumption of the lemma holds if $A \to B$ is flat, see Lemma 52.3.

053S **Lemma 52.2.** *Let $A \to B$ be a local homomorphism of local rings, and $g \in \mathfrak{m}_B$. Assume*

(1) *A and B are domains and $A \subset B$,*
(2) *B is essentially of finite type over A,*
(3) *g is not contained in any minimal prime over $\mathfrak{m}_A B$, and*
(4) *$\dim(B/\mathfrak{m}_A B) + trdeg_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = trdeg_A(B)$.*

Then $A \subset B/gB$, i.e., the generic point of $\mathrm{Spec}(A)$ *is in the image of the morphism* $\mathrm{Spec}(B/gB) \to \mathrm{Spec}(A)$.

Proof. Note that the two assertions are equivalent by Algebra, Lemma 30.6. To start the proof let C be an A-algebra of finite type and $\mathfrak{q}$ a prime of C such that $B = C_{\mathfrak{q}}$. Of course we may assume that C is a domain and that $g \in C$. After replacing C by a localization we see that $\dim(C/\mathfrak{m}_A C) = \dim(B/\mathfrak{m}_A B) + \mathrm{trdeg}_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$, see Morphisms, Lemma 28.1. Setting K equal to the fraction field of A we see by the same reference that $\dim(C \otimes_A K) = \mathrm{trdeg}_A(B)$. Hence assumption (4) means that the generic and closed fibres of the morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ have the same dimension.

Suppose that the lemma is false. Then $(B/gB) \otimes_A K = 0$. This means that $g \otimes 1$ is invertible in $B \otimes_A K = C_\mathfrak{q} \otimes_A K$. As $C_\mathfrak{q}$ is a limit of principal localizations we conclude that $g \otimes 1$ is invertible in $C_h \otimes_A K$ for some $h \in C$, $h \notin \mathfrak{q}$. Thus after replacing C by C_h we may assume that $(C/gC) \otimes_A K = 0$. We do one more replacement of C to make sure that the minimal primes of $C/\mathfrak{m}_A C$ correspond one-to-one with the minimal primes of $B/\mathfrak{m}_A B$. At this point we apply Lemma 52.1 to $X = \text{Spec}(C) \to \text{Spec}(A) = S$ and the locally closed subscheme $Z = \text{Spec}(C/gC)$. Since $Z_K = \emptyset$ we see that $Z \otimes \kappa(\mathfrak{m}_A)$ has to contain an irreducible component of $X \otimes \kappa(\mathfrak{m}_A) = \text{Spec}(C/\mathfrak{m}_A C)$. But this contradicts the assumption that g is not contained in any prime minimal over $\mathfrak{m}_A B$. The lemma follows. □

053T **Lemma 52.3.** *Let $A \to B$ be a local homomorphism of local rings. Assume*

1. *A and B are domains and $A \subset B$,*
2. *B is essentially of finite type over A, and*
3. *B is flat over A.*

Then we have

$$\dim(B/\mathfrak{m}_A B) + trdeg_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = trdeg_A(B).$$

Proof. Let C be an A-algebra of finite type and $\mathfrak{q}$ a prime of C such that $B = C_\mathfrak{q}$. We may assume C is a domain. We have $\dim_\mathfrak{q}(C/\mathfrak{m}_A C) = \dim(B/\mathfrak{m}_A B) + \text{trdeg}_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$, see Morphisms, Lemma 28.1. Setting K equal to the fraction field of A we see by the same reference that $\dim(C \otimes_A K) = \text{trdeg}_A(B)$. Thus we are really trying to prove that $\dim_\mathfrak{q}(C/\mathfrak{m}_A C) = \dim(C \otimes_A K)$. Choose a valuation ring A' in K dominating A, see Algebra, Lemma 50.2. Set $C' = C \otimes_A A'$. Choose a prime $\mathfrak{q}'$ of C' lying over $\mathfrak{q}$; such a prime exists because

$$C'/\mathfrak{m}_{A'} C' = C/\mathfrak{m}_A C \otimes_{\kappa(\mathfrak{m}_A)} \kappa(\mathfrak{m}_{A'})$$

which proves that $C/\mathfrak{m}_A C \to C'/\mathfrak{m}_{A'} C'$ is faithfully flat. This also proves that $\dim_\mathfrak{q}(C/\mathfrak{m}_A C) = \dim_{\mathfrak{q}'}(C'/\mathfrak{m}_{A'} C')$, see Algebra, Lemma 116.6. Note that $B' = C'_{\mathfrak{q}'}$ is a localization of $B \otimes_A A'$. Hence B' is flat over A'. The generic fibre $B' \otimes_{A'} K$ is a localization of $B \otimes_A K$. Hence B' is a domain. If we prove the lemma for $A' \subset B'$, then we get the equality $\dim_{\mathfrak{q}'}(C'/\mathfrak{m}_{A'} C') = \dim(C' \otimes_{A'} K)$ which implies the desired equality $\dim_\mathfrak{q}(C/\mathfrak{m}_A C) = \dim(C \otimes_A K)$ by what was said above. This reduces the lemma to the case where A is a valuation ring.

Let $A \subset B$ be as in the lemma with A a valuation ring. As before write $B = C_\mathfrak{q}$ for some domain C of finite type over A. By Algebra, Lemma 125.9 we obtain $\dim(C/\mathfrak{m}_A C) = \dim(C \otimes_A K)$ and we win. □

053U **Lemma 52.4.** *Let $f : X \to S$ be a morphism of schemes. Let $x \rightsquigarrow x'$ be a specialization of points in X. Set $s = f(x)$ and $s' = f(x')$. Assume*

1. *x' is a closed point of $X_{s'}$, and*
2. *f is locally of finite type.*

Then the set

$$\{x_1 \in X \textit{ such that } f(x_1) = s \textit{ and } x_1 \textit{ is closed in } X_s \textit{ and } x \rightsquigarrow x_1 \rightsquigarrow x'\}$$

is dense in the closure of x in X_s.

Proof. We apply Schemes, Lemma 20.4 to the specialization $x \rightsquigarrow x'$. This produces a morphism $\varphi : \text{Spec}(B) \to X$ where B is a valuation ring such that φ maps the generic point to x and the closed point to x'. We may also assume that $\kappa(x)$ is the fraction field of B. Let $A = B \cap \kappa(s)$. Note that this is a valuation ring (see Algebra, Lemma 50.7) which dominates the image of $\mathcal{O}_{S,s'} \to \kappa(s)$. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[rd] & X_A \arrow[r] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}(A) \arrow[r] & S \end{tikzcd}$$

The generic (resp. closed) point of B maps to a point x_A (resp. x'_A) of X_A lying over the generic (resp. closed) point of $\mathrm{Spec}(A)$. Note that x'_A is a closed point of the special fibre of X_A by Morphisms, Lemma 20.4. Note that the generic fibre of $X_A \to \mathrm{Spec}(A)$ is isomorphic to X_s. Thus we have reduced the lemma to the case where S is the spectrum of a valuation ring, $s = \eta \in S$ is the generic point, and $s' \in S$ is the closed point.

We will prove the lemma by induction on $\dim_x(X_\eta)$. If $\dim_x(X_\eta) = 0$, then there are no other points of X_η specializing to x and x is closed in its fibre, see Morphisms, Lemma 20.6, and the result holds. Assume $\dim_x(X_\eta) > 0$.

Let $X' \subset X$ be the reduced induced scheme structure on the irreducible closed subscheme $\overline{\{x\}}$ of X, see Schemes, Definition 12.5. To prove the lemma we may replace X by X' as this only decreases $\dim_x(X_\eta)$. Hence we may also assume that X is an integral scheme and that x is its generic point. In addition, we may replace X by an affine neighbourhood of x'. Thus we have $X = \mathrm{Spec}(B)$ where $A \subset B$ is a finite type extension of domains. Note that in this case $\dim_x(X_\eta) = \dim(X_\eta) = \dim(X_{s'})$, and that in fact $X_{s'}$ is equidimensional, see Algebra, Lemma 125.9.

Let $W \subset X_\eta$ be a proper closed subset (this is the subset we want to "avoid"). As X_s is of finite type over a field we see that W has finitely many irreducible components $W = W_1 \cup \ldots \cup W_n$. Let $\mathfrak{q}_j \subset B$, $j = 1, \ldots, r$ be the corresponding prime ideals. Let $\mathfrak{q} \subset B$ be the maximal ideal corresponding to the point x'. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s \subset B$ be the minimal primes lying over $\mathfrak{m}_A B$. There are finitely many as these correspond to the irreducible components of the Noetherian scheme $X_{s'}$. Moreover, each of these irreducible components has dimension > 0 (see above) hence we see that $\mathfrak{p}_i \neq \mathfrak{q}$ for all i. Now, pick an element $g \in \mathfrak{q}$ such that $g \notin \mathfrak{q}_j$ for all j and $g \notin \mathfrak{p}_i$ for all i, see Algebra, Lemma 15.2. Denote $Z \subset X$ the locally principal closed subscheme defined by g. Let $Z_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta}$, $n \geq 0$ be the decomposition of the generic fibre of Z into irreducible components (finitely many as the generic fibre is Noetherian). Denote $Z_i \subset X$ the closure of $Z_{i,\eta}$. After replacing X by a smaller affine neighbourhood we may assume that $x' \in Z_i$ for each $i = 1, \ldots, n$. By construction $Z \cap X_{s'}$ does not contain any irreducible component of $X_{s'}$. Hence by Lemma 52.1 we conclude that $Z_\eta \neq \emptyset$! In other words $n \geq 1$. Letting $x_1 \in Z_1$ be the generic point we see that $x_1 \rightsquigarrow x'$ and $f(x_1) = \eta$. Also, by construction $Z_{1,\eta} \cap W_j \subset W_j$ is a proper closed subset. Hence every irreducible component of $Z_{1,\eta} \cap W_j$ has codimension ≥ 2 in X_η whereas $\mathrm{codim}(Z_{1,\eta}, X_\eta) = 1$ by Algebra, Lemma 60.11. Thus $W \cap Z_{1,\eta}$ is a proper closed subset. At this point we see that the induction hypothesis applies to $Z_1 \to S$ and the specialization $x_1 \rightsquigarrow x'$. This produces a closed point x_2 of $Z_{1,\eta}$ not contained in W which specializes to

x'. Thus we obtain $x \rightsquigarrow x_2 \rightsquigarrow x'$, the point x_2 is closed in X_η, and $x_2 \notin W$ as desired. $\square$

053V **Remark 52.5.** The proof of Lemma 52.4 actually shows that there exists a sequence of specializations

$$x \rightsquigarrow x_1 \rightsquigarrow x_2 \rightsquigarrow \ldots \rightsquigarrow x_d \rightsquigarrow x'$$

where all x_i are in the fibre X_s, each specialization is immediate, and x_d is a closed point of X_s. The integer $d = \text{trdeg}_{\kappa(s)}(\kappa(x)) = \dim(\overline{\{x\}})$ where the closure is taken in X_s. Moreover, the points x_i can be chosen to avoid any closed subset of X_s which does not contain the point x.

Examples, Section 39 shows that the following lemma is false if A is not assumed Noetherian.

05GT **Lemma 52.6.** *Let $\varphi : A \to B$ be a local ring map of local rings. Let $V \subset \text{Spec}(B)$ be an open subscheme which contains at least one prime not lying over $\mathfrak{m}_A$. Assume A is Noetherian, φ essentially of finite type, and $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is finite. Then there exists a $\mathfrak{q} \in V$, $\mathfrak{m}_A \neq \mathfrak{q} \cap A$ such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map.*

Proof. Since A is Noetherian and $A \to B$ is essentially of finite type, we know that B is Noetherian too. By Properties, Lemma 6.4 the topological space $\text{Spec}(B) \setminus \{\mathfrak{m}_B\}$ is Jacobson. Hence we can choose a closed point $\mathfrak{q}$ which is contained in the nonempty open

$$V \setminus \{\mathfrak{q} \subset B \mid \mathfrak{m}_A = \mathfrak{q} \cap A\}.$$

(Nonempty by assumption, open because $\{\mathfrak{m}_A\}$ is a closed subset of $\text{Spec}(A)$.) Then $\text{Spec}(B/\mathfrak{q})$ has two points, namely $\mathfrak{m}_B$ and $\mathfrak{q}$ and $\mathfrak{q}$ does not lie over $\mathfrak{m}_A$. Write $B/\mathfrak{q} = C_\mathfrak{m}$ for some finite type A-algebra C and prime ideal $\mathfrak{m}$. Then $A \to C$ is quasi-finite at $\mathfrak{m}$ by Algebra, Lemma 122.2 (2). Hence by Algebra, Lemma 123.13 we see that after replacing C by a principal localization the ring map $A \to C$ is quasi-finite. $\square$

05GU **Lemma 52.7.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $U \subset X$ be an open subscheme. Assume f locally of finite type, S locally Noetherian, x a closed point of X_s, and assume there exists a point $x' \in U$ with $x' \rightsquigarrow x$ and $f(x') \neq s$. Then there exists a closed subscheme $Z \subset X$ such that (a) $x \in Z$, (b) $f|_Z : Z \to S$ is quasi-finite at x, and (c) there exists a $z \in Z$, $z \in U$, $z \rightsquigarrow x$ and $f(z) \neq s$.*

Proof. This is a reformulation of Lemma 52.6. Namely, set $A = \mathcal{O}_{S,s}$ and $B = \mathcal{O}_{X,x}$. Denote $V \subset \text{Spec}(B)$ the inverse image of U. The ring map $f^\sharp : A \to B$ is essentially of finite type. By assumption there exists at least one point of V which does not map to the closed point of $\text{Spec}(A)$. Hence all the assumptions of Lemma 52.6 hold and we obtain a prime $\mathfrak{q} \subset B$ which does not lie over $\mathfrak{m}_A$ and such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map. Let $z \in X$ be the image of the point $\mathfrak{q}$ under the canonical morphism $\text{Spec}(B) \to X$. Set $Z = \overline{\{z\}}$ with the induced reduced scheme structure. As $z \rightsquigarrow x$ we see that $x \in Z$ and $\mathcal{O}_{Z,x} = B/\mathfrak{q}$. By construction $Z \to S$ is quasi-finite at x. $\square$

05GV **Remark 52.8.** We can use Lemma 52.6 or its variant Lemma 52.7 to give an alternative proof of Lemma 52.4 in case S is locally Noetherian. Here is a rough sketch. Namely, first replace S by the spectrum of the local ring at s'. Then we may use induction on $\dim(S)$. The case $\dim(S) = 0$ is trivial because then $s' = s$. Replace X by the reduced induced scheme structure on $\overline{\{x\}}$. Apply Lemma 52.7 to $X \to S$ and $x' \mapsto s'$ and any nonempty open $U \subset X$ containing x. This gives us a closed subscheme $x' \in Z \subset X$ a point $z \in Z$ such that $Z \to S$ is quasi-finite at x' and such that $f(z) \neq s'$. Then z is a closed point of $X_{f(z)}$, and $z \rightsquigarrow x'$. As $f(z) \neq s'$ we see $\dim(\mathcal{O}_{S,f(z)}) < \dim(S)$. Since x is the generic point of X we see $x \rightsquigarrow z$, hence $s = f(x) \rightsquigarrow f(z)$. Apply the induction hypothesis to $s \rightsquigarrow f(z)$ and $z \mapsto f(z)$ to win.

05GW **Lemma 52.9.** *Suppose that $f : X \to S$ is locally of finite type, S locally Noetherian, $x \in X$ a closed point of its fibre X_s, and $U \subset X$ an open subscheme such that $U \cap X_s = \emptyset$ and $x \in \overline{U}$, then the conclusions of Lemma 52.7 hold.*

Proof. Namely, we can reduce this to the cited lemma as follows: First we replace X and S by affine neighbourhoods of x and s. Then X is Noetherian, in particular U is quasi-compact (see Morphisms, Lemma 15.6 and Topology, Lemmas 9.2 and 12.13). Hence there exists a specialization $x' \rightsquigarrow x$ with $x' \in U$ (see Morphisms, Lemma 6.5). Note that $f(x') \neq s$. Thus we see all hypotheses of the lemma are satisfied and we win. $\square$

53. Stein factorization

03GX Stein factorization is the statement that a proper morphism $f : X \to S$ with $f_*\mathcal{O}_X = \mathcal{O}_S$ has connected fibres.

03GY **Lemma 53.1.** *Let S be a scheme. Let $f : X \to S$ be a universally closed and quasi-separated morphism. There exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[rd, "f"'] & & S' \arrow[ld, "\pi"] \\ & S & \end{tikzcd}$$

with the following properties:

1. *the morphism f' is universally closed, quasi-compact, quasi-separated, and surjective,*
2. *the morphism $\pi : S' \to S$ is integral,*
3. *we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,*
4. *we have $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$, and*
5. *S' is the normalization of S in X, see Morphisms, Definition 53.3.*

Formation of the factorization $f = \pi \circ f'$ commutes with flat base change.

Proof. By Morphisms, Lemma 41.8 the morphism f is quasi-compact. Hence the normalization S' of S in X is defined (Morphisms, Definition 53.3) and we have the factorization $X \to S' \to S$. By Morphisms, Lemma 53.11 we have (2), (4), and (5). The morphism f' is universally closed by Morphisms, Lemma 41.7. It is quasi-compact by Schemes, Lemma 21.14 and quasi-separated by Schemes, Lemma 21.13.

To show the remaining statements we may assume the base scheme S is affine, say $S = \mathrm{Spec}(R)$. Then $S' = \mathrm{Spec}(A)$ with $A = \Gamma(X, \mathcal{O}_X)$ an integral R-algebra. Thus it is clear that $f'_*\mathcal{O}_X$ is $\mathcal{O}_{S'}$ (because $f'_*\mathcal{O}_X$ is quasi-coherent, by Schemes, Lemma 24.1, and hence equal to $\widetilde{A}$). This proves (3).

Let us show that f' is surjective. As f' is universally closed (see above) the image of f' is a closed subset $V(I) \subset S' = \text{Spec}(A)$. Pick $h \in I$. Then $h|_X = f^\sharp(h)$ is a global section of the structure sheaf of X which vanishes at every point. As X is quasi-compact this means that $h|_X$ is a nilpotent section, i.e., $h^n|X = 0$ for some $n > 0$. But $A = \Gamma(X, \mathcal{O}_X)$, hence $h^n = 0$. As every element of I is nilpotent, we conclude that $V(I) = S'$ as desired.

By Cohomology of Schemes, Lemma 5.2 we see that formation of $f_*\mathcal{O}_X$ commutes with flat base change. Formation of the relative spectrum commutes with any base change by Constructions, Lemma 4.6. Thus formation of the factorization commutes with flat base change. □

0E0M **Lemma 53.2.** *In Lemma 53.1 assume in addition that f is locally of finite type. Then for $s \in S$ the fibre $\pi^{-1}(\{s\}) = \{s_1, \ldots, s_n\}$ is finite and the field extensions $\kappa(s_i)/\kappa(s)$ are finite.*

Proof. Recall that there are no specializations among the points of $\pi^{-1}(\{s\})$, see Algebra, Lemma 36.20. As f' is surjective, we find that $|X_s| \to \pi^{-1}(\{s\})$ is surjective. Observe that X_s is a quasi-separated scheme of finite type over a field (quasi-compactness was shown in the proof of the referenced lemma). Thus X_s is Noetherian (Morphisms, Lemma 15.6). A topological argument (omitted) now shows that $\pi^{-1}(\{s\})$ is finite. For each i we can pick a finite type point $x_i \in X_s$ mapping to s_i (Morphisms, Lemma 16.7). We conclude that $\kappa(s_i)/\kappa(s)$ is finite: x_i can be represented by a morphism $\text{Spec}(k_i) \to X_s$ of finite type (by our definition of finite type points) and hence $\text{Spec}(k_i) \to s = \text{Spec}(\kappa(s))$ is of finite type (as a composition of finite type morphisms), hence $k_i/\kappa(s)$ is finite (Morphisms, Lemma 16.1). □

03GZ **Lemma 53.3.** *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Then X_s is geometrically connected, if and only if for every étale neighbourhood $(U, u) \to (S, s)$ the base change $X_U \to U$ has connected fibre X_u.*

Proof. If X_s is geometrically connected, then any base change of it is connected. On the other hand, suppose that X_s is not geometrically connected. Then by Varieties, Lemma 7.11 we see that $X_s \times_{\text{Spec}(\kappa(s))} \text{Spec}(k)$ is disconnected for some finite separable field extension $k/\kappa(s)$. By Lemma 35.2 there exists an affine étale neighbourhood $(U, u) \to (S, s)$ such that $\kappa(u)/\kappa(s)$ is identified with $k/\kappa(s)$. In this case X_u is disconnected. □

03H0 **Theorem 53.4** (Stein factorization; Noetherian case)**.** *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & S' \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

with the following properties:

(1) *the morphism f' is proper with geometrically connected fibres,*
(2) *the morphism $\pi : S' \to S$ is finite,*

(3) *we have* $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,
(4) *we have* $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$, *and*
(5) S' *is the normalization of* S *in* X, *see Morphisms, Definition 53.3.*

Proof. Let $f = \pi \circ f'$ be the factorization of Lemma 53.1. Note that besides the conclusions of Lemma 53.1 we also have that f' is separated (Schemes, Lemma 21.13) and finite type (Morphisms, Lemma 15.8). Hence f' is proper. By Cohomology of Schemes, Proposition 19.1 we see that $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_S$-module. Hence we see that π is finite, i.e., (2) holds.

This proves all but the most interesting assertion, namely that all the fibres of f' are geometrically connected. It is clear from the discussion above that we may replace S by S', and we may therefore assume that S is Noetherian, affine, $f : X \to S$ is proper, and $f_*\mathcal{O}_X = \mathcal{O}_S$. Let $s \in S$ be a point of S. We have to show that X_s is geometrically connected. By Lemma 53.3 we see that it suffices to show X_u is connected for every étale neighbourhood $(U, u) \to (S, s)$. We may assume U is affine. Thus U is Noetherian (Morphisms, Lemma 15.6), the base change $f_U : X_U \to U$ is proper (Morphisms, Lemma 41.5), and that also $(f_U)_*\mathcal{O}_{X_U} = \mathcal{O}_U$ (Cohomology of Schemes, Lemma 5.2). Hence after replacing $(f : X \to S, s)$ by the base change $(f_U : X_U \to U, u)$ it suffices to prove that the fibre X_s is connected when $f_*\mathcal{O}_X = \mathcal{O}_S$. We can deduce this from Derived Categories of Schemes, Lemma 32.7 (by looking at idempotents in the structure sheaf of X_s) but we will also give a direct argument below.

Namely, we apply the theorem on formal functions, more precisely Cohomology of Schemes, Lemma 20.7. It tells us that

$$\mathcal{O}^\wedge_{S,s} = (f_*\mathcal{O}_X)^\wedge_s = \lim_n H^0(X_n, \mathcal{O}_{X_n})$$

where X_n is the nth infinitesimal neighbourhood of X_s. Since the underlying topological space of X_n is equal to that of X_s we see that if $X_s = T_1 \amalg T_2$ is a disjoint union of nonempty open and closed subschemes, then similarly $X_n = T_{1,n} \amalg T_{2,n}$ for all n. And this in turn means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent $e_{1,n}$, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$. It is clear that $e_{1,n+1}$ restricts to $e_{1,n}$ on X_n. Hence $e_1 = \lim e_{1,n}$ is a nontrivial idempotent of the limit. This contradicts the fact that $\mathcal{O}^\wedge_{S,s}$ is a local ring. Thus the assumption was wrong, i.e., X_s is connected, and we win. $\square$

03H2 **Theorem 53.5** (Stein factorization; general case)**.** *Let* S *be a scheme. Let* $f : X \to S$ *be a proper morphism. There exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[rd, "f"'] & & S' \arrow[ld, "\pi"] \\ & S & \end{tikzcd}$$

with the following properties:

(1) *the morphism* f' *is proper with geometrically connected fibres,*
(2) *the morphism* $\pi : S' \to S$ *is integral,*
(3) *we have* $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,
(4) *we have* $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$, *and*
(5) S' *is the normalization of* S *in* X, *see Morphisms, Definition 53.3.*

Proof. We may apply Lemma 53.1 to get the morphism $f' : X \to S'$. Note that besides the conclusions of Lemma 53.1 we also have that f' is separated (Schemes, Lemma 21.13) and finite type (Morphisms, Lemma 15.8). Hence f' is proper. At this point we have proved all of the statements except for the statement that f' has geometrically connected fibres.

We may assume that $S = \mathrm{Spec}(R)$ is affine. Set $R' = \Gamma(X, \mathcal{O}_X)$. Then $S' = \mathrm{Spec}(R')$. Thus we may replace S by S' and assume that $S = \mathrm{Spec}(R)$ is affine $R = \Gamma(X, \mathcal{O}_X)$. Next, let $s \in S$ be a point. Let $U \to S$ be an étale morphism of affine schemes and let $u \in U$ be a point mapping to s. Let $X_U \to U$ be the base change of X. By Lemma 53.3 it suffices to show that the fibre of $X_U \to U$ over u is connected. By Cohomology of Schemes, Lemma 5.2 we see that $\Gamma(X_U, \mathcal{O}_{X_U}) = \Gamma(U, \mathcal{O}_U)$. Hence we have to show: Given $S = \mathrm{Spec}(R)$ affine, $X \to S$ proper with $\Gamma(X, \mathcal{O}_X) = R$ and $s \in S$ is a point, the fibre X_s is connected.

To do this it suffices to show that the only idempotents $e \in H^0(X_s, \mathcal{O}_{X_s})$ are 0 and 1 (we already know that X_s is nonempty by Lemma 53.1). By Derived Categories of Schemes, Lemma 32.7 after replacing R by a principal localization we may assume e is the image of an element of R. Since $R \to H^0(X_s, \mathcal{O}_{X_s})$ factors through $\kappa(s)$ we conclude. □

Here is an application.

0AY8 **Lemma 53.6.** *Let $f : X \to S$ be a morphism of schemes. Assume*

1. *f is proper,*
2. *S is integral with generic point ξ,*
3. *S is normal,*
4. *X is reduced,*
5. *every generic point of an irreducible component of X maps to ξ,*
6. *we have $H^0(X_\xi, \mathcal{O}) = \kappa(\xi)$.*

Then $f_*\mathcal{O}_X = \mathcal{O}_S$ and f has geometrically connected fibres.

Proof. Apply Theorem 53.5 to get a factorization $X \to S' \to S$. It is enough to show that $S' = S$. This will follow from Morphisms, Lemma 54.8. Namely, S' is reduced because X is reduced (Morphisms, Lemma 53.8). The morphism $S' \to S$ is integral by the theorem cited above. Every generic point of S' lies over ξ by Morphisms, Lemma 53.9 and assumption (5). On the other hand, since S' is the relative spectrum of $f_*\mathcal{O}_X$ we see that the scheme theoretic fibre S'_ξ is the spectrum of $H^0(X_\xi, \mathcal{O})$ which is equal to $\kappa(\xi)$ by assumption. Hence S' is an integral scheme with function field equal to the function field of S. This finishes the proof. □

Here is another application.

0BUI **Lemma 53.7.** *Let $X \to S$ be a flat proper morphism of finite presentation. Let $n_{X/S}$ be the function on S counting the numbers of geometric connected components of fibres of f introduced in Lemma 28.3. Then $n_{X/S}$ is lower semi-continuous.*

Proof. Let $s \in S$. Set $n = n_{X/S}(s)$. Note that $n < \infty$ as the geometric fibre of $X \to S$ at s is a proper scheme over a field, hence Noetherian, hence has a finite number of connected components. We have to find an open neighbourhood V of s such that $n_{X/S}|_V \geq n$. Let $X \to S' \to S$ be the Stein factorization as in Theorem 53.5. By Lemma 53.2 there are finitely many points $s'_1, \ldots, s'_m \in S'$ lying over

s and the extensions $\kappa(s'_i)/\kappa(s)$ are finite. Then Lemma 42.1 tells us that after replacing S by an étale neighbourhood of s we may assume $S' = V_1 \amalg \ldots \amalg V_m$ as a scheme with $s'_i \in V_i$ and $\kappa(s'_i)/\kappa(s)$ purely inseparable. Then the schemes $X_{s'_i}$ are geometrically connected over $\kappa(s)$, hence $m = n$. The schemes $X_i = (f')^{-1}(V_i)$, $i = 1, \ldots, n$ are flat and of finite presentation over S. Hence the image of $X_i \to S$ is open (Morphisms, Lemma 25.10). Thus in a neighbourhood of s we see that $n_{X/S}$ is at least n. $\square$

0E0N **Lemma 53.8.** *Let $f : X \to S$ be a morphism of schemes. Assume*

1. *f is proper, flat, and of finite presentation, and*
2. *the geometric fibres of f are reduced.*

Then the function $n_{X/S} : S \to \mathbf{Z}$ counting the numbers of geometric connected components of fibres of f is locally constant.

Proof. By Lemma 53.7 the function $n_{X/S}$ is lower semincontinuous. For $s \in S$ consider the $\kappa(s)$-algebra

$$A = H^0(X_s, \mathcal{O}_{X_s})$$

By Varieties, Lemma 9.3 and the fact that X_s is geometrically reduced A is finite product of finite separable extensions of $\kappa(s)$. Hence $A \otimes_{\kappa(s)} \kappa(\overline{s})$ is a product of $\beta_0(s) = \dim_{\kappa(s)} H^0(E \otimes^{\mathbf{L}} \kappa(s))$ copies of $\kappa(\overline{s})$. Thus $X_{\overline{s}}$ has $\beta_0(s) = \dim_{\kappa(s)} A$ connected components. In other words, we have $n_{X/S} = \beta_0$ as functions on S. Thus $n_{X/S}$ is upper semi-continuous by Derived Categories of Schemes, Lemma 32.1. This finishes the proof. $\square$

A final application.

0CT9 **Lemma 53.9.** *Let (A, I) be a henselian pair. Let $X \to \mathrm{Spec}(A)$ be separated and of finite type. Set $X_0 = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. Let $Y \subset X_0$ be an open and closed subscheme such that $Y \to \mathrm{Spec}(A/I)$ is proper. Then there exists an open and closed subscheme $W \subset X$ which is proper over A with $W \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I) = Y$.*

A reference for the case of an adic Noetherian base is [DG67, III, Proposition 5.5.1]

Proof. We will denote $T \mapsto T_0$ the base change by $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$. By Chow's lemma (in the form of Limits, Lemma 12.1) there exists a surjective proper morphism $\varphi : X' \to X$ such that X' admits an immersion into $\mathbf{P}^n_A$. Set $Y' = \varphi^{-1}(Y)$. This is an open and closed subscheme of X'_0. Suppose the lemma holds for (X', Y'). Let $W' \subset X'$ be the open and closed subscheme proper over A such that $Y' = W'_0$. By Morphisms, Lemma 41.7 $W = \varphi(W') \subset X$ and $Q = \varphi(X' \setminus W') \subset X$ are closed subsets and by Morphisms, Lemma 41.9 W is proper over A. The image of $W \cap Q$ in $\mathrm{Spec}(A)$ is closed. Since (A, I) is henselian, if $W \cap Q$ is nonempty, then we find that $W \cap Q$ has a point lying over $\mathrm{Spec}(A/I)$. This is impossible as $W'_0 = Y' = \varphi^{-1}(Y)$. We conclude that W is an open and closed subscheme of X proper over A with $W_0 = Y$. Thus we reduce to the case described in the next paragraph.

Assume there exists an immersion $j : X \to \mathbf{P}^n_A$ over A. Let $\overline{X}$ be the scheme theoretic image of j. Since j is a quasi-compact morphism (Schemes, Lemma 21.14) we see that $j : X \to \overline{X}$ is an open immersion (Morphisms, Lemma 7.7). Hence the base change $j_0 : X_0 \to \overline{X}_0$ is an open immersion as well. Thus $j_0(Y) \subset \overline{X}_0$ is open. It is also closed by Morphisms, Lemma 41.7. Suppose that the lemma holds for $(\overline{X}, j_0(Y))$. Let $\overline{W} \subset \overline{X}$ be the corresponding open and closed subscheme proper over A such that $j_0(Y) = \overline{W}_0$. Then $T = \overline{W} \setminus j(X)$ is closed in $\overline{W}$, hence has closed image in $\mathrm{Spec}(A)$ by properness of $\overline{W}$ over A. Since (A, I) is henselian, we find that if T is nonempty, then there is a point of T mapping into $\mathrm{Spec}(A/I)$. This is impossible because $j_0(Y) = \overline{W}_0$ is contained in $j(X)$. Hence $\overline{W}$ is contained in $j(X)$ and we can set $W \subset X$ equal to the unique open and closed subscheme mapping isomorphically to $\overline{W}$ via j. Thus we reduce to the case described in the next paragraph.

Assume $X \subset \mathbf{P}^n_A$ is a closed subscheme. Then $X \to \mathrm{Spec}(A)$ is a proper morphism. Let $Z = X_0 \setminus Y$. This is an open and closed subscheme of X_0 and $X_0 = Y \amalg Z$. Let $X \to X' \to \mathrm{Spec}(A)$ be the Stein factorization as in Theorem 53.5. Let $Y' \subset X'_0$ and $Z' \subset X'_0$ be the images of Y and Z. Since the fibres of $X \to Z$ are geometrically connected, we see that $Y' \cap Z' = \emptyset$. Hence $X'_0 = Y' \amalg Z'$ as $X \to X'$ is surjective. Since $X' \to \mathrm{Spec}(A)$ is integral, we see that X' is the spectrum of an A-algebra integral over A. Recall that open and closed subsets of spectra correspond 1-to-1 with idempotents in the corresponding ring, see Algebra, Lemma 21.3. Hence by More on Algebra, Lemma 11.6 we see that we may write $X' = W' \amalg V'$ with W' and V' open and closed and with $Y' = W'_0$ and $Z' = V'_0$. Let W be the inverse image in X to finish the proof. $\square$

54. Generic flatness stratification

0H3Y We can use generic flatness to construct a stratification of the base such that a given module becomes flat over the strata.

0ASY **Lemma 54.1** (Generic flatness stratification). *Let $f : X \to S$ be a morphism of finite presentation between quasi-compact and quasi-separated schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Then there exists a $t \geq 0$ and closed subschemes*

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

such that $S_i \to S$ is defined by a finite type ideal sheaf, $S_0 \subset S$ is a thickening, and $\mathcal{F}$ pulled back to $X \times_S (S_i \setminus S_{i+1})$ is flat over $S_i \setminus S_{i+1}$.

Proof. We can find a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_0 \arrow[d] \\ S \arrow[r] & S_0 \end{tikzcd}$$

and a finitely presented $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ which pulls back to $\mathcal{F}$ such that X_0 and S_0 are of finite type over $\mathbf{Z}$. See Limits, Proposition 5.4 and Lemmas 10.1 and 10.2. Thus we may assume X and S are of finite type over $\mathbf{Z}$ and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module.

Assume X and S are of finite type over $\mathbf{Z}$ and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. In this case every quasi-coherent ideal is of finite type, hence we do not have to check the condition that S_i is cut out by a finite type ideal. Set $S_0 = S_{red}$ equal to the reduction of S. By generic flatness as stated in Morphisms, Proposition 27.2 there is a dense open $U_0 \subset S_0$ such that $\mathcal{F}$ pulled back to $X \times_S U_0$ is flat over U_0. Let $S_1 \subset S_0$ be the reduced closed subscheme whose underlying closed subset is $S \setminus U_0$. We continue in this way, provided $S_1 \neq \emptyset$, to find $S_0 \supset S_1 \supset \ldots$. Because S is Noetherian any descending chain of closed subsets stabilizes hence we see that $S_t = \emptyset$ for some $t \geq 0$. $\square$

0H3Z **Lemma 54.2.** *Let $f : X \to S$ be a morphism of finite presentation between quasi-compact and quasi-separated schemes. Then there exists a $t \geq 0$ and closed subschemes*

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

such that $S_i \to S$ is defined by a finite type ideal sheaf, $S_0 \subset S$ is a thickening, and $X \times_S (S_i \setminus S_{i+1})$ is flat over $S_i \setminus S_{i+1}$.

Proof. Apply Lemma 54.1 with $\mathcal{F} = \mathcal{O}_X$. $\square$

0H40 **Lemma 54.3.** *Let R be a Noetherian domain. Let $R \to A \to B$ be finite type ring maps. Let M be a finite A-module and let N a finite B-module. Let $M \to N$ be an A-linear map. There exists an nonzero $f \in R$ such that the cokernel of $M_f \to N_f$ is a flat R_f-module.*

Proof. By replacing M by the image of $M \to N$, we may assume $M \subset N$. Choose a filtration $0 = N_0 \subset N_1 \subset \ldots \subset N_t = N$ such that $N_i/N_{i-1} = B/\mathfrak{q}_i$ for some prime ideal $\mathfrak{q}_i \subset B$, see Algebra, Lemma 62.1. Set $M_i = M \cap N_i$. Then $Q = N/M$ has a filtration by the submodules $Q_i = N_i/M_i$. It suffices to prove Q_i/Q_{i-1} becomes flat after localizing at a nonzero element of f (since extensions of flat modules are flat by Algebra, Lemma 39.13). Since Q_i/Q_{i-1} is isomorphic to the cokernel of the map $M_i/M_{i-1} \to N_i/N_{i-1}$, we reduce to the case discussed in the next paragraph.

Assume B is a domain and $M \subset N = B$. After replacing A by the image of A in B we may assume $A \subset B$. By generic flatness, we may assume A and B are flat over R (Algebra, Lemma 118.1). It now suffices to show $M \to B$ becomes R-universally injective after replacing R by a principal localization (Algebra, Lemma 82.7). By generic freeness, we can find a nonzero $g \in A$ such that B_g is a free A_g-module (Algebra, Lemma 118.1). Thus we may choose a direct summand $M' \subset B_g$ as an A_g-module, which is finite free as an A_g-module, and such that $M \to B \to B_g$ factors through M'. Clearly, it suffices to show that $M \to M'$ becomes R-universally injective after replacing R by a principal localization.

Say $M' = A_g^{\oplus n}$. Since $M \subset M'$ is a finite A-module, we see that M is contained in $(1/g^m)A^{\oplus n}$ for some $m \geq 0$. After changing our basis for M' we may assume $M \subset A^{\oplus n}$. Then it suffices to show that $A^{\oplus n}/M$ and A_g/A become R-flat after replacing R by a principal localization. Namely, then $M' \to A^{\oplus n}$ and $A^{\oplus n} \to A_g^{\oplus n}$ are universally injective by Algebra, Lemma 39.12 and consequently so is the composition $M \to M' = A_g^{\oplus n}$.

By generic flatness (see reference above), we may assume the module $A^{\oplus n}/M$ is R-flat. For the quotient A_g/A we use the fact that

$$A_g/A = \operatorname{colim}(1/g^m)A/A \cong \operatorname{colim} A/g^m A$$

and the module $A/g^m A$ has a filtration of length m whose successive quotients are isomorphic to A/gA. Again by generic flatness we may assume A/gA is R-flat and hence each $A/g^m A$ is R-flat, and hence so is A_g/A. $\square$

Let $f : X \to Y$ be a morphism of schemes over a base scheme S. Let $Z \subset Y$ be the scheme theoretic image of f, see Morphisms, Section 6. Let $g : S' \to S$ be a morphism of schemes and let $f' : X \times_S S' \to Y \times_S S'$ be the base change of f by g. It is not always true that $Z \times_S S' \subset Y \times_S S'$ is the scheme theoretic image of f'. Let us say that *formation of the scheme theoretic image of f/S commutes with* *arbitrary base change* if for every g as above the scheme theoretic image of f' is equal to $Z \times_S S'$.

0H41 **Lemma 54.4.** *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes over S with both X and Y of finite presentation over S. Then there exists a $t \geq 0$ and closed subschemes*

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

with the following properties:

(1) *$S_i \to S$ is defined by a finite type ideal sheaf,*
(2) *$S_0 \subset S$ is a thickening, and*
(3) *with $T_i = S_i \setminus S_{i+1}$ and f_i the base change of f to T_i we have: formation of the scheme theoretic image of f_i/T_i commutes with arbitrary base change (see discussion above the lemma).*

Proof. We can find a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & S \arrow[d] \\ U \arrow[r] & V \arrow[r] & W \end{tikzcd}$$

with cartesian squares such that U, V, W are of finite type over $\mathbf{Z}$. Namely, first write S as a cofiltered limit of finite type schemes over $\mathbf{Z}$ with affine transition morphisms using Limits, Proposition 5.4 and then descend the morphism $X \to Y$ using Limits, Lemma 10.1. This reduces us to the case discussed in the next paragraph.

Assume S is Noetherian. In this case every quasi-coherent ideal is of finite type, hence we do not have to check the condition that S_i is cut out by a finite type ideal. Set $S_0 = S_{red}$ equal to the reduction of S. Let $\eta \in S_0$ be a generic point of an irreducible component of S_0. By Noetherian induction on the underlying topological space of S_0, we may assume the result holds for any closed subscheme of S_0 not containing η. Thus it suffices to show that there exists an open neighbourhood $U_0 \subset S_0$ such that the base change f_0 of f to U_0 has property (3).

Let R be a Noetherian domain. Let $f : X \to Y$ be a morphism of finite type schemes over R. By the discussion in the previous paragraph it suffices to show that after replacing R by R_g for some $g \in R$ nonzero and X, Y by their base changes to R_g, formation of the scheme theoretic image of f/R commutes with arbitrary base change.

Let $Y = V_1 \cup \ldots V_n$ be an affine open covering. Let $U_i = f^{-1}(V_i)$. If the statement is true for each of the morphisms $U_i \to V_i$ over R, then it holds for f. Namely, the scheme theoretic image of $U_i \to V_i$ is the intersection of V_i with the scheme theoretic image of $f : X \to Y$ by Morphisms, Lemma 6.3. Thus we may assume Y is affine.

Let $X = U_1 \cup \ldots U_n$ be an affine open covering. Then the scheme theoretic image of $X \to Y$ is the same as the scheme theoretic imge of $\coprod U_i \to Y$. Thus we may assume X is affine.

Say $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ and f corresponds to the R-algebra map $\varphi : A \to B$. Then the scheme theoretic image of f is $\mathrm{Spec}(A/\mathrm{Ker}(\varphi))$ and similarly after base change (by an affine morphism, but it is enough to check for those). Thus formation of the scheme theoretic image commutes with base change if $\text{Ker}(\varphi \otimes_R R') = \text{Ker}(\varphi) \otimes_R R'$ for all ring maps $R \to R'$.

After replacing R, A, B by R_g, A_g, B_g for a suitable nonzero g in R, we may assume A and B are flat over R. By Lemma 54.3 we may also assume B/A is a flat R-module. Then $0 \to \text{Ker}(\varphi) \to A \to B \to B/A \to 0$ is an exact sequence of flat R-modules, which implies the desired base change statement. □

55. Stratifying a morphism

0H42 Let $f : X \to S$ be a finitely presented morphism of quasi-compact and quasi-separated schemes. In Section 54 we have seen that we can stratify S such that X is flat over the strata. In this section look for stratifications of both S and X such that we obtain smooth strata; this won't quite work and we'll need a base change by finite locally free morphisms as well.

0H43 **Lemma 55.1.** *Let $f : X \to S$ be a morphism of schemes of finite presentation. Let $\eta \in S$ be a generic point of an irreducible component of S. Assume S is reduced. Then there exist*

1. *an open subscheme $U \subset S$ containing η,*
2. *a surjective, universally injective, finite locally free morphism $V \to U$,*
3. *a $t \geq 0$ and closed subschemes*

$$X \times_S V \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_i \to X \times_S V$ is defined by a finite type ideal sheaf, $Z_0 \subset X \times_S V$ is a thickening, and such that the morphism $Z_i \setminus Z_{i+1} \to V$ is smooth.

Proof. It is clear that we may replace S by an open neighbourhood of η and X by the restriction to this open. Thus we may assume $S = \text{Spec}(A)$ where A is a reduced ring and η corresponds to a minimal prime ideal $\mathfrak{p}$. Recall that the local ring $\mathcal{O}_{S,\eta} = A_\mathfrak{p}$ is equal to $\kappa(\mathfrak{p})$ in this case, see Algebra, Lemma 25.1.

Apply Varieties, Lemma 25.11 to the scheme X_η over $k = \kappa(\eta)$. Denote k'/k the purely inseparable field extension this produces. In the next paragraph we reduce to the case $k' = k$. (This step corresponds to finding the morphism $V \to U$ in the statement of the lemma; in particular we can take $V = U$ if the characteristic of $\kappa(\mathfrak{p})$ is zero.)

If the characteristic of $k = \kappa(\mathfrak{p})$ is zero, then $k' = k$. If the characteristic of $k = \kappa(\mathfrak{p})$ is $p > 0$, then p maps to zero in $A_\mathfrak{p} = \kappa(\mathfrak{p})$. Hence after replacing A by a principal localization (i.e., shrinking S) we may assume $p = 0$ in A. If $k' \neq k$, then there exists an $\beta \in k'$, $\beta \notin k$ such that $\beta^p \in k$. After replacing A by a principal localization we may assume there exists an $a \in A$ such that $\beta^p = a$. Set $A' = A[x]/(x^p - a)$. Then $S' = \text{Spec}(A') \to \text{Spec}(A) = S$ is finite locally free, surjective, and universally injective. Moreover, if $\mathfrak{p}' \subset A'$ denotes the unique prime ideal lying over $\mathfrak{p}$, then $A'_{\mathfrak{p}'} = k(\beta)$ and $k'/k(\beta)$ has smaller degree. Thus after replacing S by S' and η by the point η' corresponding to $\mathfrak{p}'$ we see that the degree of k' over the residue field of η has decreased. Continuing like this, by induction we reduce to the case $k' = \kappa(\mathfrak{p}) = \kappa(\eta)$.

Thus we may assume S is affine, reduced, and that we have a $t \geq 0$ and closed subschemes

$$X_\eta \supset Z_{\eta,0} \supset Z_{\eta,1} \supset \ldots \supset Z_{\eta,t} = \emptyset$$

such that $Z_{\eta,0} = (X_\eta)_{red}$ and $Z_{\eta,i} \setminus Z_{\eta,i+1}$ is smooth over η for all i. Recall that $\kappa(\eta) = \kappa(\mathfrak p) = A_\mathfrak p$ is the filtered colimit of A_a for $a \in A$, $a \notin \mathfrak p$. See Algebra, Lemma 9.9. Thus we can descend the diagram above to a corresponding diagram over $\text{Spec}(A_a)$ for some $a \in A$, $a \notin \mathfrak p$. More precisely, after replacing S by $\text{Spec}(A_a)$ we may assume we have a $t \geq 0$ and closed subschemes

$$X \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_i \to X$ is a closed immersion of finite presentation, such that $Z_0 \to X$ is a thickening, and such that $Z_i \setminus Z_{i+1}$ is smooth over S. In other words, the lemma holds. More precisely, we first use Limits, Lemma 10.1 to obtain morphisms

$$Z_t \to Z_{t-1} \to \ldots \to Z_0 \to X$$

over S, each of finite presentation, and whose base change to η produces the inclusions between the given closed subschemes above. After shrinking S further we may assume each of the morphisms is a closed immersion, see Limits, Lemma 8.5. After shrinking S we may assume $Z_0 \to X$ is surjective and hence a thickening, see Limits, Lemma 8.15. After shrinking S once more we may assume $Z_i \setminus Z_{i+1} \to S$ is smooth, see Limits, Lemma 8.9. This finishes the proof. □

0H44 **Lemma 55.2.** *Let $f : X \to S$ be a morphism of finite presentation between quasi-compact and quasi-separated schemes. Then there exists a $t \geq 0$ and closed subschemes*

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

such that

1. *$S_i \to S$ is defined by a finite type ideal sheaf,*
2. *$S_0 \subset S$ is a thickening,*
3. *for each i there exists a surjective finite locally free morphism $T_i \to S_i \setminus S_{i+1}$,*
4. *for each i there exists a $t_i \geq 0$ and closed subschemes*
$$X_i = X \times_S T_i \supset Z_{i,0} \supset Z_{i,1} \supset \ldots \supset Z_{i,t_i} = \emptyset$$
such that $Z_{i,j} \to X_i$ is defined by a finite type ideal sheaf, $Z_{i,0} \subset X_i$ is a thickening, and such that the morphism $Z_{i,j} \setminus Z_{i,j+1} \to T_i$ is smooth.

Proof. We can find a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_0 \arrow[d] \\ S \arrow[r] & S_0 \end{tikzcd}$$

such that X_0 and S_0 are of finite type over $\mathbf{Z}$. See Limits, Proposition 5.4 and Lemma 10.1. Thus we may assume X and S are of finite type over $\mathbf{Z}$. Namely, a solution of the problem posed by the lemma for $X_0 \to S_0$ will base change to a solution over S; details omitted.

Assume X and S are of finite type over $\mathbf{Z}$. In this case every quasi-coherent ideal is of finite type, hence we do not have to check the condition that S_i is cut out by a finite type ideal. Set $S_0 = S_{red}$ equal to the reduction of S. Let $\eta \in S_0$ be a generic point of an irreducible component. By Lemma 55.1 we can find an open subscheme $U \subset S_0$, a surjective, universally injective, finite locally free morphism $V \to U$, a $t_0 \geq 0$ and closed subschemes

$$X \times_S V \supset Z_{0,0} \supset Z_{0,1} \supset \ldots \supset Z_{0,t_0} = \emptyset$$

such that $Z_{0,i} \to X \times_S V$ is defined by a finite type ideal sheaf, $Z_{0,0} \subset X \times_S V$ is a thickening, and such that the morphism $Z_{0,i} \setminus Z_{0,i+1} \to V$ is smooth. Then we let $S_1 \subset S_0$ be the reduced induced subscheme structure on $S_0 \setminus U$. By Noetherian induction on the underlying topological space of S, we may assume that the lemma holds for $X \times_S S_1 \to S_1$. This produces $t \geq 1$ and

$$S_1 = S_1 \supset S_2 \supset \ldots \supset S_t = \emptyset$$

and t_i and $Z_{i,j}$ as in the statement of the lemma. This proves the lemma. □

56. Improving morphisms of relative dimension one

0GK6 We can make any curve be smooth and projective after extending the ground field, compactifying, and normalizing. This also implies results about finite type morphisms whose generic fibres have dimension 1.

0GK7 **Lemma 56.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\eta \in S$ be a generic point of an irreducible component of S. Assume f is separated, of finite presentation, and $\dim(X_\eta) \leq 1$. Then there exists a commutative diagram*

$$\begin{tikzcd}
\overline{Y}_1 \amalg \ldots \amalg \overline{Y}_n \arrow[dr] & Y_1 \amalg \ldots \amalg Y_n \arrow[l, "j"'] \arrow[r, "\nu"'] \arrow[d] & X_V \arrow[r] \arrow[d] & X_U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& T_1 \amalg \ldots \amalg T_n \arrow[r] & V \arrow[r] & U \arrow[r] & S
\end{tikzcd}$$

of schemes with the following properties:

1. *$U \subset X$ is an open neighbourhood of η,*
2. *$V \to U$ is a finite, surjective, universally injective morphism,*
3. *$X_U = U \times_S X$ and $X_V = V \times_S X$ are the base changes,*
4. *ν is finite, surjective, and there is an open $W \subset X_V$ such that*
   - (a) *W is dense in all fibres of $X_V \to V$,*
   - (b) *$\nu^{-1}(W) \cap Y_i$ is dense in all fibres of $Y_i \to T_i$, and*
   - (c) *$\nu^{-1}(W) \to W$ is a thickening,*
5. *j is an open immersion,*
6. *$T_i \to V$ is finite étale,*
7. *$Y_i \to T_i$ is surjective and smooth,*
8. *$\overline{Y}_i \to T_i$ is smooth, proper, with geometrically connected fibres of dimension ≤ 1.*

Proof. It is clear that we may replace S by an open neighbourhood of η and X by the restriction to this open. Moreover, we may replace S by its reduction and X by the base change to this reduction. Thus we may assume $S = \mathrm{Spec}(A)$ where A is a reduced ring and η corresponds to a minimal prime ideal $\mathfrak{p}$. Recall that the local ring $\mathcal{O}_{S,\eta} = A_\mathfrak{p}$ is equal to $\kappa(\mathfrak{p})$ in this case, see Algebra, Lemma 25.1.

Apply Varieties, Lemma 43.7 to the scheme X_η over $k = \kappa(\eta)$. Denote k'/k the purely inseparable field extension this produces. In the next paragraph we reduce to the case $k' = k$. (This step corresponds to finding the morphism $V \to U$ in the statement of the lemma; in particular we can take $V = U$ if the characteristic of $\kappa(\mathfrak p)$ is zero.)

If the characteristic of $k = \kappa(\mathfrak p)$ is zero, then $k' = k$. If the characteristic of $k = \kappa(\mathfrak p)$ is $p > 0$, then p maps to zero in $A_{\mathfrak p} = \kappa(\mathfrak p)$. Hence after replacing A by a principal localization (i.e., shrinking S) we may assume $p = 0$ in A. If $k' \neq k$, then there exists an $\beta \in k'$, $\beta \notin k$ such that $\beta^p \in k$. After replacing A by a principal localization we may assume there exists an $a \in A$ such that $\beta^p = a$. Set $A' = A[x]/(x^p - a)$. Then $S' = \mathrm{Spec}(A') \to \mathrm{Spec}(A) = S$ is finite, surjective, and universally injective. Moreover, if $\mathfrak p' \subset A'$ denotes the unique prime ideal lying over $\mathfrak p$, then $A'_{\mathfrak p'} = k(\beta)$ and $k'/k(\beta)$ has smaller degree. Thus after replacing S by S' and η by the point η' corresponding to $\mathfrak p'$ we see that the degree of k' over the residue field of η has decreased. Continuing like this, by induction we reduce to the case $k' = \kappa(\mathfrak p) = \kappa(\eta)$.

Thus we may assume S is affine, reduced, and that we have a diagram

$$\begin{tikzcd}
\overline{Y}_{1,\eta} \amalg \ldots \amalg \overline{Y}_{n,\eta} \arrow[dr] & Y_{1,\eta} \amalg \ldots \amalg Y_{n,\eta} \arrow[l, "j"'] \arrow[r, "\nu"'] \arrow[d] & X_\eta \arrow[d] \\
& \mathrm{Spec}(k_1) \amalg \ldots \amalg \mathrm{Spec}(k_n) \arrow[r] & \eta
\end{tikzcd}$$

of schemes with the following properties:

(1) ν is the normalization of X_η,
(2) j is an open immersion with dense image,
(3) $k_i/\kappa(\eta)$ is a finite separable extension for $i = 1, \ldots, n$,
(4) $\overline{Y}_{i,\eta}$ is smooth, projective, and geometrically irreducible of dimension ≤ 1 over k_i.

Recall that $\kappa(\eta) = \kappa(\mathfrak p) = A_{\mathfrak p}$ is the filtered colimit of A_a for $a \in A$, $a \notin \mathfrak p$. See Algebra, Lemma 9.9. Thus we can descend the diagram above to a corresponding diagram over $\mathrm{Spec}(A_a)$ for some $a \in A$, $a \notin \mathfrak p$. More precisely, after replacing S by $\mathrm{Spec}(A_a)$ we may assume we have a commutative diagram

$$\begin{tikzcd}
\overline{Y}_1 \amalg \ldots \amalg \overline{Y}_n \arrow[dr] & Y_1 \amalg \ldots \amalg Y_n \arrow[l, "j"'] \arrow[r, "\nu"'] \arrow[d] & X \arrow[d] \\
& T_1 \amalg \ldots \amalg T_n \arrow[r] & S
\end{tikzcd}$$

of schemes whose base change to η is the diagram above with the following properties

(1) ν is a finite, surjective morphism,
(2) j is an open immersion,
(3) $T_i \to S$ is finite étale for $i = 1, \ldots, n$,
(4) $Y_i \to T_i$ is smooth and surjective,
(5) $\overline{Y}_i \to T_i$ is smooth and proper and has geometrically connected fibres of dimension ≤ 1.

For this we first use Limits, Lemma 10.1 to obtain the diagram base changing to the previous diagram. Then we use Limits, Lemmas 8.10, 8.9, 8.3, 4.13, 8.12, 13.1, and 8.15 to obtain ν finite, surjective, j open immersion, $T_i \to S$ finite étale, $Y_i \to T$ smooth, $\overline{Y}_i \to T_i$ proper and smooth. Since Y_i cannot be empty, since smooth morphisms are open, and since $T_i \to S$ is finite étale, after shrinking S we may assume $Y_i \to T_i$ is surjective. Finally, the fibre of $\overline{Y}_i \to T_i$ over the unique point $\eta_i = \mathrm{Spec}(k_i)$ of T_i lying over η is geometrically connected. Hence by another shrinking we may assume the same thing is true for all fibres, see Lemma 53.8.

It remains to prove the existence of an open $W \subset X$ satisfying (a), (b), and (c). Since $\nu_\eta : \coprod Y_{i,\eta} \to X_\eta$ is the normalization morphism, we know by Varieties, Lemma 27.1 there exists a dense open $W_\eta \subset X_\eta$ such that $\nu^{-1}(W_\eta) \to W_\eta$ is equal to the inclusion of the reduction of W_η into W_η. Let $W \subset X$ be a quasi-compact open whose fibre over η is the open W_η we just found. After replacing $A = \Gamma(S, \mathcal{O}_S)$ by another localization we may assume $\nu^{-1}(W) \to W$ is a closed immersion, see Limits, Lemma 8.5. Since ν is also surjective we conclude $\nu^{-1}(W) \to W$ is a thickening. Set $W_i = \nu^{-1}(W) \cap Y_i$. Shrinking S once more we can assume $W_i \to T_i$ is surjective for all i (same argument as above). Then we find that $W_i \subset Y_i$ is dense in all fibres of $Y_i \to T_i$ as $Y_i \to T_i$ has geometrically irreducible fibres. Since ν is finite and surjective, it then follows that $W = \nu(\nu^{-1}(W))$ is dense in all fibres of $X \to S$ too. $\square$

57. Descending separated locally quasi-finite morphisms

02W7 In this section we show that "separated locally quasi-finite morphisms satisfy descent for fppf-coverings". See Descent, Definition 36.1 for terminology. This is in the marvellous (for many reasons) paper by Raynaud and Gruson hidden in the proof of [GR71, Lemma 5.7.1]. It can also be found in [Mur95], and [ABD$^+$66, Exposé X, Lemma 5.4] under the additional hypothesis that the morphism is locally of finite presentation. Here is the formal statement.

02W8 **Lemma 57.1.** *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fppf covering, see Topologies, Definition 7.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is separated and locally quasi-finite, then the descent datum is effective.*

Proof. Being separated and being locally quasi-finite are properties of morphisms of schemes which are preserved under any base change, see Schemes, Lemma 21.12 and Morphisms, Lemma 20.13. Hence Descent, Lemma 36.2 applies and it suffices to prove the statement of the lemma in case the fppf-covering is given by a single $\{X \to S\}$ flat surjective morphism of finite presentation of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $\pi : V \to X$ is separated and locally quasi-finite.

Let $W^1 \subset V$ be any affine open. Consider $W = \text{pr}_1(\varphi(W^1 \times_S X)) \subset V$. Here is a picture

$$\begin{tikzcd}
W^1 \times_S X \arrow[rrrrrr] \arrow[dddd] \arrow[dr] & & & & & & \varphi(W^1 \times_S X) \arrow[dl] \arrow[dddd] \\
& V \times_S X \arrow[rrrr, "\varphi"] \arrow[ddd] \arrow[dr] & & & & X \times_S V \arrow[dl] \arrow[ddd] & \\
& & X \times_S X \arrow[rr, "1"] \arrow[dd, "\text{pr}_0"'] & & X \times_S X \arrow[dd, "\text{pr}_1"] & & \\
& & & & & & \\
W^1 \arrow[r] & V \arrow[r] & X & & X & V \arrow[l] & W \arrow[l]
\end{tikzcd}$$

Ok, and now since $X \to S$ is flat and of finite presentation it is universally open (Morphisms, Lemma 25.10). Hence we conclude that W is open. Moreover, it is also clearly the case that W is quasi-compact, and $W^1 \subset W$. Moreover, we note that $\varphi(W \times_S X) = X \times_S W$ by the cocycle condition for φ. Hence we obtain a new descent datum (W, φ') by restricting φ to $W \times_S X$. Note that the morphism $W \to X$ is quasi-compact, separated and locally quasi-finite. This implies that it is separated and quasi-finite by definition. Hence it is quasi-affine by Lemma 43.2. Thus by Descent, Lemma 38.1 we see that the descent datum (W, φ') is effective.

In other words, we find that there exists an open covering $V = \bigcup W_i$ by quasi-compact opens W_i which are stable for the descent morphism φ. Moreover, for each such quasi-compact open $W \subset V$ the corresponding descent data (W, φ') is effective. This means the original descent datum is effective by glueing the schemes obtained from descending the opens W_i, see Descent, Lemma 35.13. $\square$

58. Relative finite presentation

05GX Let $R \to A$ be a finite type ring map. Let M be an A-module. In More on Algebra, Section 80 we defined what it means for M to be finitely presented relative to R. We also proved this notion has good localization properties and glues. Hence we can define the corresponding global notion as follows.

05H1 **Definition 58.1.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *finitely presented relative to S* or *of finite presentation relative to S* if there exists an affine open covering $S = \bigcup V_i$ and for every i an affine open covering $f^{-1}(V_i) = \bigcup_j U_{ij}$ such that $\mathcal{F}(U_{ij})$ is a $\mathcal{O}_X(U_{ij})$-module of finite presentation relative to $\mathcal{O}_S(V_i)$.

Note that this implies that $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module. If $X \to S$ is just locally of finite type, then $\mathcal{F}$ may be of finite presentation relative to S, without $X \to S$ being locally of finite presentation. We will see that $X \to S$ is locally of finite presentation if and only if $\mathcal{O}_X$ is of finite presentation relative to S.

09T7 **Lemma 58.2.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is of finite presentation relative to S,*
2. *for every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is finitely presented relative to $\mathcal{O}_S(V)$.*

Moreover, if this is true, then for every open subschemes $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the restriction $\mathcal{F}|_U$ is of finite presentation relative to V.

Proof. The final statement is clear from the equivalence of (1) and (2). It is also clear that (2) implies (1). Assume (1) holds. Let $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup U_{ij}$ be affine open coverings as in Definition 58.1. Let $U \subset X$ and $V \subset S$ be as in (2). By More on Algebra, Lemma 80.8 it suffices to find a standard open covering $U = \bigcup U_k$ of U such that $\mathcal{F}(U_k)$ is finitely presented relative to $\mathcal{O}_S(V)$. In other words, for every $u \in U$ it suffices to find a standard affine open $u \in U' \subset U$ such that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V)$. Pick i such that $f(u) \in V_i$ and then pick j such that $u \in U_{ij}$. By Schemes, Lemma 11.5 we can find $v \in V' \subset V \cap V_i$ which is standard affine open in V' and V_i. Then $f^{-1}V' \cap U$, resp. $f^{-1}V' \cap U_{ij}$ are standard affine opens of U, resp. U_{ij}. Applying the lemma again we can find $u \in U' \subset f^{-1}V' \cap U \cap U_{ij}$ which is standard affine open in both $f^{-1}V' \cap U$ and $f^{-1}V' \cap U_{ij}$. Thus U' is also a standard affine open of U and U_{ij}. By More on Algebra, Lemma 80.4 the assumption that $\mathcal{F}(U_{ij})$ is finitely presented relative to $\mathcal{O}_S(V_i)$ implies that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V_i)$. Since $\mathcal{O}_X(U') = \mathcal{O}_X(U') \otimes_{\mathcal{O}_S(V_i)} \mathcal{O}_S(V')$ we see from More on Algebra, Lemma 80.5 that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V')$. Applying More on Algebra, Lemma 80.4 again we conclude that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V)$. This finishes the proof. □

09T8 **Lemma 58.3.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

1. *If f is locally of finite presentation, then $\mathcal{F}$ is of finite presentation relative to S if and only if $\mathcal{F}$ is of finite presentation.*
2. *The morphism f is locally of finite presentation if and only if $\mathcal{O}_X$ is of finite presentation relative to S.*

Proof. Follows immediately from the definitions, see discussion following More on Algebra, Definition 80.2. □

09T9 **Lemma 58.4.** *Let $\pi : X \to Y$ be a finite morphism of schemes locally of finite type over a base scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is of finite presentation relative to S if and only if $\pi_*\mathcal{F}$ is of finite presentation relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 80.3 into the language of schemes. □

09TA **Lemma 58.5.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $S' \to S$ be a morphism of schemes, set $X' = X \times_S S'$ and denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to X'. If $\mathcal{F}$ is of finite presentation relative to S, then $\mathcal{F}'$ is of finite presentation relative to S'.*

Proof. Translation of the result of More on Algebra, Lemma 80.5 into the language of schemes. □

09TB **Lemma 58.6.** *Let $X \to Y \to S$ be morphisms of schemes which are locally of finite type. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $f : X \to Y$ is locally of finite presentation and $\mathcal{G}$ of finite presentation relative to S, then $f^*\mathcal{G}$ is of finite presentation relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 80.6 into the language of schemes. ☐

09TC **Lemma 58.7.** *Let $X \to Y \to S$ be morphisms of schemes which are locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $Y \to S$ is locally of finite presentation and $\mathcal{F}$ is of finite presentation relative to Y, then $\mathcal{F}$ is of finite presentation relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 80.7 into the language of schemes. ☐

09TD **Lemma 58.8.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules.*

1. *If $\mathcal{F}', \mathcal{F}''$ are finitely presented relative to S, then so is $\mathcal{F}$.*
2. *If $\mathcal{F}'$ is a finite type $\mathcal{O}_X$-module and $\mathcal{F}$ is finitely presented relative to S, then $\mathcal{F}''$ is finitely presented relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 80.9 into the language of schemes. ☐

09TE **Lemma 58.9.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}, \mathcal{F}'$ be quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{F} \oplus \mathcal{F}'$ is finitely presented relative to S, then so are $\mathcal{F}$ and $\mathcal{F}'$.*

Proof. Translation of the result of More on Algebra, Lemma 80.10 into the language of schemes. ☐

59. Relative pseudo-coherence

09UH This section is the analogue of More on Algebra, Section 81 for schemes. We strongly urge the reader to take a look at that section first. Although we have developed the material in this section and the material on pseudo-coherent complexes in Cohomology, Sections 46, 47, 48, and 49 for arbitrary complexes of $\mathcal{O}_X$-modules, if X is a scheme then working exclusively with objects in $D_{QCoh}(\mathcal{O}_X)$ greatly simplifies many of the lemmmas and arguments, often reducing the problem at hand immediately to the algebraic counterpart. Moreover, one of the first thing we do is to show that being relatively pseudo-coherent implies the cohomology sheaves are quasi-coherent, see Lemma 59.3. Hence, on a first reading we suggest the reader work exclusively with objects in $D_{QCoh}(\mathcal{O}_X)$.

09VC **Lemma 59.1.** *Let $X \to S$ be a finite type morphism of affine schemes. Let E be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. The following are equivalent*

1. *for some closed immersion $i : X \to \mathbf{A}^n_S$ the object Ri_*E of $D(\mathcal{O}_{\mathbf{A}^n_S})$ is m-pseudo-coherent, and*
2. *for all closed immersions $i : X \to \mathbf{A}^n_S$ the object Ri_*E of $D(\mathcal{O}_{\mathbf{A}^n_S})$ is m-pseudo-coherent.*

Proof. Say $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$. Let i correspond to the surjection $\alpha : R[x_1, \ldots, x_n] \to A$ and let $X \to \mathbf{A}^m_S$ correspond to $\beta : R[y_1, \ldots, y_m] \to A$. Choose

$f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{tikzcd} R[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[r, "y_j \mapsto f_j"'] \arrow[d, "x_i \mapsto g_i"] & R[x_1, \ldots, x_n] \arrow[d] \\ R[y_1, \ldots, y_m] \arrow[r] & A \end{tikzcd}$$

corresponding to the commutative diagram of closed immersions

$$\begin{tikzcd} \mathbf{A}^{n+m}_S & \mathbf{A}^n_S \arrow[l] \\ \mathbf{A}^m_S \arrow[u] & X \arrow[l] \arrow[u] \end{tikzcd}$$

Thus it suffices to show that under a closed immersion

$$f : \mathbf{A}^m_S \to \mathbf{A}^{n+m}_S$$

an object E of $D(\mathcal{O}_{\mathbf{A}^m_S})$ is m-pseudo-coherent if and only if Rf_*E is m-pseudo-coherent. This follows from Derived Categories of Schemes, Lemma 12.5 and the fact that $f_*\mathcal{O}_{\mathbf{A}^m_S}$ is a pseudo-coherent $\mathcal{O}_{\mathbf{A}^{n+m}_S}$-module. The pseudo-coherence of $f_*\mathcal{O}_{\mathbf{A}^m_S}$ is straightforward to prove directly, but it also follows from Derived Categories of Schemes, Lemma 10.2 and More on Algebra, Lemma 81.3. $\square$

Recall that if $f : X \to S$ is a morphism of scheme which is locally of finite type, then for every pair of affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$, the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite type (Morphisms, Lemma 15.2). Hence there always exist closed immersions $U \to \mathbf{A}^n_V$ and the following definition makes sense.

09UI **Definition 59.2.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D(\mathcal{O}_X)$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Fix $m \in \mathbf{Z}$.

(1) We say E is *m-pseudo-coherent relative to S* if there exists an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the equivalent conditions of Lemma 59.1 are satisfied for each of the pairs $(U_{ij} \to V_i, E|_{U_{ij}})$.
(2) We say E is *pseudo-coherent relative to S* if E is m-pseudo-coherent relative to S for all $m \in \mathbf{Z}$.
(3) We say $\mathcal{F}$ is *m-pseudo-coherent relative to S* if $\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is m-pseudo-coherent relative to S.
(4) We say $\mathcal{F}$ is *pseudo-coherent relative to S* if $\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is pseudo-coherent relative to S.

If X is quasi-compact and E is m-pseudo-coherent relative to S for some m, then E is bounded above. If E is pseudo-coherent relative to S, then E has quasi-coherent cohomology sheaves.

0CSU **Lemma 59.3.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. If E in $D(\mathcal{O}_X)$ is m-pseudo-coherent relative to S, then $H^i(E)$ is a quasi-coherent $\mathcal{O}_X$-module for $i > m$. If E is pseudo-coherent relative to S, then E is an object of $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Choose an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the equivalent conditions of Lemma 59.1 are satisfied for each of the pairs $(U_{ij} \to V_i, E|_{U_{ij}})$. Since being quasi-coherent is local on X, we may assume that there exists an closed immersion $i : X \to \mathbf{A}^n_S$ such that Ri_*E is m-pseudo-coherent on $\mathbf{A}^n_S$. By Derived Categories of Schemes, Lemma 10.1 this means that $H^q(Ri_*E)$ is quasi-coherent for $q > m$. Since i_* is an exact functor, we have $i_*H^q(E) = H^q(Ri_*E)$ is quasi-coherent on $\mathbf{A}^n_S$. By Morphisms, Lemma 4.1 this implies that $H^q(E)$ is quasi-coherent as desired (strictly speaking it implies there exists some quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $i_*\mathcal{F} = i_*H^q(E)$ and then Modules, Lemma 13.4 tells us that $\mathcal{F} \cong H^q(E)$ hence the result). $\square$

Next, we prove the condition of relative pseudo-coherence localizes well.

09VD **Lemma 59.4.** *Let S be an affine scheme. Let $V \subset S$ be a standard open. Let $X \to V$ be a finite type morphism of affine schemes. Let $U \subset X$ be an affine open. Let E be an object of $D(\mathcal{O}_X)$. If the equivalent conditions of Lemma 59.1 are satisfied for the pair $(X \to V, E)$, then the equivalent conditions of Lemma 59.1 are satisfied for the pair $(U \to S, E|_U)$.*

Proof. Write $S = \mathrm{Spec}(R)$, $V = D(f)$, $X = \mathrm{Spec}(A)$, and $U = D(g)$. Assume the equivalent conditions of Lemma 59.1 are satisfied for the pair $(X \to V, E)$.

Choose $R_f[x_1, \ldots, x_n] \to A$ surjective. Write $R_f = R[x_0]/(fx_0 - 1)$. Then $R[x_0, x_1, \ldots, x_n] \to A$ is surjective, and $R_f[x_1, \ldots, x_n]$ is pseudo-coherent as an $R[x_0, \ldots, x_n]$-module. Thus we have

$$X \to \mathbf{A}^n_V \to \mathbf{A}^{n+1}_S$$

and we can apply Derived Categories of Schemes, Lemma 12.5 to conclude that the pushforward E' of E to $\mathbf{A}^{n+1}_S$ is m-pseudo-coherent.

Choose an element $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to $g \in A$. Consider the surjection $R[x_0, \ldots, x_{n+1}] \to R[x_0, \ldots, x_n, 1/g']$. We obtain

$$\begin{tikzcd} X & U \arrow[l] \arrow[d] \arrow[dr] & \\ \mathbf{A}^{n+1}_S \arrow[from=u] & D(g') \arrow[l] \arrow[r] & \mathbf{A}^{n+2}_S \end{tikzcd}$$

where the lower left arrow is an open immersion and the lower right arrow is a closed immersion. We conclude as before that the pushforward of $E'|_{D(g')}$ to $\mathbf{A}^{n+2}_S$ is m-pseudo-coherent. Since this is also the pushforward of $E|_U$ to $\mathbf{A}^{n+2}_S$ we conclude the lemma is true. $\square$

09VE **Lemma 59.5.** *Let $X \to S$ be a finite type morphism of affine schemes. Let E be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Let $X = \bigcup U_i$ be a standard affine open covering. The following are equivalent*

(1) *the equivalent conditions of Lemma 59.1 hold for the pairs $(U_i \to S, E|_{U_i})$,*
(2) *the equivalent conditions of Lemma 59.1 hold for the pair $(X \to S, E)$.*

Proof. The implication (2) $\Rightarrow$ (1) is Lemma 59.4. Assume (1). Say $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ and $U_i = D(f_i)$. Write $1 = \sum f_i g_i$ in A. Consider the surjections

$$R[x_i, y_i, z_i] \to R[x_i, y_i, z_i]/(\sum y_i z_i - 1) \to A.$$

which sends y_i to f_i and z_i to g_i. Note that $R[x_i, y_i, z_i]/(\sum y_i z_i - 1)$ is pseudo-coherent as an $R[x_i, y_i, z_i]$-module. Thus it suffices to prove that the pushforward of E to $T = \text{Spec}(R[x_i, y_i, z_i]/(\sum y_i z_i - 1))$ is m-pseudo-coherent, see Derived Categories of Schemes, Lemma 12.5. For each i_0 it suffices to prove the restriction of this pushforward to $W_{i_0} = \text{Spec}(R[x_i, y_i, z_i, 1/y_{i_0}]/(\sum y_i z_i - 1))$ is m-pseudo-coherent. Note that there is a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U_{i_0} \arrow[l] \arrow[d] \\ T & W_{i_0} \arrow[l] \end{tikzcd}$$

which implies that the pushforward of E to T restricted to W_{i_0} is the pushforward of $E|_{U_{i_0}}$ to W_{i_0}. Since $R[x_i, y_i, z_i, 1/y_{i_0}]/(\sum y_i z_i - 1)$ is isomorphic to a polynomial ring over R this proves what we want. □

09UJ **Lemma 59.6.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D(\mathcal{O}_X)$. Fix $m \in \mathbf{Z}$. The following are equivalent*

1. *E is m-pseudo-coherent relative to S,*
2. *for every affine opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the equivalent conditions of Lemma 59.1 are satisfied for the pair $(U \to V, E|_U)$.*

Moreover, if this is true, then for every open subschemes $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the restriction $E|_U$ is m-pseudo-coherent relative to V.

Proof. The final statement is clear from the equivalence of (1) and (2). It is also clear that (2) implies (1). Assume (1) holds. Let $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup U_{ij}$ be affine open coverings as in Definition 59.2. Let $U \subset X$ and $V \subset S$ be as in (2). By Lemma 59.5 it suffices to find a standard open covering $U = \bigcup U_k$ of U such that the equivalent conditions of Lemma 59.1 are satisfied for the pairs $(U_k \to V, E|_{U_k})$. In other words, for every $u \in U$ it suffices to find a standard affine open $u \in U' \subset U$ such that the equivalent conditions of Lemma 59.1 are satisfied for the pair $(U' \to V, E|_{U'})$. Pick i such that $f(u) \in V_i$ and then pick j such that $u \in U_{ij}$. By Schemes, Lemma 11.5 we can find $v \in V' \subset V \cap V_i$ which is standard affine open in V' and V_i. Then $f^{-1}V' \cap U$, resp. $f^{-1}V' \cap U_{ij}$ are standard affine opens of U, resp. U_{ij}. Applying the lemma again we can find $u \in U' \subset f^{-1}V' \cap U \cap U_{ij}$ which is standard affine open in both $f^{-1}V' \cap U$ and $f^{-1}V' \cap U_{ij}$. Thus U' is also a standard affine open of U and U_{ij}. By Lemma 59.4 the assumption that the equivalent conditions of Lemma 59.1 are satisfied for the pair $(U_{ij} \to V_i, E|_{U_{ij}})$ implies that the equivalent conditions of Lemma 59.1 are satisfied for the pair $(U' \to V, E|_{U'})$. □

For objects of the derived category whose cohomology sheaves are quasi-coherent, we can relate relative m-pseudo-coherence to the notion defined in More on Algebra, Definition 81.4. We will use the fact that for an affine scheme $U = \text{Spec}(A)$ the functor $R\Gamma(U, -)$ induces an equivalence between $D_{QCoh}(\mathcal{O}_U)$ and $D(A)$, see Derived Categories of Schemes, Lemma 3.5. This functor is compatible with pullbacks: if E is an object of $D_{QCoh}(\mathcal{O}_U)$ and $A \to B$ is a ring map corresponding to a morphism of affine schemes $g : V = \text{Spec}(B) \to \text{Spec}(A) = U$, then $R\Gamma(V, Lg^*E) = R\Gamma(U, E) \otimes_A^{\mathbf{L}} B$. See Derived Categories of Schemes, Lemma 3.8.

09VF **Lemma 59.7.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Fix $m \in \mathbf{Z}$. The following are equivalent*

1. *E is m-pseudo-coherent relative to S,*
2. *there exists an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the complex of $\mathcal{O}_X(U_{ij})$-modules $R\Gamma(U_{ij}, E)$ is m-pseudo-coherent relative to $\mathcal{O}_S(V_i)$, and*
3. *for every affine opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the complex of $\mathcal{O}_X(U)$-modules $R\Gamma(U, E)$ is m-pseudo-coherent relative to $\mathcal{O}_S(V)$.*

Proof. Let U and V be as in (2) and choose a closed immersion $i : U \to \mathbf{A}^n_V$. A formal argument, using Lemma 59.6, shows it suffices to prove that $Ri_*(E|_U)$ is m-pseudo-coherent if and only if $R\Gamma(U, E)$ is m-pseudo-coherent relative to $\mathcal{O}_S(V)$. Say $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(R)$, and $\mathbf{A}^n_V = \mathrm{Spec}(R[x_1, \ldots, x_n]$. By the remarks preceding the lemma, $E|_U$ is quasi-isomorphic to the complex of quasi-coherent sheaves on U associated to the object $R\Gamma(U, E)$ of $D(A)$. Note that $R\Gamma(U, E) = R\Gamma(\mathbf{A}^n_V, Ri_*(E|_U))$ as i is a closed immersion (and hence i_* is exact). Thus Ri_*E is associated to $R\Gamma(U, E)$ viewed as an object of $D(R[x_1, \ldots, x_n])$. We conclude as m-pseudo-coherence of $Ri_*(E|_U)$ is equivalent to m-pseudo-coherence of $R\Gamma(U, E)$ in $D(R[x_1, \ldots, x_n])$ by Derived Categories of Schemes, Lemma 10.2 which is equivalent to $R\Gamma(U, E)$ is m-pseudo-coherent relative to $R = \mathcal{O}_S(V)$ by definition. □

09VG **Lemma 59.8.** *Let $i : X \to Y$ morphism of schemes locally of finite type over a base scheme S. Assume that i induces a homeomorphism of X with a closed subset of Y. Let E be an object of $D(\mathcal{O}_X)$. Then E is m-pseudo-coherent relative to S if and only if Ri_*E is m-pseudo-coherent relative to S.*

Proof. By Morphisms, Lemma 45.4 the morphism i is affine. Thus we may assume S, Y, and X are affine. Say $S = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(A)$, and $X = \mathrm{Spec}(B)$. The condition means that $A/\mathrm{rad}(A) \to B/\mathrm{rad}(B)$ is surjective; here $\mathrm{rad}(A)$ and $\mathrm{rad}(B)$ denote the Jacobson radical of A and B. As B is of finite type over A, we can find $b_1, \ldots, b_m \in \mathrm{rad}(B)$ which generate B as an A-algebra. Say $b_j^N = 0$ for all j. Consider the diagram of rings

$$\begin{tikzcd}
B & R[x_i, y_j]/(y_j^N) \arrow[l] & R[x_i, y_j] \arrow[l] \\
A \arrow[u] & R[x_i] \arrow[l] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

which translates into a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & \mathbf{A}^{n+m}_S \arrow[dl] \\
Y \arrow[r] & \mathbf{A}^n_S &
\end{tikzcd}$$

of affine schemes. By Lemma 59.6 we see that E is m-pseudo-coherent relative to S if and only if its pushforward to $\mathbf{A}^{n+m}_S$ is m-pseudo-coherent. By Derived Categories of Schemes, Lemma 12.5 we see that this is true if and only if its pushforward to T is m-pseudo-coherent. The same lemma shows that this holds if and only if the pushforward to $\mathbf{A}^n_S$ is m-pseudo-coherent. Again by Lemma 59.6 this holds if and only if Ri_*E is m-pseudo-coherent relative to S. □

09UK **Lemma 59.9.** *Let $\pi : X \to Y$ be a finite morphism of schemes locally of finite type over a base scheme S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then E is m-pseudo-coherent relative to S if and only if $R\pi_*E$ is m-pseudo-coherent relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 81.5 into the language of schemes. Observe that $R\pi_*$ indeed maps $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ by Derived Categories of Schemes, Lemma 4.1. To do the translation use Lemma 59.6. $\square$

09UL **Lemma 59.10.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let (E, E', E'') be a distinguished triangle of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$.*

1. *If E is $(m+1)$-pseudo-coherent relative to S and E' is m-pseudo-coherent relative to S then E'' is m-pseudo-coherent relative to S.*
2. *If E, E'' are m-pseudo-coherent relative to S, then E' is m-pseudo-coherent relative to S.*
3. *If E' is $(m+1)$-pseudo-coherent relative to S and E'' is m-pseudo-coherent relative to S, then E is $(m+1)$-pseudo-coherent relative to S.*

Moreover, if two out of three of E, E', E'' are pseudo-coherent relative to S, the so is the third.

Proof. Immediate from Lemma 59.6 and Cohomology, Lemma 47.4. $\square$

09UM **Lemma 59.11.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then*

1. *$\mathcal{F}$ is m-pseudo-coherent relative to S for all $m > 0$,*
2. *$\mathcal{F}$ is 0-pseudo-coherent relative to S if and only if $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module,*
3. *$\mathcal{F}$ is (-1)-pseudo-coherent relative to S if and only if $\mathcal{F}$ is quasi-coherent and finitely presented relative to S.*

Proof. Part (1) is immediate from the definition. To see part (3) we may work locally on X (both properties are local). Thus we may assume X and S are affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$. Then we see that $\mathcal{F}$ is (-1)-pseudo-coherent relative to S if and only if $i_*\mathcal{F}$ is (-1)-pseudo-coherent, which is true if and only if $i_*\mathcal{F}$ is an $\mathcal{O}_{\mathbf{A}^n_S}$-module of finite presentation, see Cohomology, Lemma 47.9. A module of finite presentation is quasi-coherent, see Modules, Lemma 11.2. By Morphisms, Lemma 4.1 we see that $\mathcal{F}$ is quasi-coherent if and only if $i_*\mathcal{F}$ is quasi-coherent. Having said this part (3) follows. The proof of (2) is similar but less involved. $\square$

09UN **Lemma 59.12.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let E, K be objects of $D(\mathcal{O}_X)$. If $E \oplus K$ is m-pseudo-coherent relative to S so are E and K.*

Proof. Follows from Cohomology, Lemma 47.6 and the definitions. $\square$

09UP **Lemma 59.13.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let $\mathcal{F}^\bullet$ be a (locally) bounded above complex of $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $(m-i)$-pseudo-coherent relative to S for all i. Then $\mathcal{F}^\bullet$ is m-pseudo-coherent relative to S.*

Proof. Follows from Cohomology, Lemma 47.7 and the definitions. $\square$

09UQ **Lemma 59.14.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$. If E is (locally) bounded above and $H^i(E)$ is $(m-i)$-pseudo-coherent relative to S for all i, then E is m-pseudo-coherent relative to S.*

Proof. Follows from Cohomology, Lemma 47.8 and the definitions. □

09UR **Lemma 59.15.** *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$ which is m-pseudo-coherent relative to S. Let $S' \to S$ be a morphism of schemes. Set $X' = X \times_S S'$ and denote E' the derived pullback of E to X'. If S' and X are Tor independent over S, then E' is m-pseudo-coherent relative to S'.*

Proof. The problem is local on X and X' hence we may assume X, S, S', and X' are affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$ and denote $i' : X' \to \mathbf{A}^n_{S'}$ the base change to S'. Denote $g : X' \to X$ and $g' : \mathbf{A}^n_{S'} \to \mathbf{A}^n_S$ the projections, so $E' = Lg^*E$. Since X and S' are tor-independent over S, the base change map (Cohomology, Remark 28.3) induces an isomorphism

$$Ri'_*(Lg^*E) = L(g')^*Ri_*E$$

Namely, for a point $x' \in X'$ lying over $x \in X$ the base change map on stalks at x' is the map

$$E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{A}^n_S,x}} \mathcal{O}_{\mathbf{A}^n_{S'},x'} \longrightarrow E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

coming from the closed immersions i and i'. Note that the source is quasi-isomorphic to a localization of $E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ which is isomorphic to the target as $\mathcal{O}_{X',x'}$ is isomorphic to (the same) localization of $\mathcal{O}_{X,x} \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ by assumption. We conclude the lemma holds by an application of Cohomology, Lemma 47.3. □

09US **Lemma 59.16.** *Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base S. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_Y)$. Assume*

1. *$\mathcal{O}_X$ is pseudo-coherent relative to Y[12], and*
2. *E is m-pseudo-coherent relative to S.*

Then Lf^*E is m-pseudo-coherent relative to S.

Proof. The problem is local on X. Thus we may assume X, Y, and S are affine. Arguing as in the proof of More on Algebra, Lemma 81.13 we can find a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[d, "f"'] & \mathbf{A}^d_Y \arrow[r, "j"'] \arrow[dl, "p"] & \mathbf{A}^{n+d}_S \arrow[dl] \\ Y \arrow[r] & \mathbf{A}^n_S & \end{tikzcd}$$

Observe that

$$Ri_*Lf^*E = Ri_*Li^*Lp^*E = Lp^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{A}^n_Y}} Ri_*\mathcal{O}_X$$

by Cohomology, Lemma 54.4. By assumption and the fact that Y is affine, we can represent $Ri_*\mathcal{O}_X = i_*\mathcal{O}_X$ by a complexes of finite free $\mathcal{O}_{\mathbf{A}^n_Y}$-modules $\mathcal{F}^\bullet$, with $\mathcal{F}^q = 0$ for $q > 0$ (details omitted; use Derived Categories of Schemes, Lemma 10.2 and More on Algebra, Lemma 81.7). By assumption E is bounded above, say

[12] This means f is pseudo-coherent, see Definition 60.2.

$H^q(E) = 0$ for $q > a$. Represent E by a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Y$-modules with $\mathcal{E}^q = 0$ for $q > a$. Then the derived tensor product above is represented by $\text{Tot}(p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^\bullet)$.

Since j is a closed immersion, the functor j_* is exact and Rj_* is computed by applying j_* to any representing complex of sheaves. Thus we have to show that $j_*\text{Tot}(p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^\bullet)$ is m-pseudo-coherent as a complex of $\mathcal{O}_{\mathbf{A}^{n+m}_S}$-modules. Note that $\text{Tot}(p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^\bullet)$ has a filtration by subcomplexes with successive quotients the complexes $p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^q[-q]$. Note that for $q \ll 0$ the complexes $p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^q[-q]$ have zero cohomology in degrees $\leq m$ and hence are m-pseudo-coherent. Hence, applying Lemma 59.10 and induction, it suffices to show that $p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^q[-q]$ is pseudo-coherent relative to S for all q. Note that $\mathcal{F}^q = 0$ for $q > 0$. Since also $\mathcal{F}^q$ is finite free this reduces to proving that $p^*\mathcal{E}^\bullet$ is m-pseudo-coherent relative to S which follows from Lemma 59.15 for instance. $\square$

09UT **Lemma 59.17.** *Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base S. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$. Assume $\mathcal{O}_Y$ is pseudo-coherent relative to S*[13]*. Then the following are equivalent*

1. *E is m-pseudo-coherent relative to Y, and*
2. *E is m-pseudo-coherent relative to S.*

Proof. The question is local on X, hence we may assume X, Y, and S are affine. Arguing as in the proof of More on Algebra, Lemma 81.13 we can find a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[d, "f"'] & \mathbf{A}^m_Y \arrow[r, "j"'] \arrow[dl, "p"] & \mathbf{A}^{n+m}_S \arrow[dl] \\ Y \arrow[r] & \mathbf{A}^n_S & \end{tikzcd}$$

The assumption that $\mathcal{O}_Y$ is pseudo-coherent relative to S implies that $\mathcal{O}_{\mathbf{A}^m_Y}$ is pseudo-coherent relative to $\mathbf{A}^m_S$ (by flat base change; this can be seen by using for example Lemma 59.15). This in turn implies that $j_*\mathcal{O}_{\mathbf{A}^n_Y}$ is pseudo-coherent as an $\mathcal{O}_{\mathbf{A}^{n+m}_S}$-module. Then the equivalence of the lemma follows from Derived Categories of Schemes, Lemma 12.5. $\square$

09UU **Lemma 59.18.** *Let*

$$\begin{tikzcd} X \arrow[rr, "i"'] \arrow[dr] & & P \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of schemes. Assume i is a closed immersion and $P \to S$ flat and locally of finite presentation. Let E be an object of $D(\mathcal{O}_X)$. Then the following are equivalent

1. *E is m-pseudo-coherent relative to S,*
2. *Ri_*E is m-pseudo-coherent relative to S, and*
3. *Ri_*E is m-pseudo-coherent on P.*

[13]This means $Y \to S$ is pseudo-coherent, see Definition 60.2.

Proof. The equivalence of (1) and (2) is Lemma 59.9. The equivalence of (2) and (3) follows from Lemma 59.17 applied to $\text{id} : P \to P$ provided we can show that $\mathcal{O}_P$ is pseudo-coherent relative to S. This follows from More on Algebra, Lemma 82.4 and the definitions. $\square$

60. Pseudo-coherent morphisms

067X Avoid reading this section at all cost. If you need some of this material, first take a look at the corresponding algebra sections, see More on Algebra, Sections 64, 81, and 82. For now the only thing you need to know is that a ring map $A \to B$ is pseudo-coherent if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a resolution by finite free $A[x_1, \ldots, x_n]$-modules.

067Y **Lemma 60.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *there exist an affine open covering $S = \bigcup V_j$ and for each j an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ij})$ is a pseudo-coherent ring map,*
(2) *for every pair of affine opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is pseudo-coherent, and*
(3) *f is locally of finite type and $\mathcal{O}_X$ is pseudo-coherent relative to S.*

Proof. To see the equivalence of (1) and (2) it suffices to check conditions (1)(a), (b), (c) of Morphisms, Definition 14.1 for the property of being a pseudo-coherent ring map. These properties follow (using localization is flat) from More on Algebra, Lemmas 81.12, 81.11, and 81.16.

If (1) holds, then f is locally of finite type as a pseudo-coherent ring map is of finite type by definition. Moreover, (1) implies via Lemma 59.7 and the definitions that $\mathcal{O}_X$ is pseudo-coherent relative to S. Conversely, if (3) holds, then we see that for every U and V as in (2) the ring $\mathcal{O}_X(U)$ is of finite type over $\mathcal{O}_S(V)$ and $\mathcal{O}_X(U)$ is as a module pseudo-coherent relative to $\mathcal{O}_S(V)$, see Lemmas 59.6 and 59.7. This is the definition of a pseudo-coherent ring map, hence (2) and (1) hold. $\square$

067Z **Definition 60.2.** A morphism of schemes $f : X \to S$ is called *pseudo-coherent* if the equivalent conditions of Lemma 60.1 are satisfied. In this case we also say that X is pseudo-coherent over S.

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

0680 **Lemma 60.3.** *A flat base change of a pseudo-coherent morphism is pseudo-coherent.*

Proof. This translates into the following algebra result: Let $A \to B$ be a pseudo-coherent ring map. Let $A \to A'$ be flat. Then $A' \to B \otimes_A A'$ is pseudo-coherent. This follows from the more general More on Algebra, Lemma 81.12. $\square$

0681 **Lemma 60.4.** *A composition of pseudo-coherent morphisms of schemes is pseudo-coherent.*

Proof. This translates into the following algebra result: If $A \to B \to C$ are composable pseudo-coherent ring maps then $A \to C$ is pseudo-coherent. This follows from either More on Algebra, Lemma 81.13 or More on Algebra, Lemma 81.15. $\square$

0682 **Lemma 60.5.** *A pseudo-coherent morphism is locally of finite presentation.*

Proof. Immediate from the definitions. □

0695 **Lemma 60.6.** *A flat morphism which is locally of finite presentation is pseudo-coherent.*

Proof. This follows from the fact that a flat ring map of finite presentation is pseudo-coherent (and even perfect), see More on Algebra, Lemma 82.4. □

0683 **Lemma 60.7.** *Let $f : X \to Y$ be a morphism of schemes pseudo-coherent over a base scheme S. Then f is pseudo-coherent.*

Proof. This translates into the following algebra result: If $R \to A \to B$ are composable ring maps and $R \to A$, $R \to B$ pseudo-coherent, then $R \to B$ is pseudo-coherent. This follows from More on Algebra, Lemma 81.15. □

0AVX **Lemma 60.8.** *Let $f : X \to S$ be a finite morphism of schemes. Then f is pseudo-coherent if and only if $f_*\mathcal{O}_X$ is pseudo-coherent as an $\mathcal{O}_S$-module.*

Proof. Translated into algebra this lemma says the following: If $R \to A$ is a finite ring map, then $R \to A$ is pseudo-coherent as a ring map (which means by definition that A as an A-module is pseudo-coherent relative to R) if and only if A is pseudo-coherent as an R-module. This follows from the more general More on Algebra, Lemma 81.5. □

0684 **Lemma 60.9.** *Let $f : X \to S$ be a morphism of schemes. If S is locally Noetherian, then f is pseudo-coherent if and only if f is locally of finite type.*

Proof. This translates into the following algebra result: If $R \to A$ is a finite type ring map with R Noetherian, then $R \to A$ is pseudo-coherent if and only if $R \to A$ is of finite type. To see this, note that a pseudo-coherent ring map is of finite type by definition. Conversely, if $R \to A$ is of finite type, then we can write $A = R[x_1, \ldots, x_n]/I$ and it follows from More on Algebra, Lemma 64.17 that A is pseudo-coherent as an $R[x_1, \ldots, x_n]$-module, i.e., $R \to A$ is a pseudo-coherent ring map. □

0696 **Lemma 60.10.** *The property $\mathcal{P}(f) =$"f is pseudo-coherent" is fpqc local on the base.*

Proof. We will use the criterion of Descent, Lemma 22.4 to prove this. By Definition 60.2 being pseudo-coherent is Zariski local on the base. By Lemma 60.3 being pseudo-coherent is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 22.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an A-algebra. If $C \otimes_A B$ is pseudo-coherent as an $B[x_1, \ldots, x_n]$-module, then C is pseudo-coherent as a $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 64.15. □

0697 **Lemma 60.11.** *Let $A \to B$ be a flat ring map of finite presentation. Let $I \subset B$ be an ideal. Then $A \to B/I$ is pseudo-coherent if and only if I is pseudo-coherent as a B-module.*

Proof. Choose a presentation $B = A[x_1, \ldots, x_n]/J$. Note that B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module because $A \to B$ is a pseudo-coherent ring map by Lemma 60.6. Note that $A \to B/I$ is pseudo-coherent if and only if B/I is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module. By More on Algebra, Lemma 64.11 we see this is equivalent to the condition that B/I is pseudo-coherent as an B-module. This proves the lemma as the short exact sequence $0 \to I \to B \to B/I \to 0$ shows that I is pseudo-coherent if and only if B/I is (see More on Algebra, Lemma 64.6). $\square$

The following lemma will be obsoleted by the stronger Lemma 60.13.

0698 **Lemma 60.12.** *The property $\mathcal{P}(f) =$"f is pseudo-coherent" is syntomic local on the source.*

Proof. We will use the criterion of Descent, Lemma 26.4 to prove this. It follows from Lemmas 60.6 and 60.4 that being pseudo-coherent is preserved under precomposing with flat morphisms locally of finite presentation, in particular under precomposing with syntomic morphisms (see Morphisms, Lemmas 30.7 and 30.6). It is clear from Definition 60.2 that being pseudo-coherent is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 26.4 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ pseudo-coherent. Then $X \to Y$ is pseudo-coherent. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma 14.1. Choose a closed immersion $X \to \mathbf{A}^n_Y$. By Algebra, Lemma 136.18 we can find an affine open covering $X' = \bigcup_{i=1,\ldots,n} X'_i$ and syntomic morphisms $W_i \to \mathbf{A}^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product diagrams

$$\begin{tikzcd} X'_i \arrow[r] \arrow[d] & W_i \arrow[d] \\ X \arrow[r] & \mathbf{A}^n_Y \end{tikzcd}$$

After replacing X' by $\coprod X'_i$ and setting $W = \coprod W_i$ we obtain a fibre product diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & W \arrow[d, "h"] \\ X \arrow[r] & \mathbf{A}^n_Y \end{tikzcd}$$

with $W \to \mathbf{A}^n_Y$ flat and of finite presentation and $X' \to Y$ still pseudo-coherent. Since $W \to \mathbf{A}^n_Y$ is open (see Morphisms, Lemma 25.10) and $X' \to X$ is surjective we can find $f \in \Gamma(\mathbf{A}^n_Y, \mathcal{O})$ such that $X \subset D(f) \subset \text{Im}(h)$. Write $Y = \text{Spec}(R)$, $X = \text{Spec}(A)$, $X' = \text{Spec}(A')$ and $W = \text{Spec}(B)$, $A = R[x_1, \ldots, x_n]/I$ and $A' = B/IB$. Then $R \to A'$ is pseudo-coherent. Picture

$$\begin{tikzcd} A' = B/IB & & B \arrow[ll] \\ A = R[x_1, \ldots, x_n]/I \arrow[u] & & R[x_1, \ldots, x_n] \arrow[ll] \arrow[u] \end{tikzcd}$$

By Lemma 60.11 we see that IB is pseudo-coherent as a B-module. The ring map $R[x_1, \ldots, x_n]_f \to B_f$ is faithfully flat by our choice of f above. This implies that $I_f \subset R[x_1, \ldots, x_n]_f$ is pseudo-coherent, see More on Algebra, Lemma 64.15. Applying Lemma 60.11 one more time we see that $R \to A$ is pseudo-coherent. $\square$

0699 **Lemma 60.13.** *The property $\mathcal{P}(f) =$"f is pseudo-coherent" is fppf local on the source.*

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{g_i : X_i \to X\}$ be an fppf covering such that each composition $f \circ g_i$ is pseudo-coherent. According to Lemma 48.2 there exist

(1) a Zariski open covering $X = \bigcup U_j$,
(2) surjective finite locally free morphisms $W_j \to U_j$,
(3) Zariski open coverings $W_j = \bigcup_k W_{j,k}$,
(4) surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$

such that the fppf covering $\{h_{j,k} : T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Denote $\psi_{j,k} : T_{j,k} \to X_{\alpha(j,k)}$ the morphisms that witness the fact that $\{T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Note that $T_{j,k} \to X$ is a flat, locally finitely presented morphism, so both X_i and $T_{j,k}$ are pseudo-coherent over X by Lemma 60.6. Hence $\psi_{j,k} : T_{j,k} \to X_i$ is pseudo-coherent, see Lemma 60.7. Hence $T_{j,k} \to S$ is pseudo coherent as the composition of $\psi_{j,k}$ and $f \circ g_{\alpha(j,k)}$, see Lemma 60.4. Thus we see we have reduced the lemma to the case of a Zariski open covering (which is OK) and the case of a covering given by a single surjective finite locally free morphism which we deal with in the following paragraph.

Assume that $X' \to X \to S$ is a sequence of morphisms of schemes with $X' \to X$ surjective finite locally free and $X' \to Y$ pseudo-coherent. Our goal is to show that $X \to S$ is pseudo-coherent. Note that by Descent, Lemma 14.3 the morphism $X \to S$ is locally of finite presentation. It is clear that the problem reduces to the case that X', X and S are affine and $X' \to X$ is free of some rank $r > 0$. The corresponding algebra problem is the following: Suppose $R \to A \to A'$ are ring maps such that $R \to A'$ is pseudo-coherent, $R \to A$ is of finite presentation, and $A' \cong A^{\oplus r}$ as an A-module. Goal: Show $R \to A$ is pseudo-coherent. The assumption that $R \to A'$ is pseudo-coherent means that A' as an A'-module is pseudo-coherent relative to R. By More on Algebra, Lemma 81.5 this implies that A' as an A-module is pseudo-coherent relative to R. Since $A' \cong A^{\oplus r}$ as an A-module we see that A as an A-module is pseudo-coherent relative to R, see More on Algebra, Lemma 81.8. This by definition means that $R \to A$ is pseudo-coherent and we win. $\square$

61. Perfect morphisms

0685 In order to understand the material in this section you have to understand the material of the section on pseudo-coherent morphisms just a little bit. For now the only thing you need to know is that a ring map $A \to B$ is perfect if and only if it is pseudo-coherent and B has finite tor dimension as an A-module.

0686 **Lemma 61.1.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. The following are equivalent*

(1) *there exist an affine open covering $S = \bigcup V_j$ and for each j an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ij})$ is a perfect ring map, and*
(2) *for every pair of affine opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is perfect.*

Proof. Assume (1) and let U, V be as in (2). It follows from Lemma 60.1 that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is pseudo-coherent. Hence it suffices to prove that the property of a ring map being "of finite tor dimension" satisfies conditions (1)(a), (b), (c) of Morphisms, Definition 14.1. These properties follow from More on Algebra, Lemmas 66.11, 66.14, and 66.16. Some details omitted. □

0687 **Definition 61.2.** A morphism of schemes $f : X \to S$ is called *perfect* if the equivalent conditions of Lemma 61.1 are satisfied. In this case we also say that X is perfect over S.

Note that a perfect morphism is in particular pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

0688 **Lemma 61.3.** *A flat base change of a perfect morphism is perfect.*

Proof. This translates into the following algebra result: Let $A \to B$ be a perfect ring map. Let $A \to A'$ be flat. Then $A' \to B \otimes_A A'$ is perfect. This result for pseudo-coherent ring maps we have seen in Lemma 60.3. The corresponding fact for finite tor dimension follows from More on Algebra, Lemma 66.14. □

0689 **Lemma 61.4.** *A composition of perfect morphisms of schemes is perfect.*

Proof. This translates into the following algebra result: If $A \to B \to C$ are composable perfect ring maps then $A \to C$ is perfect. We have seen this is the case for pseudo-coherent in Lemma 60.4 and its proof. By assumption there exist integers n, m such that B has tor dimension $\leq n$ over A and C has tor dimension $\leq m$ over B. Then for any A-module M we have

$$M \otimes_A^{\mathbf{L}} C = (M \otimes_A^{\mathbf{L}} B) \otimes_B^{\mathbf{L}} C$$

and the spectral sequence of More on Algebra, Example 62.4 shows that $\text{Tor}_p^A(M, C) = 0$ for $p > n + m$ as desired. □

068A **Lemma 61.5.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *f is flat and perfect, and*
2. *f is flat and locally of finite presentation.*

Proof. The implication (2) $\Rightarrow$ (1) is More on Algebra, Lemma 82.4. The converse follows from the fact that a pseudo-coherent morphism is locally of finite presentation, see Lemma 60.5. □

068B **Lemma 61.6.** *Let $f : X \to S$ be a morphism of schemes. Assume S is regular and f is locally of finite type. Then f is perfect.*

Proof. See More on Algebra, Lemma 82.5. □

068C **Lemma 61.7.** *A regular immersion of schemes is perfect. A Koszul-regular immersion of schemes is perfect.*

Proof. Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 21.2, it suffices to prove the second statement. This translates into the following algebraic statement: Suppose that $I \subset A$ is an ideal generated by a Koszul-regular sequence $f_1, \ldots, f_r$ of A. Then $A \to A/I$ is a perfect ring map. Since $A \to A/I$ is surjective this is a presentation of A/I by a polynomial algebra over A. Hence it suffices to see that A/I is pseudo-coherent as an A-module and has finite tor dimension. By definition of a Koszul sequence the Koszul complex

$K(A, f_1, \ldots, f_r)$ is a finite free resolution of A/I. Hence A/I is a perfect complex of A-modules and we win. $\square$

068D **Lemma 61.8.** *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ perfect. Then $f : X \to Y$ is perfect.

Proof. We can factor f as the composition

$$X \longrightarrow X \times_S Y \longrightarrow Y$$

where the first morphism is the map $i = (1, f)$ and the second morphism is the projection. Since $Y \to S$ is flat, see Morphisms, Lemma 34.9, we see that $X \times_S Y \to Y$ is perfect by Lemma 61.3. As $Y \to S$ is smooth, also $X \times_S Y \to X$ is smooth, see Morphisms, Lemma 34.5. Hence i is a section of a smooth morphism, therefore i is a regular immersion, see Divisors, Lemma 22.8. This implies that i is perfect, see Lemma 61.7. We conclude that f is perfect because the composition of perfect morphisms is perfect, see Lemma 61.4. $\square$

069A **Remark 61.9.** It is not true that a morphism between schemes X, Y perfect over a base S is perfect. An example is $S = \mathrm{Spec}(k)$, $X = \mathrm{Spec}(k)$, $Y = \mathrm{Spec}(k[x]/(x^2)$ and $X \to Y$ the unique S-morphism.

069B **Lemma 61.10.** *The property $\mathcal{P}(f) =$"f is perfect" is fpqc local on the base.*

Proof. We will use the criterion of Descent, Lemma 22.4 to prove this. By Definition 61.2 being perfect is Zariski local on the base. By Lemma 61.3 being perfect is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 22.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an A-algebra. If $C \otimes_A B$ is perfect as an $B[x_1, \ldots, x_n]$-module, then C is perfect as a $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 74.13. $\square$

069C **Lemma 61.11.** *Let $f : X \to S$ be a pseudo-coherent morphism of schemes. The following are equivalent*

1. *f is perfect,*
2. *$\mathcal{O}_X$ locally has finite tor dimension as a sheaf of $f^{-1}\mathcal{O}_S$-modules, and*
3. *for all $x \in X$ the ring $\mathcal{O}_{X,x}$ has finite tor dimension as an $\mathcal{O}_{S,f(x)}$-module.*

Proof. The problem is local on X and S. Hence we may assume that $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and f corresponds to a pseudo-coherent ring map $A \to B$.

If (1) holds, then B has finite tor dimension d as A-module. Then $B_\mathfrak{q}$ has tor dimension d as an $A_\mathfrak{p}$-module for all primes $\mathfrak{q} \subset B$ with $\mathfrak{p} = A \cap \mathfrak{q}$, see More on Algebra, Lemma 66.15. Then $\mathcal{O}_X$ has tor dimension d as a sheaf of $f^{-1}\mathcal{O}_S$-modules by Cohomology, Lemma 48.5. Thus (1) implies (2).

By Cohomology, Lemma 48.5 (2) implies (3).

Assume (3). We cannot use More on Algebra, Lemma 66.15 to conclude as we are not given that the tor dimension of $B_\mathfrak{q}$ over $A_\mathfrak{p}$ is bounded independent of $\mathfrak{q}$. Choose a presentation $A[x_1, \ldots, x_n] \to B$. Then B is pseudo-coherent as a $A[x_1, \ldots, x_n]$-module. Let $\mathfrak{q} \subset A[x_1, \ldots, x_n]$ be a prime ideal lying over $\mathfrak{p} \subset A$. Then either $B_\mathfrak{q}$ is zero or by assumption it has finite tor dimension as an $A_\mathfrak{p}$-module. Since the fibres of $A \to A[x_1, \ldots, x_n]$ have finite global dimension, we can apply More on Algebra, Lemma 77.5 to $A_\mathfrak{p} \to A[x_1, \ldots, x_n]_\mathfrak{q}$ to see that $B_\mathfrak{q}$ is a perfect $A[x_1, \ldots, x_n]_\mathfrak{q}$-module. Hence B is a perfect $A[x_1, \ldots, x_n]$-module by More on Algebra, Lemma 77.3. Thus $A \to B$ is a perfect ring map by definition. □

0G2E **Lemma 61.12.** *Let $i : Z \to X$ be a perfect closed immersion of schemes. Then $i_*\mathcal{O}_Z$ is a perfect $\mathcal{O}_X$-module, i.e., it is a perfect object of $D(\mathcal{O}_X)$.*

Proof. This is more or less immediate from the definition. Namely, let $U = \text{Spec}(A)$ be an affine open of X. Then $i^{-1}(U) = \text{Spec}(A/I)$ for some ideal $I \subset A$ and A/I has a finite resolution by finite projective A-modules by More on Algebra, Lemma 82.2. Hence $i_*\mathcal{O}_Z|_U$ can be represented by a finite length complex of finite locally free $\mathcal{O}_U$-modules. This is what we had to show, see Cohomology, Section 49. □

0B6G **Lemma 61.13.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a perfect proper morphism of schemes. Let $E \in D(\mathcal{O}_X)$ be perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$.*

Proof. We claim that Derived Categories of Schemes, Lemma 27.1 applies. Conditions (1) and (2) are immediate. Condition (3) is local on X. Thus we may assume X and S affine and E represented by a strictly perfect complex of $\mathcal{O}_X$-modules. Thus it suffices to show that $\mathcal{O}_X$ has finite tor dimension as a sheaf of $f^{-1}\mathcal{O}_S$-modules. This is equivalent to being perfect by Lemma 61.11. □

069D **Lemma 61.14.** *The property $\mathcal{P}(f) =$"f is perfect" is fppf local on the source.*

Proof. Let $\{g_i : X_i \to X\}_{i \in I}$ be an fppf covering of schemes and let $f : X \to S$ be a morphism such that each $f \circ g_i$ is perfect. By Lemma 60.13 we conclude that f is pseudo-coherent. Hence by Lemma 61.11 it suffices to check that $\mathcal{O}_{X,x}$ is an $\mathcal{O}_{S,f(x)}$-module of finite tor dimension for all $x \in X$. Pick $i \in I$ and $x_i \in X_i$ mapping to x. Then we see that $\mathcal{O}_{X_i,x_i}$ has finite tor dimension over $\mathcal{O}_{S,f(x)}$ and that $\mathcal{O}_{X,x} \to \mathcal{O}_{X_i,x_i}$ is faithfully flat. The desired conclusion follows from More on Algebra, Lemma 66.17. □

09RK **Lemma 61.15.** *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Assume*

1. *X is locally Noetherian,*
2. *$j \circ i$ is a regular immersion, and*
3. *i is perfect.*

Then i and j are regular immersions.

Proof. Since X (and hence Y) is locally Noetherian all 4 types of regular immersions agree, and moreover we may check whether a morphism is a regular immersion on the level of local rings, see Divisors, Lemma 20.8. Thus the result follows from Divided Power Algebra, Lemma 7.5. □

62. Local complete intersection morphisms

068E In Divisors, Section 21 we have defined 4 different types of regular immersions: regular, Koszul-regular, H_1-regular, and quasi-regular. In this section we consider morphisms $f : X \to S$ which locally on X factor as

$$\begin{tikzcd} X \arrow[rr, "i"'] \arrow[dr] & & \mathbf{A}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where i is a $*$-regular immersion for $* \in \{\emptyset, Koszul, H_1, quasi\}$. However, we don't know how to prove that this condition is independent of the factorization if $* = \emptyset$, i.e., when we require i to be a regular immersion. On the other hand, we want a local complete intersection morphism to be perfect, which is only going to be true if $* = Koszul$ or $* = \emptyset$. Hence we will define a *local complete intersection morphism* or *Koszul morphism* to be a morphism of schemes $f : X \to S$ that locally on X has a factorization as above with i a Koszul-regular immersion. To see that this works we first prove this is independent of the chosen factorizations.

069E **Lemma 62.1.** *Let S be a scheme. Let U, P, P' be schemes over S. Let $u \in U$. Let $i : U \to P$, $i' : U \to P'$ be immersions over S. Assume P and P' smooth over S. Then the following are equivalent*

1. *i is a Koszul-regular immersion in a neighbourhood of u, and*
2. *i' is a Koszul-regular immersion in a neighbourhood of u.*

Proof. Assume i is a Koszul-regular immersion in a neighbourhood of u. Consider the morphism $j = (i, i') : U \to P \times_S P' = P''$. Since $P'' = P \times_S P' \to P$ is smooth, it follows from Divisors, Lemma 22.9 that j is a Koszul-regular immersion, whereupon it follows from Divisors, Lemma 22.12 that i' is a Koszul-regular immersion. $\square$

Before we state the definition, let us make the following simple remark. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $x \in X$. Then there exist an open neighbourhood $U \subset X$ and a factorization of $f|_U$ as the composition of an immersion $i : U \to \mathbf{A}^n_S$ followed by the projection $\mathbf{A}^n_S \to S$ which is smooth. Picture

$$\begin{tikzcd} X \arrow[dr] & U \arrow[l] \arrow[r, "i"'] \arrow[d] & \mathbf{A}^n_S = P \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

In fact you can do this with any affine open neighbourhood U of x in X, see Morphisms, Lemma 39.2.

069F **Definition 62.2.** Let $f : X \to S$ be a morphism of schemes.

1. Let $x \in X$. We say that f is *Koszul at x* if f is of finite type at x and there exists an open neighbourhood and a factorization of $f|_U$ as $\pi \circ i$ where $i : U \to P$ is a Koszul-regular immersion and $\pi : P \to S$ is smooth.
2. We say f is a *Koszul morphism*, or that f is a *local complete intersection morphism* if f is Koszul at every point.

We have seen above that the choice of the factorization $f|_U = \pi \circ i$ is irrelevant, i.e., given a factorization of $f|_U$ as an immersion i followed by a smooth morphism π, whether or not i is Koszul regular in a neighbourhood of x is an intrinsic property of f at x. Let us record this here explicitly as a lemma so that we can refer to it

069G **Lemma 62.3.** *Let $f : X \to S$ be a local complete intersection morphism. Let P be a scheme smooth over S. Let $U \subset X$ be an open subscheme and $i : U \to P$ an immersion of schemes over S. Then i is a Koszul-regular immersion.*

Proof. This is the defining property of a local complete intersection morphism. See discussion above. □

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

069H **Lemma 62.4.** *Let $f : X \to S$ be a local complete intersection morphism. Then*

1. *f is locally of finite presentation,*
2. *f is pseudo-coherent, and*
3. *f is perfect.*

Proof. Since a perfect morphism is pseudo-coherent (because a perfect ring map is pseudo-coherent) and a pseudo-coherent morphism is locally of finite presentation (because a pseudo-coherent ring map is of finite presentation) it suffices to prove the last statement. Being perfect is a local property, hence we may assume that f factors as $\pi \circ i$ where π is smooth and i is a Koszul-regular immersion. A Koszul-regular immersion is perfect, see Lemma 61.7. A smooth morphism is perfect as it is flat and locally of finite presentation, see Lemma 61.5. Finally a composition of perfect morphisms is perfect, see Lemma 61.4. □

07DB **Lemma 62.5.** *Let $f : X = \mathrm{Spec}(B) \to S = \mathrm{Spec}(A)$ be a morphism of affine schemes. Then f is a local complete intersection morphism if and only if $A \to B$ is a local complete intersection homomorphism, see More on Algebra, Definition 33.2.*

Proof. Follows immediately from the definitions. □

Beware that a base change of a Koszul morphism is not Koszul in general.

069I **Lemma 62.6.** *A flat base change of a local complete intersection morphism is a local complete intersection morphism.*

Proof. Omitted. Hint: This is true because a base change of a smooth morphism is smooth and a flat base change of a Koszul-regular immersion is a Koszul-regular immersion, see Divisors, Lemma 21.3. □

069J **Lemma 62.7.** *A composition of local complete intersection morphisms is a local complete intersection morphism.*

Proof. Let $g : Y \to S$ and $f : X \to Y$ be local complete intersection morphisms. Let $x \in X$ and set $y = f(x)$. Choose an open neighbourhood $V \subset Y$ of y and a factorization $g|_V = \pi \circ i$ for some Koszul-regular immersion $i : V \to P$ and smooth morphism $\pi : P \to S$. Next choose an open neighbourhood U of $x \in X$ and a factorization $f|_U = \pi' \circ i'$ for some Koszul-regular immersion $i' : U \to P'$

and smooth morphism $\pi' : P' \to Y$. In fact, we may assume that $P' = \mathbf{A}^n_V$, see discussion preceding and following Definition 62.2. Picture:

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "i'"'] & P' = \mathbf{A}^n_V \arrow[d] & \\ Y \arrow[d] & & V \arrow[ll] \arrow[r, "i"'] & P \arrow[d] \\ S & & & S \arrow[lll] \end{tikzcd}$$

Set $P'' = \mathbf{A}^n_P$. Then $U \to P' \to P''$ is a Koszul-regular immersion as a composition of Koszul-regular immersions, namely i' and the flat base change of i via $P'' \to P$, see Divisors, Lemma 21.3 and Divisors, Lemma 21.7. Also $P'' \to P \to S$ is smooth as a composition of smooth morphisms, see Morphisms, Lemma 34.4. Hence we conclude that $X \to S$ is Koszul at x as desired. $\square$

069K **Lemma 62.8.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *f is flat and a local complete intersection morphism, and*
2. *f is syntomic.*

Proof. Working affine locally this is More on Algebra, Lemma 33.5. We also give a more geometric proof.

Assume (2). By Morphisms, Lemma 30.10 for every point x of X there exist affine open neighbourhoods U of x and V of $f(x)$ such that $f|_U : U \to V$ is standard syntomic. This means that $U = \mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to V = \mathrm{Spec}(R)$ where $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R. By Algebra, Lemma 136.12 the sequence $f_1, \ldots, f_c$ is a regular sequence in each local ring $R[x_1, \ldots, x_n]_\mathfrak{q}$ for every prime $\mathfrak{q} \supset (f_1, \ldots, f_c)$. Consider the Koszul complex $K_\bullet = K_\bullet(R[x_1, \ldots, x_n], f_1, \ldots, f_c)$ with homology groups $H_i = H_i(K_\bullet)$. By More on Algebra, Lemma 30.2 we see that $(H_i)_\mathfrak{q} = 0$, $i > 0$ for every $\mathfrak{q}$ as above. On the other hand, by More on Algebra, Lemma 28.6 we see that H_i is annihilated by $(f_1, \ldots, f_c)$. Hence we see that $H_i = 0$, $i > 0$ and $f_1, \ldots, f_c$ is a Koszul-regular sequence. This proves that $U \to V$ factors as a Koszul-regular immersion $U \to \mathbf{A}^n_V$ followed by a smooth morphism as desired.

Assume (1). Then f is a flat and locally of finite presentation (Lemma 62.4). Hence, according to Morphisms, Lemma 30.10 it suffices to show that the local rings $\mathcal{O}_{X_s,x}$ are local complete intersection rings. Choose, locally on X, a factorization $f = \pi \circ i$ for some Koszul-regular immersion $i : X \to P$ and smooth morphism $\pi : P \to S$. Note that $X \to P$ is a relative quasi-regular immersion over S, see Divisors, Definition 22.2. Hence according to Divisors, Lemma 22.4 we see that $X \to P$ is a regular immersion and the same remains true after any base change. Thus each fibre is a regular immersion, whence all the local rings of all the fibres of X are local complete intersections. $\square$

069L **Lemma 62.9.** *A regular immersion of schemes is a local complete intersection morphism. A Koszul-regular immersion of schemes is a local complete intersection morphism.*

Proof. Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 21.2, it suffices to prove the second statement. The second statement follows immediately from the definition. □

069M **Lemma 62.10.** *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism.

Proof. Immediate from the definitions. □

0E9K **Lemma 62.11.** *Let $f : X \to Y$ be a morphism of schemes. If f is locally of finite type and X and Y are regular, then f is a local complete intersection morphism.*

Proof. We may assume there is a factorization $X \to \mathbf{A}^n_Y \to Y$ where the first arrow is an immersion. As Y is regular also $\mathbf{A}^n_Y$ is regular by Algebra, Lemma 163.10. Hence $X \to \mathbf{A}^n_Y$ is a regular immersion by Divisors, Lemma 21.12. □

The following lemma is of a different nature.

09RL **Lemma 62.12.** *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume

(1) *S is locally Noetherian,*
(2) *$Y \to S$ is locally of finite type,*
(3) *$f : X \to Y$ is perfect,*
(4) *$X \to S$ is a local complete intersection morphism.*

Then $X \to Y$ is a local complete intersection morphism and $Y \to S$ is Koszul at $f(x)$ for all $x \in X$.

Proof. In the course of this proof all schemes will be locally Noetherian and all rings will be Noetherian. We will use without further mention that regular sequences and Koszul regular sequences agree in this setting, see More on Algebra, Lemma 30.7. Moreover, whether an ideal (resp. ideal sheaf) is regular may be checked on local rings (resp. stalks), see Algebra, Lemma 68.6 (resp. Divisors, Lemma 20.8)

The question is local. Hence we may assume S, X, Y are affine. In this situation we may choose a commutative diagram

$$\begin{tikzcd} \mathbf{A}^{n+m}_S \arrow[d] & X \arrow[l] \arrow[d] \\ \mathbf{A}^n_S \arrow[d] & Y \arrow[l] \arrow[dl] \\ S & \end{tikzcd}$$

whose horizontal arrows are closed immersions. Let $x \in X$ be a point and consider the corresponding commutative diagram of local rings

$$\begin{tikzcd} J \arrow[r] & \mathcal{O}_{\mathbf{A}^{n+m}_S, x} \arrow[r] & \mathcal{O}_{X,x} \\ I \arrow[r] \arrow[u] & \mathcal{O}_{\mathbf{A}^n_S, f(x)} \arrow[r] \arrow[u] & \mathcal{O}_{Y,f(x)} \arrow[u] \end{tikzcd}$$

where J and I are the kernels of the horizontal arrows. Since $X \to S$ is a local complete intersection morphism, the ideal J is generated by a regular sequence. Since $X \to Y$ is perfect the ring $\mathcal{O}_{X,x}$ has finite tor dimension over $\mathcal{O}_{Y,f(x)}$. Hence we may apply Divided Power Algebra, Lemma 7.6 to conclude that I and J/I are generated by regular sequences. By our initial remarks, this finishes the proof. $\square$

0FJ2 **Lemma 62.13.** *Let*

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume S is locally Noetherian, $Y \to S$ is locally of finite type, Y is regular, and $X \to S$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism and $Y \to S$ is Koszul at $f(x)$ for all $x \in X$.

Proof. This is a special case of Lemma 62.12 in view of Lemma 61.6 (and Morphisms, Lemma 15.8). $\square$

0FK1 **Lemma 62.14.** *Let $i : X \to Y$ be an immersion. If*

(1) *i is perfect,*
(2) *Y is locally Noetherian, and*
(3) *the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free,*

then i is a regular immersion.

Proof. Translated into algebra, this is Divided Power Algebra, Proposition 11.3. $\square$

0FV6 **Lemma 62.15.** *Let $f : X \to Y$ be a local complete intersection homomorphism. Then the naive cotangent complex $NL_{X/Y}$ is a perfect object of $D(\mathcal{O}_X)$ of tor-amplitude in $[-1, 0]$.*

Proof. Translated into algebra this is More on Algebra, Lemma 85.4. To do the translation use Lemmas 62.5 and 13.2 as well as Derived Categories of Schemes, Lemmas 3.5, 10.4 and 10.7. ☐

0FK2 **Lemma 62.16.** *Let $f : X \to Y$ be a perfect morphism of locally Noetherian schemes. The following are equivalent*

(1) *f is a local complete intersection morphism,*
(2) *$NL_{X/Y}$ has tor-amplitude in $[-1, 0]$, and*
(3) *$NL_{X/Y}$ is perfect with tor-amplitude in $[-1, 0]$.*

Proof. Translated into algebra this is Divided Power Algebra, Lemma 11.4. To do the translation use Lemmas 62.5 and 13.2 as well as Derived Categories of Schemes, Lemmas 3.5, 10.4 and 10.7. ☐

0FK3 **Lemma 62.17.** *Let $f : X \to Y$ be a flat morphism of finite presentation. The following are equivalent*

(1) *f is a local complete intersection morphism,*
(2) *f is syntomic,*
(3) *$NL_{X/Y}$ has tor-amplitude in $[-1, 0]$, and*
(4) *$NL_{X/Y}$ is perfect with tor-amplitude in $[-1, 0]$.*

Proof. Translated into algebra this is Divided Power Algebra, Lemma 11.5. To do the translation use Lemmas 62.5 and 13.2 as well as Derived Categories of Schemes, Lemmas 3.5, 10.4 and 10.7. ☐

The following lemma gives a characterization of smooth morphisms as flat morphisms whose diagonal is perfect.

0FDP **Lemma 62.18.** *Let $f : X \to Y$ be a finite type morphism of locally Noetherian schemes. Denote $\Delta : X \to X \times_Y X$ the diagonal morphism. The following are equivalent*

(1) *f is smooth,*
(2) *f is flat and $\Delta : X \to X \times_Y X$ is a regular immersion,*
(3) *f is flat and $\Delta : X \to X \times_Y X$ is a local complete intersection morphism,*
(4) *f is flat and $\Delta : X \to X \times_Y X$ is perfect.*

Proof. Assume (1). Then f is flat by Morphisms, Lemma 34.9. The projections $X \times_Y X \to X$ are smooth by Morphisms, Lemma 34.5. Hence the diagonal is a section to a smooth morphism and hence a regular immersion, see Divisors, Lemma 22.8. Hence (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is Lemma 62.9. The implication (3) $\Rightarrow$ (4) is Lemma 62.4. The interesting implication (4) $\Rightarrow$ (1) follows immediately from Divided Power Algebra, Lemma 10.2. ☐

069N **Lemma 62.19.** *The property $\mathcal{P}(f) =$"f is a local complete intersection morphism" is fpqc local on the base.*

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{S_i \to S\}$ be an fpqc covering of S. Assume that each base change $f_i : X_i \to S_i$ of f is a local complete intersection morphism. Note that this implies in particular that f is locally of finite type, see Lemma 62.4 and Descent, Lemma 23.10. Let $x \in X$. Choose an open neighbourhood U of x and an immersion $j : U \to \mathbf{A}^n_S$ over S (see discussion preceding Definition 62.2). We have to show that j is a Koszul-regular immersion. Since f_i is a local complete intersection morphism, we see that the base change

$j_i : U \times_S S_i \to \mathbf{A}^n_{S_i}$ is a Koszul-regular immersion, see Lemma 62.3. Because $\{\mathbf{A}^n_{S_i} \to \mathbf{A}^n_S\}$ is a fpqc covering we see from Descent, Lemma 23.32 that j is a Koszul-regular immersion as desired. $\square$

069P **Lemma 62.20.** *The property $\mathcal{P}(f) =$"f is a local complete intersection morphism" is syntomic local on the source.*

Proof. We will use the criterion of Descent, Lemma 26.4 to prove this. It follows from Lemmas 62.8 and 62.7 that being a local complete intersection morphism is preserved under precomposing with syntomic morphisms. It is clear from Definition 62.2 that being a local complete intersection morphism is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 26.4 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ a local complete intersection morphism. Then $X \to Y$ is a local complete intersection morphism. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma 14.1. Choose a closed immersion $X \to \mathbf{A}^n_Y$. By Algebra, Lemma 136.18 we can find an affine open covering $X' = \bigcup_{i=1,\ldots,n} X'_i$ and syntomic morphisms $W_i \to \mathbf{A}^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product diagrams

$$\begin{tikzcd} X'_i \arrow[r] \arrow[d] & W_i \arrow[d] \\ X \arrow[r] & \mathbf{A}^n_Y \end{tikzcd}$$

After replacing X' by $\coprod X'_i$ and setting $W = \coprod W_i$ we obtain a fibre product diagram of affine schemes

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & W \arrow[d, "h"] \\ X \arrow[r] & \mathbf{A}^n_Y \end{tikzcd}$$

with $h : W \to \mathbf{A}^n_Y$ syntomic and $X' \to Y$ still a local complete intersection morphism. Since $W \to \mathbf{A}^n_Y$ is open (see Morphisms, Lemma 25.10) and $X' \to X$ is surjective we see that X is contained in the image of $W \to \mathbf{A}^n_Y$. Choose a closed immersion $W \to \mathbf{A}^{n+m}_Y$ over $\mathbf{A}^n_Y$. Now the diagram looks like

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "h"] & \mathbf{A}^{n+m}_Y \arrow[ld] \\ X \arrow[r] & \mathbf{A}^n_Y & \end{tikzcd}$$

Because h is syntomic and hence a local complete intersection morphism (see above) the morphism $W \to \mathbf{A}^{n+m}_Y$ is a Koszul-regular immersion. Because $X' \to Y$ is a local complete intersection morphism the morphism $X' \to \mathbf{A}^{n+m}_Y$ is a Koszul-regular immersion. We conclude from Divisors, Lemma 21.8 that $X' \to W$ is a Koszul-regular immersion. Hence, since being a Koszul-regular immersion is fpqc local on the target (see Descent, Lemma 23.32) we conclude that $X \to \mathbf{A}^n_Y$ is a Koszul-regular immersion which is what we had to show. $\square$

06B8 **Lemma 62.21.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume both X and Y are flat and locally of finite presentation over S. Then the set*

$$\{x \in X \mid f \text{ Koszul at } x\}.$$

is open in X and its formation commutes with arbitrary base change $S' \to S$.

Proof. The set is open by definition (see Definition 62.2). Let $S' \to S$ be a morphism of schemes. Set $X' = S' \times_S X$, $Y' = S' \times_S Y$, and denote $f' : X' \to Y'$ the base change of f. Let $x' \in X'$ be a point such that f' is Koszul at x'. Denote $s' \in S'$, $x \in X$, $y' \in Y'$, $y \in Y$, $s \in S$ the image of x'. Note that f is locally of finite presentation, see Morphisms, Lemma 21.11. Hence we may choose an affine neighbourhood $U \subset X$ of x and an immersion $i : U \to \mathbf{A}^n_Y$. Denote $U' = S' \times_S U$ and $i' : U' \to \mathbf{A}^n_{Y'}$ the base change of i. The assumption that f' is Koszul at x' implies that i' is a Koszul-regular immersion in a neighbourhood of x', see Lemma 62.3. The scheme X' is flat and locally of finite presentation over S' as a base change of X (see Morphisms, Lemmas 25.8 and 21.4). Hence i' is a relative H_1-regular immersion over S' in a neighbourhood of x' (see Divisors, Definition 22.2). Thus the base change $i'_{s'} : U'_{s'} \to \mathbf{A}^n_{Y'_{s'}}$ is a H_1-regular immersion in an open neighbourhood of x', see Divisors, Lemma 22.1 and the discussion following Divisors, Definition 22.2. Since $s' = \mathrm{Spec}(\kappa(s')) \to \mathrm{Spec}(\kappa(s)) = s$ is a surjective flat universally open morphism (see Morphisms, Lemma 23.4) we conclude that the base change $i_s : U_s \to \mathbf{A}^n_{Y_s}$ is an H_1-regular immersion in a neighbourhood of x, see Descent, Lemma 23.32. Finally, note that $\mathbf{A}^n_Y$ is flat and locally of finite presentation over S, hence Divisors, Lemma 22.7 implies that i is a (Koszul-)regular immersion in a neighbourhood of x as desired. $\square$

06B9 **Lemma 62.22.** *Let $f : X \to Y$ be a local complete intersection morphism of schemes. Then f is unramified if and only if f is formally unramified and in this case the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free on X.*

Proof. The first assertion follows immediately from Lemma 6.8 and the fact that a local complete intersection morphism is locally of finite type. To compute the conormal sheaf of f we choose, locally on X, a factorization of f as $f = p \circ i$ where $i : X \to V$ is a Koszul-regular immersion and $V \to Y$ is smooth. By Lemma 11.13 we see that $\mathcal{C}_{X/Y}$ is a locally direct summand of $\mathcal{C}_{X/V}$ which is finite locally free as i is a Koszul-regular (hence quasi-regular) immersion, see Divisors, Lemma 21.5. $\square$

06BA **Lemma 62.23.** *Let $Z \to Y \to X$ be formally unramified morphisms of schemes. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 7.12 is short exact.

Proof. The question is local on Z hence we may assume there exists a factorization $Z \to \mathbf{A}^n_Y \to Y$ of the morphism $Z \to Y$. Then we get a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"'] \arrow[d, equal] & \mathbf{A}^n_Y \arrow[r] \arrow[d] & \mathbf{A}^n_X \arrow[d] \\ Z \arrow[r, "i"] & Y \arrow[r] & X \end{tikzcd}$$

As $Z \to Y$ is a local complete intersection morphism, we see that $Z \to \mathbf{A}^n_Y$ is a Koszul-regular immersion. Hence by Divisors, Lemma 21.6 the sequence

$$0 \to (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X} \to \mathcal{C}_{Z/\mathbf{A}^n_X} \to \mathcal{C}_{Z/\mathbf{A}^n_Y} \to 0$$

is exact and locally split. Note that $i^*\mathcal{C}_{Y/X} = (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X}$ by Lemma 7.7 and note that the diagram

$$\begin{tikzcd} (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X} \arrow[r] & \mathcal{C}_{Z/\mathbf{A}^n_X} \\ i^*\mathcal{C}_{Y/X} \arrow[r] \arrow[u, "\cong"] & \mathcal{C}_{Z/X} \arrow[u] \end{tikzcd}$$

is commutative. Hence the lower horizontal arrow is a locally split injection. This proves the lemma. □

63. Exact sequences of differentials and conormal sheaves

06BB In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to a pair of composable morphisms of schemes.

Let $g : Z \to Y$ and $f : Y \to X$ be morphisms of schemes.

(1) There is a canonical exact sequence
$$g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,$$
see Morphisms, Lemma 32.9. If $g : Z \to Y$ is smooth or more generally formally smooth, then this sequence is a short exact sequence, see Morphisms, Lemma 34.16 or see Lemma 11.11.

(2) If g is an immersion or more generally formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,$$
see Morphisms, Lemma 32.15 or see Lemma 7.10. If $f \circ g : Z \to X$ is smooth or more generally formally smooth, then this sequence is a short exact sequence, see Morphisms, Lemma 34.17 or see Lemma 11.12.

(3) If g and $f \circ g$ are immersions or more generally formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to 0,$$
see Morphisms, Lemma 32.18 or see Lemma 7.11. If $f : Y \to X$ is smooth or more generally formally smooth, then this sequence is a short exact sequence, see Morphisms, Lemma 34.18 or see Lemma 11.13.

(4) If g and f are immersions or more generally formally unramified, then there is a canonical exact sequence
$$g^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0.$$
see Morphisms, Lemma 31.5 or see Lemma 7.12. If $g : Z \to Y$ is a regular immersion[14] or more generally a local complete intersection morphism, then this sequence is a short exact sequence, see Divisors, Lemma 21.6 or see Lemma 62.23.

[14]It suffices for g to be a H_1-regular immersion. Observe that an immersion which is a local complete intersection morphism is Koszul regular.

64. Weakly étale morphisms

094N A ring homomorphism $A \to B$ is weakly étale if both $A \to B$ and $B \otimes_A B \to B$ are flat, see More on Algebra, Definition 104.1. The analogous notion for morphisms of schemes is the following.

094P **Definition 64.1.** A morphism of schemes $X \to Y$ is *weakly étale* or *absolutely flat* if both $X \to Y$ and the diagonal morphism $X \to X \times_Y X$ are flat.

An étale morphism is weakly étale and conversely it turns out that a weakly étale morphism is indeed somewhat like an étale morphism. For example, if $X \to Y$ is weakly étale, then $L_{X/Y} = 0$, as follows from Cotangent, Lemma 8.4. We will prove a very precise result relating weakly étale morphisms to étale morphisms later (see Pro-étale Cohomology, Section 9). In this section we stick with the basics.

094Q **Lemma 64.2.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

1. *$X \to Y$ is weakly étale, and*
2. *for every $x \in X$ the ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is weakly étale.*

Proof. Observe that under both assumptions (1) and (2) the morphism f is flat. Thus we may assume f is flat. Let $x \in X$ with image $y = f(x)$ in Y. There are canonical maps of rings

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X \times_Y X, \Delta_{X/Y}(x)} \longrightarrow \mathcal{O}_{X,x}$$

where the first map is a localization (hence flat) and the second map is a surjection. Condition (1) means that the second arrow is flat for all x. Condition (2) is that the composition is flat for all x. Thus the equivalence by Algebra, Lemma 39.18 part (2). $\square$

094R **Lemma 64.3.** *Let $X \to Y$ be a morphism of schemes such that $X \to X \times_Y X$ is flat. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over Y, then $\mathcal{F}$ is flat over X.*

Proof. Let $x \in X$ with image $y = f(x)$ in Y. Since $X \to X \times_Y X$ is flat, we see that $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is flat. Hence the result follows from More on Algebra, Lemma 104.2 and the definitions. $\square$

094S **Lemma 64.4.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *The morphism f is weakly étale.*
2. *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is weakly étale.*
3. *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is weakly étale.*
4. *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of weakly étale, for all $j \in J, i \in I_j$.*

Moreover, if f is weakly étale then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is weakly-étale.

Proof. Suppose given open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$. Then $U \times_V U \subset X \times_Y X$ is open (Schemes, Lemma 17.3) and the diagonal $\Delta_{U/V}$ of $f|_U : U \to V$ is the restriction $\Delta_{X/Y}|_U : U \to U \times_V U$. Since flatness is a local property of morphisms of schemes (Morphisms, Lemma 25.3) the final statement of the lemma is follows as well as the equivalence of (1) and (3). If X and Y are affine, then $X \to Y$ is weakly étale if and only if $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is weakly étale (use again Morphisms, Lemma 25.3). Thus (1) and (3) are also equivalent to (2) and (4). $\square$

094T **Lemma 64.5.** *Let $X \to Y \to Z$ be morphisms of schemes.*

1. *If $X \to X \times_Y X$ and $Y \to Y \times_Z Y$ are flat, then $X \to X \times_Z X$ is flat.*
2. *If $X \to Y$ and $Y \to Z$ are weakly étale, then $X \to Z$ is weakly étale.*

Proof. Part (1) follows from the factorization

$$X \to X \times_Y X \to X \times_Z X$$

of the diagonal of X over Z, the fact that

$$X \times_Y X = (X \times_Z X) \times_{(Y \times_Z Y)} Y,$$

the fact that a base change of a flat morphism is flat, and the fact that the composition of flat morphisms is flat (Morphisms, Lemmas 25.8 and 25.6). Part (2) follows from part (1) and the fact (just used) that the composition of flat morphisms is flat. $\square$

094U **Lemma 64.6.** *Let $X \to Y$ and $Y' \to Y$ be morphisms of schemes and let $X' = Y' \times_Y X$ be the base change of X.*

1. *If $X \to X \times_Y X$ is flat, then $X' \to X' \times_{Y'} X'$ is flat.*
2. *If $X \to Y$ is weakly étale, then $X' \to Y'$ is weakly étale.*

Proof. Assume $X \to X \times_Y X$ is flat. The morphism $X' \to X' \times_{Y'} X'$ is the base change of $X \to X \times_Y X$ by $Y' \to Y$. Hence it is flat by Morphisms, Lemmas 25.8. This proves (1). Part (2) follows from (1) and the fact (just used) that the base change of a flat morphism is flat. $\square$

094V **Lemma 64.7.** *Let $X \to Y \to Z$ be morphisms of schemes. Assume that $X \to Y$ is flat and surjective and that $X \to X \times_Z X$ is flat. Then $Y \to Y \times_Z Y$ is flat.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \times_Z X \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

The top horizontal arrow is flat and the vertical arrows are flat. Hence X is flat over $Y \times_Z Y$. By Morphisms, Lemma 25.13 we see that Y is flat over $Y \times_Z Y$. $\square$

094W **Lemma 64.8.** *Let $f : X \to Y$ be a weakly étale morphism of schemes. Then f is formally unramified, i.e., $\Omega_{X/Y} = 0$.*

Proof. Recall that f is formally unramified if and only if $\Omega_{X/Y} = 0$ by Lemma 6.7. Via Lemma 64.4 and Morphisms, Lemma 32.5 this follows from the case of rings which is More on Algebra, Lemma 104.12. $\square$

094X **Lemma 64.9.** *Let $f : X \to Y$ be a morphism of schemes. Then $X \to Y$ is weakly étale in each of the following cases*

(1) *$X \to Y$ is a flat monomorphism,*
(2) *$X \to Y$ is an open immersion,*
(3) *$X \to Y$ is flat and unramified,*
(4) *$X \to Y$ is étale.*

Proof. If (1) holds, then $\Delta_{X/Y}$ is an isomorphism (Schemes, Lemma 23.2), hence certainly f is weakly étale. Case (2) is a special case of (1). The diagonal of an unramified morphism is an open immersion (Morphisms, Lemma 35.13), hence flat. Thus a flat unramified morphism is weakly étale. An étale morphism is flat and unramified (Morphisms, Lemma 36.5), hence (4) follows from (3). $\square$

094Y **Lemma 64.10.** *Let $f : X \to Y$ be a morphism of schemes. If Y is reduced and f weakly étale, then X is reduced.*

Proof. Via Lemma 64.4 this follows from the case of rings which is More on Algebra, Lemma 104.8. $\square$

The following lemma uses a nontrivial result about weakly étale ring maps.

094Z **Lemma 64.11.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

(1) *f is weakly étale, and*
(2) *for $x \in X$ the local ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces an isomorphism on strict henselizations.*

Proof. Let $x \in X$ be a point with image $y = f(x)$ in Y. Choose a separable algebraic closure κ^{sep} of $\kappa(x)$. Let $\mathcal{O}_{X,x}^{sh}$ be the strict henselization corresponding to κ^{sep} and $\mathcal{O}_{Y,y}^{sh}$ the strict henselization relative to the separable algebraic closure of $\kappa(y)$ in κ^{sep}. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r] & \mathcal{O}_{X,x}^{sh} \\ \mathcal{O}_{Y,y} \arrow[r] \arrow[u] & \mathcal{O}_{Y,y}^{sh} \arrow[u] \end{tikzcd}$$

local homomorphisms of local rings, see Algebra, Lemma 155.10. Since the strict henselization is a filtered colimit of étale ring maps, More on Algebra, Lemma 104.14 shows the horizontal maps are weakly étale. Moreover, the horizontal maps are faithfully flat by More on Algebra, Lemma 45.1.

Assume f weakly étale. By Lemma 64.2 the left vertical arrow is weakly étale. By More on Algebra, Lemmas 104.9 and 104.11 the right vertical arrow is weakly étale. By More on Algebra, Theorem 104.24 we conclude the right vertical map is an isomorphism.

Assume $\mathcal{O}_{Y,y}^{sh} \to \mathcal{O}_{X,x}^{sh}$ is an isomorphism. Then $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}^{sh}$ is weakly étale. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}^{sh}$ is faithfully flat we conclude that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is weakly étale by More on Algebra, Lemma 104.10. Thus (2) implies (1) by Lemma 64.2. $\square$

0950 **Lemma 64.12.** *Let $f : X \to Y$ be a morphism of schemes. If Y is a normal scheme and f weakly étale, then X is a normal scheme.*

Proof. By More on Algebra, Lemma 45.6 a scheme S is normal if and only if for all $s \in S$ the strict henselization of $\mathcal{O}_{S,s}$ is a normal domain. Hence the lemma follows from Lemma 64.11. $\square$

0951 **Lemma 64.13.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. If X, Y are weakly étale over S, then f is weakly étale.*

Proof. We will use Morphisms, Lemmas 25.8 and 25.6 without further mention. Write $X \to Y$ as the composition $X \to X \times_S Y \to Y$. The second morphism is flat as the base change of the flat morphism $X \to S$. The first is the base change of the flat morphism $Y \to Y \times_S Y$ by the morphism $X \times_S Y \to Y \times_S Y$, hence flat. Thus $X \to Y$ is flat. The morphism $X \times_Y X \to X \times_S X$ is an immersion. Thus Lemma 64.3 implies, that since X is flat over $X \times_S X$ it follows that X is flat over $X \times_Y X$. $\square$

The following is a scheme theoretic generalization of the observation that a field extension that is simultaneously separable and purely inseparable must be an isomorphism.

0F6V **Lemma 64.14.** *Let $f : X \to Y$ be a morphism of schemes. If f is weakly étale and a universal homeomorphism, it is an isomorphism.*

Proof. Since f is a universal homeomorphism, the diagonal $\Delta : X \to X \times_Y X$ is a surjective closed immersion by Morphisms, Lemmas 45.4 and 10.2. Since Δ is also flat, we see that Δ must be an isomorphism by Morphisms, Lemma 26.1. In other words, f is a monomorphism (Schemes, Lemma 23.2). Since f is a universal homeomorphism it is certainly quasi-compact. Hence by Descent, Lemma 25.1 we find that f is an isomorphism. $\square$

The following is a weakly étale generalization of Étale Morphisms, Lemma 14.3.

0F6W **Lemma 64.15.** *Let $U \to X$ be a weakly étale morphism of schemes where X is a scheme in characteristic p. Then the relative Frobenius $F_{U/X} : U \to U \times_{X,F_X} X$ is an isomorphism.*

Proof. The morphism $F_{U/X}$ is a universal homeomorphism by Varieties, Lemma 36.6. The morphism $F_{U/X}$ is weakly étale as a morphism between schemes weakly étale over X by Lemma 64.13. Hence $F_{U/X}$ is an isomorphism by Lemma 64.14. $\square$

65. Reduced fibre theorem

09IJ In this section we discuss the simplest kind of theorem of the kind advertised by the title. Although the proof of the result is kind of laborious, in essence it follows in a straightforward manner from Epp's result on eliminating ramification, see More on Algebra, Theorem 115.18.

Let A be a Dedekind domain with fraction field K. Let X be a scheme flat and of finite type over A. Let L be a finite extension of K. Let B be the integral closure of A in L. Then B is a Dedekind domain (Algebra, Lemma 120.18). Let $X_B = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ be the base change. Then $X_B \to \mathrm{Spec}(B)$ is of finite type (Morphisms, Lemma 15.4). Hence X_B is Noetherian (Morphisms, Lemma

15.6). Thus the normalization $\nu : Y \to X_B$ exists (see Morphisms, Definition 54.1 and the discussion following). Picture

09IK (65.0.1)
$$\begin{tikzcd} Y \arrow[r, "\nu"'] \arrow[dr] & X_B \arrow[r] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

We sometimes call Y the *normalized base change* of X. In general the morphism ν may not be finite. But if A is a Nagata ring (a condition that is virtually always satisfied in practice) then ν is finite and Y is of finite type over B, see Morphisms, Lemmas 54.11 and 18.1.

Taking the normalized base change commutes with composition. More precisely, if $M/L/K$ are finite extensions of fields with integral closures $A \subset B \subset C$ then the normalized base change Z of $Y \to \mathrm{Spec}(B)$ relative to M/L is equal to the normalized base change of $X \to \mathrm{Spec}(A)$ relative to M/K.

09IL **Theorem 65.1.** *Let A be a Dedekind ring with fraction field K. Let X be a scheme flat and of finite type over A. Assume A is a Nagata ring. There exists a finite extension L/K such that the normalized base change Y is smooth over* $\mathrm{Spec}(B)$ *at all generic points of all fibres.*

Proof. During the proof we will repeatedly use that formation of the set of points where a (flat, finitely presented) morphism like $X \to \mathrm{Spec}(A)$ is smooth commutes with base change, see Morphisms, Lemma 34.15.

We first choose a finite extension L/K such that $(X_L)_{red}$ is geometrically reduced over L, see Varieties, Lemma 6.11. Since $Y \to (X_B)_{red}$ is birational we see applying Varieties, Lemma 6.8 that Y_L is geometrically reduced over L as well. Hence $Y_L \to \mathrm{Spec}(L)$ is smooth on a dense open $V \subset Y_L$ by Varieties, Lemma 25.7. Thus the smooth locus $U \subset Y$ of the morphism $Y \to \mathrm{Spec}(B)$ is open (by Morphisms, Definition 34.1) and is dense in the generic fibre. Replacing A by B and X by Y we reduce to the case treated in the next paragraph.

Assume X is normal and the smooth locus $U \subset X$ of $X \to \mathrm{Spec}(A)$ is dense in the generic fibre. This implies that U is dense in all but finitely many fibres, see Lemma 24.3. Let $x_1, \ldots, x_r \in X \setminus U$ be the finitely many generic points of irreducible components of $X \setminus U$ which are moreover generic points of irreducible components of fibres of $X \to \mathrm{Spec}(A)$. Set $\mathcal{O}_i = \mathcal{O}_{X,x_i}$. Let A_i be the localization of A at the maximal ideal corresponding to the image of x_i in $\mathrm{Spec}(A)$. By More on Algebra, Proposition 116.8 there exist finite extensions K_i/K which are solutions for the extension of discrete valuation rings $A_i \to \mathcal{O}_i$. Let L/K be a finite extension dominating all of the extensions K_i/K. Then L/K is still a solution for $A_i \to \mathcal{O}_i$ by More on Algebra, Lemma 116.1.

Consider the diagram (65.0.1) with the extension L/K we just produced. Note that $U_B \subset X_B$ is smooth over B, hence normal (for example use Algebra, Lemma 163.9). Thus $Y \to X_B$ is an isomorphism over U_B. Let $y \in Y$ be a generic point of an irreducible component of a fibre of $Y \to \mathrm{Spec}(B)$ lying over the maximal ideal $\mathfrak{m} \subset B$. Assume that $y \notin U_B$. Then y maps to one of the points x_i. It follows that $\mathcal{O}_{Y,y}$ is a local ring of the integral closure of $\mathcal{O}_i$ in $R(X) \otimes_K L$ (details omitted). Hence because L/K is a solution for $A_i \to \mathcal{O}_i$ we see that $B_\mathfrak{m} \to \mathcal{O}_{Y,y}$ is formally smooth in the $\mathfrak{m}_y$-adic topology (this is the definition of being a "solution"). In other words, $\mathfrak{m}\mathcal{O}_{Y,y} = \mathfrak{m}_y$ and the residue field extension is separable, see More on Algebra, Lemma 111.5. Hence the local ring of the fibre at y is $\kappa(y)$. This implies the fibre is smooth over $\kappa(\mathfrak{m})$ at y for example by Algebra, Lemma 140.5. This finishes the proof. □

0BRQ **Lemma 65.2** (Variant over curves). *Let $f : X \to S$ be a flat, finite type morphism of schemes. Assume S is Nagata, integral with function field K, and regular of dimension 1. Then there exists a finite extension L/K such that in the diagram*

$$\begin{tikzcd} Y \arrow[r, "\nu"'] \arrow[rd, "g"'] & X \times_S T \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & T \arrow[r] & S \end{tikzcd}$$

the morphism g is smooth at all generic points of fibres. Here T is the normalization of S in $\text{Spec}(L)$ and $\nu : Y \to X \times_S T$ is the normalization.

Proof. Choose a finite affine open covering $S = \bigcup \text{Spec}(A_i)$. Then K is equal to the fraction field of A_i for all i. Let $X_i = X \times_S \text{Spec}(A_i)$. Choose L_i/K as in Theorem 65.1 for the morphism $X_i \to \text{Spec}(A_i)$. Let $B_i \subset L_i$ be the integral closure of A_i and let Y_i be the normalized base change of X to B_i. Let L/K be a finite extension dominating each L_i. Let $T_i \subset T$ be the inverse image of $\text{Spec}(A_i)$. For each i we get a commutative diagram

$$\begin{tikzcd} g^{-1}(T_i) \arrow[r] \arrow[d] & Y_i \arrow[r] \arrow[d] & X \times_S \text{Spec}(A_i) \arrow[d] \\ T_i \arrow[r] & \text{Spec}(B_i) \arrow[r] & \text{Spec}(A_i) \end{tikzcd}$$

and in fact the left hand square is a normalized base change as discussed at the beginning of the section. In the proof of Theorem 65.1 we have seen that the smooth locus of $Y \to T$ contains the inverse image in $g^{-1}(T_i)$ of the set of points where Y_i is smooth over B_i. This proves the lemma. □

0BRR **Lemma 65.3** (Variant with separable extension). *Let A be a Dedekind ring with fraction field K. Let X be a scheme flat and of finite type over A. Assume A is a Nagata ring and that for every generic point η of an irreducible component of X the field extension $\kappa(\eta)/K$ is separable. Then there exists a finite separable extension L/K such that the normalized base change Y is smooth over* $\text{Spec}(B)$ *at all generic points of all fibres.*

Proof. This is proved in exactly the same manner as Theorem 65.1 with a few minor modifications. The most important change is to use More on Algebra, Lemma 116.9 instead of More on Algebra, Proposition 116.8. During the proof we will repeatedly use that formation of the set of points where a (flat, finitely presented) morphism like $X \to \text{Spec}(A)$ is smooth commutes with base change, see Morphisms, Lemma 34.15.

Since X is flat over A every generic point η of X maps to the generic point of $\text{Spec}(A)$. After replacing X by its reduction we may assume X is reduced. In this case X_K is geometrically reduced over K by Varieties, Lemma 6.8. Hence $X_K \to \text{Spec}(K)$ is smooth on a dense open by Varieties, Lemma 25.7. Thus the smooth locus $U \subset X$ of the morphism $X \to \mathrm{Spec}(A)$ is open (by Morphisms, Definition 34.1) and is dense in the generic fibre. This reduces us to the situation of the following paragraph.

Assume X is normal and the smooth locus $U \subset X$ of $X \to \mathrm{Spec}(A)$ is dense in the generic fibre. This implies that U is dense in all but finitely many fibres, see Lemma 24.3. Let $x_1, \ldots, x_r \in X \setminus U$ be the finitely many generic points of irreducible components of $X \setminus U$ which are moreover generic points of irreducible components of fibres of $X \to \mathrm{Spec}(A)$. Set $\mathcal{O}_i = \mathcal{O}_{X,x_i}$. Observe that the fraction field of $\mathcal{O}_i$ is the residue field of a generic point of X. Let A_i be the localization of A at the maximal ideal corresponding to the image of x_i in $\mathrm{Spec}(A)$. We may apply More on Algebra, Lemma 116.9 and we find finite separable extensions K_i/K which are solutions for $A_i \to \mathcal{O}_i$. Let L/K be a finite separable extension dominating all of the extensions K_i/K. Then L/K is still a solution for $A_i \to \mathcal{O}_i$ by More on Algebra, Lemma 116.1.

Consider the diagram (65.0.1) with the extension L/K we just produced. Note that $U_B \subset X_B$ is smooth over B, hence normal (for example use Algebra, Lemma 163.9). Thus $Y \to X_B$ is an isomorphism over U_B. Let $y \in Y$ be a generic point of an irreducible component of a fibre of $Y \to \mathrm{Spec}(B)$ lying over the maximal ideal $\mathfrak{m} \subset B$. Assume that $y \notin U_B$. Then y maps to one of the points x_i. It follows that $\mathcal{O}_{Y,y}$ is a local ring of the integral closure of $\mathcal{O}_i$ in $R(X) \otimes_K L$ (details omitted). Hence because L/K is a solution for $A_i \to \mathcal{O}_i$ we see that $B_\mathfrak{m} \to \mathcal{O}_{Y,y}$ is formally smooth (this is the definition of being a "solution"). In other words, $\mathfrak{m}\mathcal{O}_{Y,y} = \mathfrak{m}_y$ and the residue field extension is separable. Hence the local ring of the fibre at y is $\kappa(y)$. This implies the fibre is smooth over $\kappa(\mathfrak{m})$ at y for example by Algebra, Lemma 140.5. This finishes the proof. $\square$

0BRS **Lemma 65.4** (Variant with separable extensions over curves)**.** *Let $f : X \to S$ be a flat, finite type morphism of schemes. Assume S is Nagata, integral with function field K, and regular of dimension 1. Assume the field extensions $\kappa(\eta)/K$ are separable for every generic point η of an irreducible component of X. Then there exists a finite separable extension L/K such that in the diagram*

$$\begin{tikzcd} Y \arrow[r, "\nu"'] \arrow[dr, "g"'] & X \times_S T \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & T \arrow[r] & S \end{tikzcd}$$

the morphism g is smooth at all generic points of fibres. Here T is the normalization of S in $\mathrm{Spec}(L)$ *and $\nu : Y \to X \times_S T$ is the normalization.*

Proof. This follows from Lemma 65.3 in exactly the same manner that Lemma 65.2 follows from Theorem 65.1. $\square$

66. Ind-quasi-affine morphisms

0AP5 A bit of theory to be used later.

0AP6 **Definition 66.1.** A scheme X is *ind-quasi-affine* if every quasi-compact open of X is quasi-affine. Similarly, a morphism of schemes $X \to Y$ is *ind-quasi-affine* if $f^{-1}(V)$ is ind-quasi-affine for each affine open V in Y.

An example of an ind-quasi-affine scheme is an open of an affine scheme. If $X = \bigcup_{i \in I} U_i$ is a union of quasi-affine opens such that any two U_i are contained in a third, then X is ind-quasi-affine. An ind-quasi-affine scheme X is separated because any two affine opens U, V are contained in a separated open subscheme of X, namely $U \cup V$. Similarly an ind-quasi-affine morphism is separated.

0F1U **Lemma 66.2.** *For a morphism of schemes $f : X \to Y$, the following are equivalent:*

1. *f is ind-quasi-affine,*
2. *for every affine open subscheme $V \subset Y$ and every quasi-compact open subscheme $U \subset f^{-1}(V)$, the induced morphism $U \to V$ is quasi-affine.*
3. *for some cover $\{V_j\}_{j \in J}$ of Y by quasi-compact and quasi-separated open subschemes $V_j \subset Y$, every $j \in J$, and every quasi-compact open subscheme $U \subset f^{-1}(V_j)$, the induced morphism $U \to V_j$ is quasi-affine.*
4. *for every quasi-compact and quasi-separated open subscheme $V \subset Y$ and every quasi-compact open subscheme $U \subset f^{-1}(V)$, the induced morphism $U \to V$ is quasi-affine.*

In particular, the property of being an ind-quasi-affine morphism is Zariski local on the base.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from the definitions and Morphisms, Lemma 13.3. For (2) $\Rightarrow$ (4), let U and V be as in (4). By Schemes, Lemma 21.14, the induced morphism $U \to V$ is quasi-compact. Thus, for every affine open $V' \subset V$, the fiber product $V' \times_V U$ is quasi-compact, so, by (2), the induced map $V' \times_V U \to V'$ is quasi-affine. Thus, $U \to V$ is also quasi-affine by Morphisms, Lemma 13.3. This argument also gives (3) $\Rightarrow$ (4): indeed, keeping the same notation, those affine opens $V' \subset V$ that lie in one of the V_j cover V, so one needs to argue that the quasi-compact map $V' \times_V U \to V'$ is quasi-affine. However, by (3), the composition $V' \times_V U \to V' \to V_j$ is quasi-affine and, by Schemes, Lemma 21.13, the map $V' \to V_j$ is quasi-separated. Thus, $V' \times_V U \to V'$ is quasi-affine by Morphisms, Lemma 13.8. The final implications (4) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are evident. $\square$

0F1V **Lemma 66.3.** *The property of being an ind-quasi-affine morphism is stable under composition.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be ind-quasi-affine morphisms. Let $V \subset Z$ and $U \subset f^{-1}(g^{-1}(V))$ be quasi-compact opens such that V is also quasi-separated. The image $f(U)$ is a quasi-compact subset of $g^{-1}(V)$, so it is contained in some quasi-compact open $W \subset g^{-1}(V)$ (a union of finitely many affines). We obtain a factorization $U \to W \to V$. The map $W \to V$ is quasi-affine by Lemma 66.2, so, in particular, W is quasi-separated. Then, by Lemma 66.2 again, $U \to W$ is quasi-affine as well. Consequently, by Morphisms, Lemma 13.4, the composition $U \to V$ is also quasi-affine, and it remains to apply Lemma 66.2 once more. $\square$

0F1W **Lemma 66.4.** *Any quasi-affine morphism is ind-quasi-affine. Any immersion is ind-quasi-affine.*

Proof. The first assertion is immediate from the definitions. In particular, affine morphisms, such as closed immersions, are ind-quasi-affine. Thus, by Lemma 66.3, it remains to show that an open immersion is ind-quasi-affine. This, however, is immediate from the definitions. $\square$

0F1X **Lemma 66.5.** *If $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes such that $g \circ f$ is ind-quasi-affine, then f is ind-quasi-affine.*

Proof. By Lemma 66.2, we may work Zariski locally on Z and then on Y, so we lose no generality by assuming that Z, and then also Y, is affine. Then any quasi-compact open of X is quasi-affine, so Lemma 66.2 gives the claim. $\square$

0AP7 **Lemma 66.6.** *The property of being ind-quasi-affine is stable under base change.*

Proof. Let $f : X \to Y$ be an ind-quasi-affine morphism. For checking that every base change of f is ind-quasi-affine, by Lemma 66.2, we may work Zariski locally on Y, so we assume that Y is affine. Furthermore, we may also assume that in the base change morphism $Z \to Y$ the scheme Z is affine, too. The base change $X \times_Y Z \to X$ is an affine morphism, so, by Lemmas 66.3 and 66.4, the map $X \times_Y Z \to Y$ is ind-quasi-affine. Then, by Lemma 66.5, the base change $X \times_Y Z \to Z$ is ind-quasi-affine, as desired. $\square$

0AP8 **Lemma 66.7.** *The property of being ind-quasi-affine is fpqc local on the base.*

Proof. The stability of ind-quasi-affineness under base change supplied by Lemma 66.6 gives one direction. For the other, let $f : X \to Y$ be a morphism of schemes and let $\{g_i : Y_i \to Y\}$ be an fpqc covering such that the base change $f_i : X_i \to Y_i$ is ind-quasi-affine for all i. We need to show f is ind-quasi-affine.

By Lemma 66.2, we may work Zariski locally on Y, so we assume that Y is affine. Then we use stability under base change ensured by Lemma 66.6 to refine the cover and assume that it is given by a single affine, faithfully flat morphism $g : Y' \to Y$. For any quasi-compact open $U \subset X$, its Y'-base change $U \times_Y Y' \subset X \times_Y Y'$ is also quasi-compact. It remains to observe that, by Descent, Lemma 23.20, the map $U \to Y$ is quasi-affine if and only if so is $U \times_Y Y' \to Y'$. $\square$

0AP9 **Lemma 66.8.** *A separated locally quasi-finite morphism of schemes is ind-quasi-affine.*

Proof. Let $f : X \to Y$ be a separated locally quasi-finite morphism of schemes. Let $V \subset Y$ be affine and $U \subset f^{-1}(V)$ quasi-compact open. We have to show U is quasi-affine. Since $U \to V$ is a separated quasi-finite morphism of schemes, this follows from Zariski's Main Theorem. See Lemma 43.2. $\square$

67. Pushouts in the category of schemes, II

0ECH This section is a continuation of Section 14. In this section we construct pushouts of $Y \leftarrow Z \to X$ where $Z \to X$ is a closed immersion and $Z \to Y$ is integral and an additional condition is satisfied. Please see the detailed discussion in [Fer03].

0ECI **Situation 67.1.** Here S is a scheme and $i : Z \to X$ and $j : Z \to Y$ are morphisms of schemes over S. We assume

(1) i is a closed immersion,
(2) j is an integral morphism of schemes,
(3) for $y \in Y$ there exists an affine open $U \subset X$ with $j^{-1}(\{y\}) \subset i^{-1}(U)$.

0ECJ **Lemma 67.2.** *In Situation 67.1 then for $y \in Y$ there exist affine opens $U \subset X$ and $V \subset Y$ with $i^{-1}(U) = j^{-1}(V)$ and $y \in V$.*

Proof. Let $y \in Y$. Choose an affine open $U \subset X$ such that $j^{-1}(\{y\}) \subset i^{-1}(U)$ (possible by assumption). Choose an affine open $V \subset Y$ neighbourhood of y such that $j^{-1}(V) \subset i^{-1}(U)$. This is possible because $j : Z \to Y$ is a closed morphism (Morphisms, Lemma 44.7) and $i^{-1}(U)$ contains the fibre over y. Since j is integral, the scheme theoretic fibre Z_y is the spectrum of an algebra integral over a field. By Limits, Lemma 11.6 we can find an $\overline{f} \in \Gamma(i^{-1}(U), \mathcal{O}_{i^{-1}(U)})$ such that $Z_y \subset D(\overline{f}) \subset j^{-1}(V)$. Since $i|_{i^{-1}(U)} : i^{-1}(U) \to U$ is a closed immersion of affines, we can choose an $f \in \Gamma(U, \mathcal{O}_U)$ whose restriction to $i^{-1}(U)$ is $\overline{f}$. After replacing U by the principal open $D(f) \subset U$ we find affine opens $y \in V \subset Y$ and $U \subset X$ with

$$j^{-1}(\{y\}) \subset i^{-1}(U) \subset j^{-1}(V)$$

Now we (in some sense) repeat the argument. Namely, we choose $g \in \Gamma(V, \mathcal{O}_V)$ such that $y \in D(g)$ and $j^{-1}(D(g)) \subset i^{-1}(U)$ (possible by the same argument as above). Then we can pick $f \in \Gamma(U, \mathcal{O}_U)$ whose restriction to $i^{-1}(U)$ is the pullback of g by $i^{-1}(U) \to V$ (again possible by the same reason as above). Then we finally have affine opens $y \in V' = D(g) \subset V \subset Y$ and $U' = D(f) \subset U \subset X$ with $j^{-1}(V') = i^{-1}(V')$. $\square$

0E25 **Proposition 67.3.** *In Situation 67.1 the pushout $Y \amalg_Z X$ exists in the category of schemes. Picture* [Fer03, Theorem 7.1 part iii]

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "j"'] & X \arrow[d, "a"] \\ Y \arrow[r, "b"] & Y \amalg_Z X \end{tikzcd}$$

The diagram is a fibre square, the morphism a is integral, the morphism b is a closed immersion, and

$$\mathcal{O}_{Y \amalg_Z X} = b_*\mathcal{O}_Y \times_{c_*\mathcal{O}_Z} a_*\mathcal{O}_X$$

as sheaves of rings where $c = a \circ i = b \circ j$.

Proof. As a topological space we set $Y \amalg_Z X$ equal to the pushout of the diagram in the category of topological spaces (Topology, Section 29). This is just the pushout of the underlying sets (Topology, Lemma 29.1) endowed with the quotient topology. On $Y \amalg_Z X$ we have the maps of sheaves of rings

$$b_*\mathcal{O}_Y \longrightarrow c_*\mathcal{O}_Z \longleftarrow a_*\mathcal{O}_X$$

and we can define

$$\mathcal{O}_{Y \amalg_Z X} = b_*\mathcal{O}_Y \times_{c_*\mathcal{O}_Z} a_*\mathcal{O}_X$$

as the fibre product in the category of sheaves of rings. To prove that we obtain a scheme we have to show that every point has an affine open neighbourhood. This is clear for points not in the image of c as the image of c is a closed subset whose complement is isomorphic as a ringed space to $(Y \setminus j(Z)) \amalg (X \setminus i(Z))$.

A point in the image of c corresponds to a unique $y \in Y$ in the image of j. By Lemma 67.2 we find affine opens $U \subset X$ and $V \subset Y$ with $y \in V$ and $i^{-1}(U) = j^{-1}(V)$. Since the construction of the first paragraph is clearly compatible with restriction to compatible open subschemes, to prove that it produces a scheme we may assume X, Y, and Z are affine.

If $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $Z = \mathrm{Spec}(C)$ are affine, then More on Algebra, Lemma 6.2 shows that $Y \amalg_Z X = \mathrm{Spec}(B \times_C A)$ as topological spaces. To finish the proof that $Y \times_Z X$ is a scheme, it suffices to show that on $\mathrm{Spec}(B \times_C A)$ the structure sheaf is the fibre product of the pushforwards. This follows by applying More on Algebra, Lemma 5.3 to principal affine opens of $\mathrm{Spec}(B \times_C A)$.

The discussion above shows the scheme $Y \amalg_X Z$ has an affine open covering $Y \amalg_X Z = \bigcup W_i$ such that $U_i = a^{-1}(W_i)$, $V_i = b^{-1}(W_i)$, and $\Omega_i = c^{-1}(W_i)$ are affine open in X, Y, and Z. Thus a and b are affine. Moreover, if A_i, B_i, C_i are the rings corresponding to U_i, V_i, Ω_i, then $A_i \to C_i$ is surjective and W_i corresponds to $A_i \times_{C_i} B_i$ which surjects onto B_i. Hence b is a closed immersion. The ring map $A_i \times_{C_i} B_i \to A_i$ is integral by More on Algebra, Lemma 6.3 hence a is integral. The diagram is cartesian because

$$C_i \cong B_i \otimes_{B_i \times_{C_i} A_i} A_i$$

This follows as $B_i \times_{C_i} A_i \to B_i$ and $A_i \to C_i$ are surjective maps whose kernels are the same.

Finally, we can apply Lemmas 14.1 and 14.2 to conclude our construction is a pushout in the category of schemes. $\square$

0E26 **Lemma 67.4.** *In Situation 67.1. If X and Y are separated, then the pushout $Y \amalg_Z X$ (Proposition 67.3) is separated. Same with "separated over S", "quasi-separated", and "quasi-separated over S".*

Proof. The morphism $Y \amalg X \to Y \amalg_Z X$ is surjective and universall closed. Thus we may apply Morphisms, Lemma 41.11. $\square$

0E27 **Lemma 67.5.** *In Situation 67.1 assume S is a locally Noetherian scheme and X, Y, and Z are locally of finite type over S. Then the pushout $Y \amalg_Z X$ (Proposition 67.3) is locally of finite type over S.*

Proof. Looking on affine opens we recover the result of More on Algebra, Lemma 5.1. $\square$

0ECK **Lemma 67.6.** *In Situation 67.1 suppose given a commutative diagram*

$$\begin{tikzcd} Y' \arrow[d, "g"] & Z' \arrow[l, "j'"] \arrow[r, "i'"] \arrow[d, "h"] & X' \arrow[d, "f"] \\ Y & Z \arrow[l] \arrow[r] & X \end{tikzcd}$$

with cartesian squares and f, g, h separated and locally quasi-finite. Then

1. *the pushouts $Y \amalg_Z X$ and $Y' \amalg_{Z'} X'$ exist,*
2. *$Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is separated and locally quasi-finite, and*
3. *the squares*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y' \amalg_{Z'} X' \arrow[d] & X' \arrow[l] \arrow[d] \\ Y \arrow[r] & Y \amalg_Z X & X \arrow[l] \end{tikzcd}$$

are cartesian.

Proof. The pushout $Y \amalg_Z X$ exists by Proposition 67.3. To see that the pushout $Y' \amalg_{Z'} X'$ exists, we check condition (3) of Situation 67.1 holds for (X', Y', Z', i', j'). Namely, let $y' \in Y'$ and denote $y \in Y$ the image. Choose $U \subset X$ affine open with $i(j^{-1}(y)) \subset U$. Choose a quasi-compact open $U' \subset X'$ contained in $f^{-1}(U)$ containing the quasi-compact subset $i'((j')^{-1}(\{y'\}))$. By Lemma 66.8 we see that U' is quasi-affine. Since $Z'_{y'}$ is the spectrum of an algebra integral over a field, we can apply Limits, Lemma 11.6 and we find there exists an affine open subscheme of U' containing $i'((j')^{-1}(\{y'\}))$ as desired.

Having verified existence we check the other assertions. Affine locally we are exactly in the situation of More on Algebra, Lemma 7.7 with $B \to D$ and $A' \to C'$ locally quasi-finite[15]. In particular, the morphism $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is locally of finite type. The squares in of the diagram are cartesian by More on Algebra, Lemma 6.4. Since being locally quasi-finite can be checked on fibres (Morphisms, Lemma 20.6) we conclude that $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is locally quasi-finite.

We still have to check $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is separated. Observe that $Y' \amalg X' \to Y' \amalg_{Z'} X'$ is universally closed and surjective by Proposition 67.3. Since also the morphism $Y' \amalg X' \to Y \amalg_Z X$ is separated (as it factors as $Y' \amalg X' \to Y \amalg X \to Y \amalg_Z X$) we conclude by Morphisms, Lemma 41.11. □

0ECL **Lemma 67.7.** *In Situation 67.1 the category of schemes flat, separated, and locally quasi-finite over the pushout $Y \amalg_Z X$ is equivalent to the category of $(X', Y', Z', i', j', f, g, h)$ as in Lemma 67.6 with f, g, h flat. Similarly with "flat" replaced with "étale".*

Proof. If we start with $(X', Y', Z', i', j', f, g, h)$ as in Lemma 67.6 with f, g, h flat or étale, then $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is flat or étale by More on Algebra, Lemma 7.7.

For the converse, let $W \to Y \amalg_Z X$ be a separated and locally quasi-finite morphism. Set $X' = W \times_{Y \amalg_Z X} X$, $Y' = W \times_{Y \amalg_Z X} Y$, and $Z' = W \times_{Y \amalg_Z X} Z$ with obvious morphisms i', j', f, g, h. Form the pushout $Y' \amalg_{Z'} X'$. We obtain a morphism

$$Y' \amalg_{Z'} X' \longrightarrow W$$

of schemes over $Y \amalg_X Z$ by the universal property of the pushout. If we do not assume that $W \to Y \amalg_Z X$ is flat, then in general this morphism won't be an isomorphism. (In fact, More on Algebra, Lemma 6.5 shows the displayed arrow is a closed immersion but not an isomorphism in general.) However, if $W \to Y \times_Z X$ is flat, then it is an isomorphism by More on Algebra, Lemma 7.7. □

Next, we discuss existence in the case where both morphisms are closed immersions.

0B7M **Lemma 67.8.** *Let $i : Z \to X$ and $j : Z \to Y$ be closed immersions of schemes. Then the pushout $Y \amalg_Z X$ exists in the category of schemes. Picture*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "j"'] & X \arrow[d, "a"] \\ Y \arrow[r, "b"] & Y \amalg_Z X \end{tikzcd}$$

The diagram is a fibre square, the morphisms a and b are closed immersions, and there is a short exact sequence

$$0 \to \mathcal{O}_{Y \amalg_Z X} \to a_*\mathcal{O}_X \oplus b_*\mathcal{O}_Y \to c_*\mathcal{O}_Z \to 0$$

where $c = a \circ i = b \circ j$.

[15] To be precise $X, Y, Z, Y \amalg_Z X, X', Y', Z', Y' \amalg_{Z'} X'$ correspond to $A', B, A, B', C', D, C, D'$.

Proof. This is a special case of Proposition 67.3. Observe that hypothesis (3) in Situation 67.1 is immediate because the fibres of j are singletons. Finally, reverse the roles of the arrows to conclude that both a and b are closed immersions. $\square$

0CYY **Lemma 67.9.** *Let $i : Z \to X$ and $j : Z \to Y$ be closed immersions of schemes. Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms of schemes and let $\varphi : X' \times_{X,i} Z \to Y' \times_{Y,j} Z$ be an isomorphism of schemes over Z. Consider the morphism*

$$h : X' \amalg_{X' \times_{X,i} Z, \varphi} Y' \longrightarrow X \amalg_Z Y$$

Then we have

1. *h is locally of finite type if and only if f and g are locally of finite type,*
2. *h is flat if and only if f and g are flat,*
3. *h is flat and locally of finite presentation if and only if f and g are flat and locally of finite presentation,*
4. *h is smooth if and only if f and g are smooth,*
5. *h is étale if and only if f and g are étale, and*
6. *add more here as needed.*

Proof. We know that the pushouts exist by Lemma 67.8. In particular we get the morphism h. Hence we may replace all schemes in sight by affine schemes. In this case the assertions of the lemma are equivalent to the corresponding assertions of More on Algebra, Lemma 7.7. $\square$

68. Relative morphisms

0BL0 In this section we prove a representability result which we will use in Fundamental Groups, Section 5 to prove a result on the category of finite étale coverings of a scheme. The material in this section is discussed in the correct generality in Criteria for Representability, Section 10.

Let S be a scheme. Let Z and X be schemes over S. Given a scheme T over S we can consider morphisms $b : T \times_S Z \to T \times_S X$ over S. Picture

0BL1 (68.0.1)
$$\begin{tikzcd} T \times_S Z \arrow[rr, "b"'] \arrow[dr] & & T \times_S X \arrow[dl] & Z \arrow[dr] & & X \arrow[dl] \\ & T \arrow[rrr] & & & S & \end{tikzcd}$$

Of course, we can also think of b as a morphism $b : T \times_S Z \to X$ such that

$$\begin{tikzcd} T \times_S Z \arrow[r] \arrow[d] \arrow[rr, bend left, "b"] & Z \arrow[dr] & X \arrow[d] \\ T \arrow[rr] & & S \end{tikzcd}$$

commutes. In this situation we can define a functor

0BL2 (68.0.2) $$Mor_S(Z, X) : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto \{b \text{ as above}\}$$

Here is a basic representability result.

05Y6 **Lemma 68.1.** *Let $Z \to S$ and $X \to S$ be morphisms of affine schemes. Assume $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(S, \mathcal{O}_S)$-module. Then $Mor_S(Z, X)$ is representable by an affine scheme over S.*

Proof. Write $S = \mathrm{Spec}(R)$. Choose a basis $\{e_1, \ldots, e_m\}$ for $\Gamma(Z, \mathcal{O}_Z)$ over R. Choose a presentation

$$\Gamma(X, \mathcal{O}_X) = R[\{x_i\}_{i \in I}]/(\{f_k\}_{k \in K}).$$

We will denote $\overline{x}_i$ the image of x_i in this quotient. Write

$$P = R[\{a_{ij}\}_{i \in I, 1 \leq j \leq m}].$$

Consider the R-algebra map

$$\Psi : R[\{x_i\}_{i \in I}] \longrightarrow P \otimes_R \Gamma(Z, \mathcal{O}_Z), \quad x_i \longmapsto \sum\nolimits_j a_{ij} \otimes e_j.$$

Write $\Psi(f_k) = \sum c_{kj} \otimes e_j$ with $c_{kj} \in P$. Finally, denote $J \subset P$ the ideal generated by the elements c_{kj}, $k \in K$, $1 \leq j \leq m$. We claim that $W = \mathrm{Spec}(P/J)$ represents the functor $Mor_S(Z, X)$.

First, note that by construction P/J is an R-algebra, hence a morphism $W \to S$. Second, by construction the map Ψ factors through $\Gamma(X, \mathcal{O}_X)$, hence we obtain an P/J-algebra homomorphism

$$P/J \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow P/J \otimes_R \Gamma(Z, \mathcal{O}_Z)$$

which determines a morphism $b_{univ} : W \times_S Z \to W \times_S X$. By the Yoneda lemma b_{univ} determines a transformation of functors $W \to Mor_S(Z, X)$ which we claim is an isomorphism. To show that it is an isomorphism it suffices to show that it induces a bijection of sets $W(T) \to Mor_S(Z, X)(T)$ over any affine scheme T.

Suppose $T = \mathrm{Spec}(R')$ is an affine scheme over S and $b \in Mor_S(Z, X)(T)$. The structure morphism $T \to S$ defines an R-algebra structure on R' and b defines an R'-algebra map

$$b^\sharp : R' \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow R' \otimes_R \Gamma(Z, \mathcal{O}_Z).$$

In particular we can write $b^\sharp(1 \otimes \overline{x}_i) = \sum \alpha_{ij} \otimes e_j$ for some $\alpha_{ij} \in R'$. This corresponds to an R-algebra map $P \to R'$ determined by the rule $a_{ij} \mapsto \alpha_{ij}$. This map factors through the quotient P/J by the construction of the ideal J to give a map $P/J \to R'$. This in turn corresponds to a morphism $T \to W$ such that b is the pullback of b_{univ}. Some details omitted. □

0BL3 **Lemma 68.2.** *Let $Z \to S$ and $X \to S$ be morphisms of schemes. If $Z \to S$ is finite locally free and $X \to S$ is affine, then $Mor_S(Z, X)$ is representable by a scheme affine over S.*

Proof. Choose an affine open covering $S = \bigcup U_i$ such that $\Gamma(Z \times_S U_i, \mathcal{O}_{Z \times_S U_i})$ is finite free over $\mathcal{O}_S(U_i)$. Let $F_i \subset Mor_S(Z, X)$ be the subfunctor which assigns to T/S the empty set if $T \to S$ does not factor through U_i and $Mor_S(Z, X)(T)$ otherwise. Then the collection of these subfunctors satisfy the conditions (2)(a), (2)(b), (2)(c) of Schemes, Lemma 15.4 which proves the lemma. Condition (2)(a) follows from Lemma 68.1 and the other two follow from straightforward arguments. □

The condition on the morphism $f : X \to S$ in the lemma below is very useful to prove statements like it. It holds if one of the following is true: X is quasi-affine, f is quasi-affine, f is quasi-projective, f is locally projective, there exists an ample invertible sheaf on X, there exists an f-ample invertible sheaf on X, or there exists an f-very ample invertible sheaf on X.

0BL4 **Lemma 68.3.** *Let $Z \to S$ and $X \to S$ be morphisms of schemes. Assume*

1. *$Z \to S$ is finite locally free, and*
2. *for all $(s, x_1, \ldots, x_d)$ where $s \in S$ and $x_1, \ldots, x_d \in X_s$ there exists an affine open $U \subset X$ with $x_1, \ldots, x_d \in U$.*

Then $Mor_S(Z, X)$ is representable by a scheme.

Proof. Consider the set I of pairs (U, V) where $U \subset X$ and $V \subset S$ are affine open and $U \to S$ factors through V. For $i \in I$ denote (U_i, V_i) the corresponding pair. Set $F_i = Mor_{V_i}(Z_{V_i}, U_i)$. It is immediate that F_i is a subfunctor of $Mor_S(Z, X)$. Then we claim that conditions (2)(a), (2)(b), (2)(c) of Schemes, Lemma 15.4 which proves the lemma.

Condition (2)(a) follows from Lemma 68.2.

To check condition (2)(b) consider T/S and $b \in Mor_S(Z, X)$. Thinking of b as a morphism $T \times_S Z \to X$ we find an open $b^{-1}(U_i) \subset T \times_S Z$. Clearly, $b \in F_i(T)$ if and only if $b^{-1}(U_i) = T \times_S Z$. Since the projection $p : T \times_S Z \to T$ is finite hence closed, the set $U_{i,b} \subset T$ of points $t \in T$ with $p^{-1}(\{t\}) \subset b^{-1}(U_i)$ is open. Then $f : T' \to T$ factors through $U_{i,b}$ if and only if $b \circ f \in F_i(T')$ and we are done checking (2)(b).

Finally, we check condition (2)(c) and this is where our condition on $X \to S$ is used. Namely, consider T/S and $b \in Mor_S(Z, X)$. It suffices to prove that every $t \in T$ is contained in one of the opens $U_{i,b}$ defined in the previous paragraph. This is equivalent to the condition that $b(p^{-1}(\{t\})) \subset U_i$ for some i where $p : T \times_S Z \to T$ is the projection and $b : T \times_S Z \to X$ is the given morphism. Since p is finite, the set $b(p^{-1}(\{t\})) \subset X$ is finite and contained in the fibre of $X \to S$ over the image s of t in S. Thus our condition on $X \to S$ exactly shows a suitable pair exists. $\square$

0BL5 **Lemma 68.4.** *Let $Z \to S$ and $X \to S$ be morphisms of schemes. Assume $Z \to S$ is finite locally free and $X \to S$ is separated and locally quasi-finite. Then $Mor_S(Z, X)$ is representable by a scheme.*

Proof. This follows from Lemmas 68.3 and 45.1. $\square$

69. Characterizing pseudo-coherent complexes, III

0CSI In this section we discuss characterizations of pseudo-coherent complexes in terms of cohomology. This is a continuation of Derived Categories of Schemes, Section 34. A basic tool will be to reduce to the case of projective space using a derived version of Chow's lemma, see Lemma 69.2.

0CTA **Lemma 69.1.** *Consider a commutative diagram of schemes*

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X' \arrow[r] & S' \end{tikzcd}$$

Let $S \to S'$ be a morphism. Denote by X and Y the base changes of X' and Y' to S. Assume $Y' \to S'$ and $Z' \to X'$ are flat. Then $X \times_S Y$ and Z' are Tor independent over $X' \times_{S'} Y'$.

Proof. The question is local, hence we may assume all schemes are affine (some details omitted). Observe that

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & X' \times_{S'} Y' \arrow[d] \\ X \arrow[r] & X' \end{tikzcd}$$

is cartesian with flat vertical arrows. Write $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$, $X' \times_{S'} Y' = \mathrm{Spec}(B')$. Then $X \times_S Y = \mathrm{Spec}(A \otimes_{A'} B')$. Write $Z' = \mathrm{Spec}(C')$. We have to show

$$\mathrm{Tor}_p^{B'}(A \otimes_{A'} B', C') = 0, \quad \text{for } p > 0$$

Since $A' \to B'$ is flat we have $A \otimes_{A'} B' = A \otimes_{A'}^{\mathbf{L}} B'$. Hence

$$(A \otimes_{A'} B') \otimes_{B'}^{\mathbf{L}} C' = (A \otimes_{A'}^{\mathbf{L}} B') \otimes_{B'}^{\mathbf{L}} C' = A \otimes_{A'}^{\mathbf{L}} C' = A \otimes_{A'} C'$$

The second equality by More on Algebra, Lemma 60.5. The last equality because $A' \to C'$ is flat. This proves the lemma. □

0CSJ **Lemma 69.2** (Derived Chow's lemma)**.** *Let A be a ring. Let X be a separated scheme of finite presentation over A. Let $x \in X$. Then there exist an open neighbourhood $U \subset X$ of x, an $n \geq 0$, an open $V \subset \mathbf{P}^n_A$, a closed subscheme $Z \subset X \times_A \mathbf{P}^n_A$, a point $z \in Z$, and an object E in $D(\mathcal{O}_{X \times_A \mathbf{P}^n_A})$ such that*

1. *$Z \to X \times_A \mathbf{P}^n_A$ is of finite presentation,*
2. *$b : Z \to X$ is an isomorphism over U and $b(z) = x$,*
3. *$c : Z \to \mathbf{P}^n_A$ is a closed immersion over V,*
4. *$b^{-1}(U) = c^{-1}(V)$, in particular $c(z) \in V$,*
5. *$E|_{X \times_A V} \cong (b, c)_* \mathcal{O}_Z|_{X \times_A V}$,*
6. *E is pseudo-coherent and supported on Z.*

Proof. We can find a finite type $\mathbf{Z}$-subalgebra $A' \subset A$ and a scheme X' separated and of finite presentation over A' whose base change to A is X. See Limits, Lemmas 10.1 and 8.6. Let $x' \in X'$ be the image of x. If we can prove the lemma for $x' \in X'/A'$, then the lemma follows for $x \in X/A$. Namely, if U', n', V', Z', z', E' provide the solution for $x' \in X'/A'$, then we can let $U \subset X$ be the inverse image of U', let $n = n'$, let $V \subset \mathbf{P}^n_A$ be the inverse image of V', let $Z \subset X \times \mathbf{P}^n$ be the scheme theoretic inverse image of Z', let $z \in Z$ be the unique point mapping to x, and let E be the derived pullback of E'. Observe that E is pseudo-coherent by Cohomology, Lemma 47.3. It only remains to check (5). To see this set $W = b^{-1}(U) = c^{-1}(V)$ and $W' = (b')^{-1}(U) = (c')^{-1}(V')$ and consider the cartesian square

$$\begin{tikzcd} W \arrow[r] \arrow[d, "{(b,c)}"'] & W' \arrow[d, "{(b',c')}"] \\ X \times_A V \arrow[r] & X' \times_{A'} V' \end{tikzcd}$$

By Lemma 69.1 the schemes $X \times_A V$ and W' are Tor independent over $X' \times_{A'} V'$. Hence the derived pullback of $(b', c')_* \mathcal{O}_{W'}$ to $X \times_A V$ is $(b, c)_* \mathcal{O}_W$ by Derived Categories of Schemes, Lemma 22.5. This also uses that $R(b', c')_* \mathcal{O}_{Z'} = (b', c')_* \mathcal{O}_{Z'}$ because (b', c') is a closed immersion and similarly for $(b, c)_* \mathcal{O}_Z$. Since $E'|_{U' \times_{A'} V'} = (b', c')_* \mathcal{O}_{W'}$ we obtain $E|_{U \times_A V} = (b, c)_* \mathcal{O}_W$ and (5) holds. This reduces us to the situation described in the next paragraph.

Assume A is of finite type over $\mathbf{Z}$. Choose an affine open neighbourhood $U \subset X$ of x. Then U is of finite type over A. Choose a closed immersion $U \to \mathbf{A}^n_A$ and denote $j : U \to \mathbf{P}^n_A$ the immersion we get by composing with the open immersion $\mathbf{A}^n_A \to \mathbf{P}^n_A$. Let Z be the scheme theoretic closure of

$$(\mathrm{id}_U, j) : U \longrightarrow X \times_A \mathbf{P}^n_A$$

Since the projection $X \times \mathbf{P}^n \to X$ is separated, we conclude from Morphisms, Lemma 6.8 that $b : Z \to X$ is an isomorphism over U. Let $z \in Z$ be the unique point lying over x.

Let $Y \subset \mathbf{P}^n_A$ be the scheme theoretic closure of j. Then it is clear that $Z \subset X \times_A Y$ is the scheme theoretic closure of $(\mathrm{id}_U, j) : U \to X \times_A Y$. As X is separated, the morphism $X \times_A Y \to Y$ is separated as well. Hence we see that $Z \to Y$ is an isomorphism over the open subscheme $j(U) \subset Y$ by the same lemma we used above. Choose $V \subset \mathbf{P}^n_A$ open with $V \cap Y = j(U)$. Then we see that (3) and (4) hold.

Because A is Noetherian we see that X and $X \times_A \mathbf{P}^n_A$ are Noetherian schemes. Hence we can take $E = (b, c)_*\mathcal{O}_Z$ in this case, see Derived Categories of Schemes, Lemma 10.3. This finishes the proof. □

0CSK **Lemma 69.3.** *Let A, $x \in X$, and U, n, V, Z, z, E be as in Lemma 69.2. For any $K \in D_{QCoh}(\mathcal{O}_X)$ we have*

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_V = R(U \to V)_*K|_U$$

where $p : X \times_A \mathbf{P}^n_A \to X$ and $q : X \times_A \mathbf{P}^n_A \to \mathbf{P}^n_A$ are the projections and where the morphism $U \to V$ is the finitely presented closed immersion $c \circ (b|_U)^{-1}$.

Proof. Since $b^{-1}(U) = c^{-1}(V)$ and since c is a closed immersion over V, we see that $c \circ (b|_U)^{-1}$ is a closed immersion. It is of finite presentation because U and V are of finite presentation over A, see Morphisms, Lemma 21.11. First we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_V = Rq'_*\left((Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V}\right)$$

where $q' : X \times_A V \to V$ is the projection because formation of total direct image commutes with localization. Set $W = b^{-1}(U) = c^{-1}(V)$ and denote $i : W \to X \times_A V$ the closed immersion $i = (b, c)|_W$. Then

$$Rq'_*\left((Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V}\right) = Rq'_*(Lp^*K|_{X \times_A V} \otimes^{\mathbf{L}} i_*\mathcal{O}_W)$$

by property (5). Since i is a closed immersion we have $i_*\mathcal{O}_W = Ri_*\mathcal{O}_W$. Using Derived Categories of Schemes, Lemma 22.1 we can rewrite this as

$$Rq'_*Ri_*Li^*Lp^*K|_{X \times_A V} = R(q' \circ i)_*Lb^*K|_W = R(U \to V)_*K|_U$$

which is what we want. □

0CSL **Lemma 69.4.** *Let A be a ring. Let X be a scheme separated and of finite presentation over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$, then K is pseudo-coherent relative to A.*

Proof. Assume $K \in D_{QCoh}(\mathcal{O}_X)$ and $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$. Let $x \in X$. We will show that K is pseudo-coherent relative to A in a neighbourhood of x and this will prove the lemma.

Choose U, n, V, Z, z, E as in Lemma 69.2. Denote $p : X \times \mathbf{P}^n \to X$ and $q : X \times \mathbf{P}^n \to \mathbf{P}^n_A$ the projections. Then for any $i \in \mathbf{Z}$ we have

$$\begin{aligned} & R\Gamma(\mathbf{P}^n_A, Rq_*(Lp^*K \otimes^{\mathbf{L}} E) \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}^n_A}(i)) \\ & = R\Gamma(X \times \mathbf{P}^n, Lp^*K \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i)) \\ & = R\Gamma(X, K \otimes^{\mathbf{L}} Rq_*(E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i))) \end{aligned}$$

by Derived Categories of Schemes, Lemma 22.1. By Derived Categories of Schemes, Lemma 30.5 the complex $Rq_*(E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i))$ is pseudo-coherent on X. Hence the assumption tells us the expression in the displayed formula is a pseudo-coherent object of $D(A)$. By Derived Categories of Schemes, Lemma 34.2 we conclude that $Rq_*(Lp^*K \otimes^{\mathbf{L}} E)$ is pseudo-coherent on $\mathbf{P}^n_A$. By Lemma 69.3 we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V} = R(U \to V)_* K|_U$$

Since $U \to V$ is a closed immersion into an open subscheme of $\mathbf{P}^n_A$ this means $K|_U$ is pseudo-coherent relative to A by Lemma 59.18. $\square$

0GES **Lemma 69.5.** *Let A be a ring. Let X be a scheme separated and of finite presentation over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every perfect $E \in D(\mathcal{O}_X)$, then K is pseudo-coherent relative to A.*

Proof. In view of Lemma 69.4, it suffices to show $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent $E \in D(\mathcal{O}_X)$. By Derived Categories of Schemes, Proposition 40.5 it follows that $K \in D^-_{QCoh}(\mathcal{O}_X)$. Now the result follows by Derived Categories of Schemes, Lemma 34.3. $\square$

0GET **Lemma 69.6.** *Let A be a ring. Let X be a scheme separated, of finite presentation, and flat over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is perfect in $D(A)$ for every perfect $E \in D(\mathcal{O}_X)$, then K is* $\text{Spec}(A)$*-perfect.*

Proof. By Lemma 69.5, K is pseudo-coherent relative to A. By Lemma 59.18, K is pseudo-coherent in $D(\mathcal{O}_X)$. By Derived Categories of Schemes, Proposition 40.6 we see that K is in $D^-(\mathcal{O}_X)$. Let $\mathfrak{p}$ be a prime ideal of A and denote $i : Y \to X$ the inclusion of the scheme theoretic fibre over $\mathfrak{p}$, i.e., Y is a scheme over $\kappa(\mathfrak{p})$. By Derived Categories of Schemes, Lemma 35.13, we will be done if we can show $Li^*(K)$ is bounded below. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Schemes, Theorem 15.3. We have

$$\begin{aligned} R\operatorname{Hom}_{\mathcal{O}_Y}(Li^*(G), Li^*(K)) & = R\Gamma(Y, Li^*(G^\vee \otimes^{\mathbf{L}} K)) \\ & = R\Gamma(X, G^\vee \otimes^{\mathbf{L}} K) \otimes_A^{\mathbf{L}} \kappa(\mathfrak{p}) \end{aligned}$$

The first equality uses that Li^* preserves perfect objects and duals and Cohomology, Lemma 50.5; we omit some details. The second equality follows from Derived Categories of Schemes, Lemma 22.5 as X is flat over A. It follows from our hypothesis that this is a perfect object of $D(\kappa(\mathfrak{p}))$. The object $Li^*(G) \in D_{perf}(\mathcal{O}_Y)$ generates $D_{QCoh}(\mathcal{O}_Y)$ by Derived Categories of Schemes, Remark 16.4. Hence Derived Categories of Schemes, Proposition 40.6 now implies that $Li^*(K)$ is bounded below and we win. $\square$

70. Descent finiteness properties of complexes

0CSM This section is the continuation of Derived Categories of Schemes, Section 12.

0CSN **Lemma 70.1.** *Let $X \to S$ be locally of finite type. Let $\{f_i : X_i \to X\}$ be an fppf covering of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent relative to S if and only if each Lf_i^*E is m-pseudo-coherent relative to S.*

Proof. Assume E is m-pseudo-coherent relative to S. The morphisms f_i are pseudo-coherent by Lemma 60.6. Hence Lf_i^*E is m-pseudo-coherent relative to S by Lemma 59.16.

Conversely, assume that Lf_i^*E is m-pseudo-coherent relative to S for each i. Pick $S = \bigcup U_j$, $W_j \to U_j$, $W_j = \bigcup W_{j,k}$, $T_{j,k} \to W_{j,k}$, and morphisms $\alpha_{j,k} : T_{j,k} \to X_{i(j,k)}$ over S as in Lemma 48.2. Since the morphism $T_{j,K} \to S$ is flat and of finite presentation, we see that $\alpha_{j,k}$ is pseudo-coherent by Lemma 60.7. Hence

$$L\alpha_{j,k}^* Lf_{i(j,k)}^* E = L(T_{i,k} \to S)^* E$$

is m-pseudo-coherent relative to S by Lemma 59.16. Now we want to descend this property through the coverings $\{T_{j,k} \to W_{j,k}\}$, $W_j = \bigcup W_{j,k}$, $\{W_j \to U_j\}$, and $S = \bigcup U_j$. Since for Zariski coverings the result is true (by the definition of m-pseudo-coherence relative to S), this means we may assume we have a single surjective finite locally free morphism $\pi : Y \to X$ such that $L\pi^*E$ is pseudo-coherent relative to S. In this case $R\pi_*L\pi^*E$ is pseudo-coherent relative to S by Lemma 59.9 (this is the first time we use that E has quasi-coherent cohomology sheaves). We have $R\pi_*L\pi^*E = E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \pi_*\mathcal{O}_Y$ for example by Derived Categories of Schemes, Lemma 22.1 and locally on X the map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ is the inclusion of a direct summand. Hence we conclude by Lemma 59.12. $\square$

0CSP **Lemma 70.2.** *Let $X \to T \to S$ be morphisms of schemes. Assume $T \to S$ is flat and locally of finite presentation and $X \to T$ locally of finite type. Let $E \in D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent relative to S if and only if E is m-pseudo-coherent relative to T.*

Proof. Locally on X we can choose a closed immersion $i : X \to \mathbf{A}^n_T$. Then $\mathbf{A}^n_T \to S$ is flat and locally of finite presentation. Thus we may apply Lemma 59.17 to see the equivalence holds. $\square$

0CSQ **Lemma 70.3.** *Let $f : X \to S$ be locally of finite type. Let $\{S_i \to S\}$ be an fppf covering of schemes. Denote $f_i : X_i \to S_i$ the base change of f and $g_i : X_i \to X$ the projection. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent relative to S if and only if each Lg_i^*E is m-pseudo-coherent relative to S_i.*

Proof. This follows formally from Lemmas 70.1 and 70.2. Namely, if E is m-pseudo-coherent relative to S, then Lg_i^*E is m-pseudo-coherent relative to S (by the first lemma), hence Lg_i^*E is m-pseudo-coherent relative to S_i (by the second). Conversely, if Lg_i^*E is m-pseudo-coherent relative to S_i, then Lg_i^*E is m-pseudo-coherent relative to S (by the second lemma), hence E is m-pseudo-coherent relative to S (by the first lemma). $\square$

71. Relatively perfect objects

0DJW This section is a continuation of the discussion in Derived Categories of Schemes, Section 35.

0DJX **Lemma 71.1.** *Let $i : X \to X'$ be a finite order thickening of schemes. Let $K' \in D(\mathcal{O}_{X'})$ be an object such that $K = Li^*K'$ is pseudo-coherent. Then K' is pseudo-coherent.*

Proof. We first prove K' has quasi-coherent cohomology sheaves. To do this, we may reduce to the case of a first order thickening, see Section 2. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the quasi-coherent sheaf of ideals cutting out X. Tensoring the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to i_*\mathcal{O}_X \to 0$$

with K' we obtain a distinguished triangle

$$K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{I} \to K' \to K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} i_*\mathcal{O}_X \to (K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{I})[1]$$

Since $i_* = Ri_*$ and since we may view $\mathcal{I}$ as a quasi-coherent $\mathcal{O}_X$-module (as we have a first order thickening) we may rewrite this as

$$i_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I}) \to K' \to i_*K \to i_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I})[1]$$

Please use Cohomology, Lemma 54.4 to identify the terms. Since K is in $D_{QCoh}(\mathcal{O}_X)$ we conclude that K' is in $D_{QCoh}(\mathcal{O}_{X'})$; this uses Derived Categories of Schemes, Lemmas 10.1, 3.9, and 4.1.

Assume K' is in $D_{QCoh}(\mathcal{O}_{X'})$. The question is local on X' hence we may assume X' is affine. Say $X' = \text{Spec}(A')$ and $X = \text{Spec}(A)$ with $A = A'/I$ and I nilpotent. Then K' comes from an object $M' \in D(A')$, see Derived Categories of Schemes, Lemma 3.5. Thus $M = M' \otimes^{\mathbf{L}}_{A'} A$ is a pseudo-coherent object of $D(A)$ by Derived Categories of Schemes, Lemma 10.2 and our assumption on K. It follows that M' is pseudo-coherent by More on Algebra, Lemma 75.4. This finishes the proof. □

0DJY **Lemma 71.2.** *Consider a cartesian diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

of schemes. Assume $X' \to Y'$ is flat and locally of finite presentation and $Y \to Y'$ is a finite order thickening. Let $E' \in D(\mathcal{O}_{X'})$. If $E = Li^*(E')$ is Y-perfect, then E' is Y'-perfect.

Proof. Recall that being Y-perfect for E means E is pseudo-coherent and locally has finite tor dimension as a complex of $f^{-1}\mathcal{O}_Y$-modules (Derived Categories of Schemes, Definition 35.1). By Lemma 71.1 we find that E' is pseudo-coherent. In particular, E' is in $D_{QCoh}(\mathcal{O}_{X'})$, see Derived Categories of Schemes, Lemma 10.1. To prove that E' locally has finite tor dimension we may work locally on X'. Hence we may assume X', S', X, S are affine, say given by rings A', R', A, R. Then we reduce to the commutative algebra version by Derived Categories of Schemes, Lemma 35.3. The commutative algebra version in More on Algebra, Lemma 83.8. □

0DJZ **Lemma 71.3.** *Let (R, I) be a pair consisting of a ring and an ideal I contained in the Jacobson radical. Set $S = \mathrm{Spec}(R)$ and $S_0 = \mathrm{Spec}(R/I)$. Let $f : X \to S$ be proper, flat, and of finite presentation. Denote $X_0 = S_0 \times_S X$. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. If the derived restriction E_0 of E to X_0 is S_0-perfect, then E is S-perfect.*

Proof. Choose a finite affine open covering $X = U_1 \cup \ldots \cup U_n$. For each i we can choose a closed immersion $U_i \to \mathbf{A}^{d_i}_S$. Set $U_{i,0} = S_0 \times_S U_i$. For each i the complex $E_0|_{U_{i,0}}$ has tor amplitude in $[a_i, b_i]$ for some $a_i, b_i \in \mathbf{Z}$. Let $x \in X$ be a point. We will show that the tor amplitude of E_x over R is in $[a_i - d_i, b_i]$ for some i. This will finish the proof as the tor amplitude can be read off from the stalks by Cohomology, Lemma 48.5.

Since f is proper $f(\overline{\{x\}})$ is a closed subset of S. Since I is contained in the Jacobson radical, we see that $f(\overline{\{x\}})$ meeting the closed subset $S_0 \subset S$. Hence there is a specialization $x \leadsto x_0$ with $x_0 \in X_0$. Pick an i with $x_0 \in U_i$, so $x_0 \in U_{i,0}$. We will fix i for the rest of the proof. Write $U_i = \mathrm{Spec}(A)$. Then A is a flat, finitely presented R-algebra which is a quotient of a polynomial R-algebra in d_i-variables. The restriction $E|_{U_i}$ corresponds (by Derived Categories of Schemes, Lemma 3.5 and 10.2) to a pseudo-coherent object K of $D(A)$. Observe that E_0 corresponds to $K \otimes_A^{\mathbf{L}} A/IA$. Let $\mathfrak{q} \subset \mathfrak{q}_0 \subset A$ be the prime ideals corresponding to $x \leadsto x_0$. Then $E_x = K_\mathfrak{q}$ and $K_\mathfrak{q}$ is a localization of $K_{\mathfrak{q}_0}$. Hence it suffices to show that $K_{\mathfrak{q}_0}$ has tor amplitude in $[a_i - d_i, b_i]$ as a complex of R-modules. Let $I \subset \mathfrak{p}_0 \subset R$ be the prime ideal corresponding to $f(x_0)$. Then we have

$$\begin{aligned} K \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p}_0) &= (K \otimes_R^{\mathbf{L}} R/I) \otimes_{R/I}^{\mathbf{L}} \kappa(\mathfrak{p}_0) \\ &= (K \otimes_A^{\mathbf{L}} A/IA) \otimes_{R/I}^{\mathbf{L}} \kappa(\mathfrak{p}_0) \end{aligned}$$

the second equality because $R \to A$ is flat. By our choice of a_i, b_i this complex has cohomology only in degrees in the interval $[a_i, b_i]$. Thus we may finally apply More on Algebra, Lemma 83.9 to $R \to A$, $\mathfrak{q}_0$, $\mathfrak{p}_0$ and K to conclude. □

72. Contracting rational curves

0E7E In this section we study proper morphisms $f : X \to Y$ whose fibres have dimension ≤ 1 having $R^1 f_* \mathcal{O}_X = 0$. To understand the title of this section, please take a look at Algebraic Curves, Sections 22, 23, and 24.

0E7F **Lemma 72.1.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $y \in Y$ be a point with $\dim(X_y) \leq 1$. If*

1. *$R^1 f_* \mathcal{O}_X = 0$, or more generally*
2. *there is a morphism $g : Y' \to Y$ such that y is in the image of g and such that $R' f'_* \mathcal{O}_{X'} = 0$ where $f' : X' \to Y'$ is the base change of f by g.*

Then $H^1(X_y, \mathcal{O}_{X_y}) = 0$.

Proof. To prove the lemma we may replace Y by an open neighbourhood of y. Thus we may assume Y is affine and that all fibres of f have dimension ≤ 1, see Morphisms, Lemma 28.4. In this case $R^1 f_* \mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type and its formation commutes with arbitrary base change, see Limits, Lemmas 19.3 and 19.2. The lemma follows immediately. □

0E7G **Lemma 72.2.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $y \in Y$ be a point with $\dim(X_y) \leq 1$ and $H^1(X_y, \mathcal{O}_{X_y}) = 0$. Then there is an open neighbourhood $V \subset Y$ of y such that $R^1f_*\mathcal{O}_X|_V = 0$ and the same is true after base change by any $Y' \to V$.*

Proof. To prove the lemma we may replace Y by an open neighbourhood of y. Thus we may assume Y is affine and that all fibres of f have dimension ≤ 1, see Morphisms, Lemma 28.4. In this case $R^1f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type and its formation commutes with arbitrary base change, see Limits, Lemmas 19.3 and 19.2. Say $Y = \text{Spec}(A)$, y corresponds to the prime $\mathfrak{p} \subset A$, and $R^1f_*\mathcal{O}_X$ corresponds to the finite A-module M. Then $H^1(X_y, \mathcal{O}_{X_y}) = 0$ means that $\mathfrak{p}M_\mathfrak{p} = M_\mathfrak{p}$ by the statement on base change. By Nakayama's lemma we conclude $M_\mathfrak{p} = 0$. Since M is finite, we find an $f \in A$, $f \notin \mathfrak{p}$ such that $M_f = 0$. Thus taking V the principal open $D(f)$ we obtain the desired result. $\square$

0E7H **Lemma 72.3.** *Let $f : X \to Y$ be a proper morphism of schemes such that $\dim(X_y) \leq 1$ and $H^1(X_y, \mathcal{O}_{X_y}) = 0$ for all $y \in Y$. Let $\mathcal{F}$ be quasi-coherent on X. Then*

(1) *$R^pf_*\mathcal{F} = 0$ for $p > 1$, and*
(2) *$R^1f_*\mathcal{F} = 0$ if there is a surjection $f^*\mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ quasi-coherent on Y.*

If Y is affine, then we also have

(3) *$H^p(X, \mathcal{F}) = 0$ for $p \notin \{0, 1\}$, and*
(4) *$H^1(X, \mathcal{F}) = 0$ if $\mathcal{F}$ is globally generated.*

Proof. The vanishing in (1) is Limits, Lemma 19.2. To prove (2) we may work locally on Y and assume Y is affine. Then $R^1f_*\mathcal{F}$ is the quasi-coherent module on Y associated to the module $H^1(X, \mathcal{F})$. Here we use that Y is affine, quasi-coherence of higher direct images (Cohomology of Schemes, Lemma 4.5), and Cohomology of Schemes, Lemma 4.6. Since Y is affine, the quasi-coherent module $\mathcal{G}$ is globally generated, and hence so is $f^*\mathcal{G}$ and $\mathcal{F}$. In this way we see that (4) implies (2). Part (3) follows from (1) as well as the remarks on quasi-coherence of direct images just made. Thus all that remains is the prove (4). If $\mathcal{F}$ is globally generated, then there is a surjection $\bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$. By part (1) and the long exact sequence of cohomology this induces a surjection on H^1. Since $H^1(X, \mathcal{O}_X) = 0$ because $R^1f_*\mathcal{O}_X = 0$ by Lemma 72.2, and since $H^1(X, -)$ commutes with direct sums (Cohomology, Lemma 19.1) we conclude. $\square$

0E7I **Lemma 72.4.** *Let $f : X \to Y$ be a proper morphism of schemes. Assume*

(1) *for all $y \in Y$ we have $\dim(X_y) \leq 1$ and $H^1(X_y, \mathcal{O}_{X_y}) = 0$, and*
(2) *$\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.*

Then $\mathcal{O}_{Y'} \to f'_*\mathcal{O}_{X'}$ is surjective for any base change $f' : X' \to Y'$ of f.

Proof. We may assume Y and Y' affine. Then we can choose a closed immersion $Y' \to Y''$ with $Y'' \to Y$ a flat morphism of affines. By flat base change (Cohomology of Schemes, Lemma 5.2) we see that the result holds for $X'' \to Y''$. Thus we may assume Y' is a closed subscheme of Y. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal cutting out Y'. Then there is a short exact sequence

$$0 \to \mathcal{I}\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$$

where we view $\mathcal{O}_{X'}$ as a quasi-coherent module on X. By Lemma 72.3 we have $H^1(X, \mathcal{I}\mathcal{O}_X) = 0$. It follows that

$$H^0(Y, \mathcal{O}_Y) \to H^0(Y, f_*\mathcal{O}_X) = H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_{X'})$$

is surjective as desired. The first arrow is surjective as Y is affine and since we assumed $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective and the second by the long exact sequence of cohomology associated to the short exact sequence above and the vanishing just proved. □

0E7J **Lemma 72.5.** *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

of morphisms of schemes. Let $s \in S$ be a point. Assume

1. *$X \to S$ is locally of finite presentation and flat at points of X_s,*
2. *f is proper,*
3. *the fibres of $f_s : X_s \to Y_s$ have dimension ≤ 1 and $R^1 f_{s,*}\mathcal{O}_{X_s} = 0$,*
4. *$\mathcal{O}_{Y_s} \to f_{s,*}\mathcal{O}_{X_s}$ is surjective.*

Then there is an open $Y_s \subset V \subset Y$ such that (a) $f^{-1}(V)$ is flat over S, (b) $\dim(X_y) \leq 1$ for $y \in V$, (c) $R^1 f_*\mathcal{O}_X|_V = 0$, (d) $\mathcal{O}_V \to f_*\mathcal{O}_X|_V$ is surjective, and (b), (c), and (d) remain true after base change by any $Y' \to V$.

Proof. Let $y \in Y$ be a point over s. It suffices to find an open neighbourhood of y with the desired properties. As a first step, we replace Y by the open V found in Lemma 72.2 so that $R^1 f_*\mathcal{O}_X$ is zero universally (the hypothesis of the lemma holds by Lemma 72.1). We also shrink Y so that all fibres of f have dimension ≤ 1 (use Morphisms, Lemma 28.4 and properness of f). Thus we may assume we have (b) and (c) with $V = Y$ and after any base change $Y' \to Y$. Thus by Lemma 72.4 it now suffices to show (d) over Y. We may still shrink Y further; for example, we may and do assume Y and S are affine.

By Theorem 15.1 there is an open subset $U \subset X$ where $X \to S$ is flat which contains X_s by hypothesis. Then $f(X \setminus U)$ is a closed subset not containing y. Thus after shrinking Y we may assume X is flat over S.

Say $S = \text{Spec}(R)$. Choose a closed immersion $Y \to Y'$ where Y' is the spectrum of a polynomial ring $R[x_e; e \in E]$ on a set E. Denote $f' : X \to Y'$ the composition of f with $Y \to Y'$. Then the hypotheses (1) – (4) as well as (b) and (c) hold for f' and s. If we we show $\mathcal{O}_{Y'} \to f'_*\mathcal{O}_X$ is surjective in an open neighbourhood of y, then the same is true for $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Thus we may assume Y is the spectrum of $R[x_e; e \in E]$.

At this point X and Y are flat over S. Then Y_s and X are tor independent over Y. We urge the reader to find their own proof, but it also follows from Lemma 69.1 applied to the square with corners X, Y, S, S and its base change by $s \to S$. Hence

$$Rf_{s,*}\mathcal{O}_{X_s} = L(Y_s \to Y)^* Rf_*\mathcal{O}_X$$

by Derived Categories of Schemes, Lemma 22.5. Because of the vanishing already established this implies $f_{s,*}\mathcal{O}_{X_s} = (Y_s \to Y)^* f_*\mathcal{O}_X$. We conclude that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of quasi-coherent $\mathcal{O}_Y$-modules whose pullback to Y_s is surjective. We claim $f_*\mathcal{O}_X$ is a finite type $\mathcal{O}_Y$-module. If true, then the cokernel $\mathcal{F}$ of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a finite type quasi-coherent $\mathcal{O}_Y$-module such that $\mathcal{F}_y \otimes \kappa(y) = 0$. By Nakayama's lemma (Algebra, Lemma 20.1) we have $\mathcal{F}_y = 0$. Thus $\mathcal{F}$ is zero in an open neighbourhood of y (Modules, Lemma 9.5) and the proof is complete.

Proof of the claim. For a finite subset $E' \subset E$ set $Y' = \text{Spec}(R[x_e; e \in E'])$. For large enough E' the morphism $f' : X \to Y \to Y'$ is proper, see Limits, Lemma 13.4. We fix E' and Y' in the following. Write $R = \text{colim}\, R_i$ as the colimit of its finite type $\mathbf{Z}$-subalgebras. Set $S_i = \text{Spec}(R_i)$ and $Y'_i = \text{Spec}(R_i[x_e; e \in E'])$. For i large enough we can find a diagram

$$\begin{tikzcd} X \arrow[r, "f'"'] \arrow[d] & Y' \arrow[r] \arrow[d] & S \arrow[d] \\ X_i \arrow[r, "f'_i"] & Y'_i \arrow[r] & S_i \end{tikzcd}$$

with cartesian squares such that X_i is flat over S_i and $X_i \to Y'_i$ is proper. See Limits, Lemmas 10.1, 8.7, and 13.1. The same argument as above shows Y' and X_i are tor independent over Y'_i and hence

$$R\Gamma(X, \mathcal{O}_X) = R\Gamma(X_i, \mathcal{O}_{X_i}) \otimes^{\mathbf{L}}_{R_i[x_e; e \in E']} R[x_e; e \in E']$$

by the same reference as above. By Cohomology of Schemes, Lemma 19.2 the complex $R\Gamma(X_i, \mathcal{O}_{X_i})$ is pseudo-coherent in the derived category of the Noetherian ring $R_i[x_e; e \in E']$ (see More on Algebra, Lemma 64.17). Hence $R\Gamma(X, \mathcal{O}_X)$ is pseudo-coherent in the derived category of $R[x_e; e \in E']$, see More on Algebra, Lemma 64.12. Since the only nonvanishing cohomology module is $H^0(X, \mathcal{O}_X)$ we conclude it is a finite $R[x_e; e \in E']$-module, see More on Algebra, Lemma 64.4. This concludes the proof. $\square$

0E7K **Lemma 72.6.** *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

of morphisms of schemes. Assume $X \to S$ is flat, f is proper, $\dim(X_y) \leq 1$ *for $y \in Y$, and $R^1 f_*\mathcal{O}_X = 0$. Then $f_*\mathcal{O}_X$ is S-flat and formation of $f_*\mathcal{O}_X$ commutes with arbitrary base change $S' \to S$.*

Proof. We may assume Y and S are affine, say $S = \text{Spec}(A)$. To show the quasi-coherent $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$ is flat relative to S it suffices to show that $H^0(X, \mathcal{O}_X)$ is flat over A (some details omitted). By Lemma 72.3 we have $H^1(X, \mathcal{O}_X \otimes_A M) = 0$ for every A-module M. Since also $\mathcal{O}_X$ is flat over A we deduce the functor $M \mapsto H^0(X, \mathcal{O}_X \otimes_A M)$ is exact. Moreover, this functor commutes with direct sums by Cohomology, Lemma 19.1. Then it is an exercise to see that $H^0(X, \mathcal{O}_X \otimes_A M) = M \otimes_A H^0(X, \mathcal{O}_X)$ functorially in M and this gives the desired flatness. Finally, if $S' \to S$ is a morphism of affines given by the ring map $A \to A'$, then in the affine case just discussed we see that

$$H^0(X \times_S S', \mathcal{O}_{X \times_S S'}) = H^0(X, \mathcal{O}_X \otimes_A A') = H^0(X, \mathcal{O}_X) \otimes_A A'$$

This shows that formation of $f_*\mathcal{O}_X$ commutes with any base change $S' \to S$. Some details omitted. $\square$

0E7L **Lemma 72.7.** *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

of morphisms of schemes. Let $s \in S$ be a point. Assume

1. *$X \to S$ is locally of finite presentation and flat at points of X_s,*
2. *$Y \to S$ is locally of finite presentation,*
3. *f is proper,*
4. *the fibres of $f_s : X_s \to Y_s$ have dimension ≤ 1 and $R^1 f_{s,*}\mathcal{O}_{X_s} = 0$,*
5. *$\mathcal{O}_{Y_s} \to f_{s,*}\mathcal{O}_{X_s}$ is an isomorphism.*

Then there is an open $Y_s \subset V \subset Y$ such that (a) V is flat over S, (b) $f^{-1}(V)$ is flat over S, (c) $\dim(X_y) \leq 1$ for $y \in V$, (d) $R^1 f_*\mathcal{O}_X|_V = 0$, (e) $\mathcal{O}_V \to f_*\mathcal{O}_X|_V$ is an isomorphism, and (a) – (e) remain true after base change of $f^{-1}(V) \to V$ by any $S' \to S$.

Proof. Let $y \in Y_s$. We may always replace Y by an open neighbourhood of y. Thus we may assume Y and S affine. We may also assume that X is flat over S, $\dim(X_y) \leq 1$ for $y \in Y$, $R^1 f_*\mathcal{O}_X = 0$ universally, and that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, see Lemma 72.5. (We won't use all of this.)

Assume S and Y affine. Write $S = \lim S_i$ as a cofiltered of affine Noetherian schemes S_i. By Limits, Lemma 10.1 there exists an element $0 \in I$ and a diagram

$$\begin{tikzcd} X_0 \arrow[rr, "f_0"] \arrow[dr] & & Y_0 \arrow[dl] \\ & S_0 & \end{tikzcd}$$

of finite type morphisms of schemes whose base change to S is the diagram of the lemma. After increasing 0 we may assume Y_0 is affine and $X_0 \to S_0$ proper, see Limits, Lemmas 13.1 and 4.13. Let $s_0 \in S_0$ be the image of s. As Y_s is affine, we see that $R^1 f_{s,*}\mathcal{O}_{X_s} = 0$ is equivalent to $H^1(X_s, \mathcal{O}_{X_s}) = 0$. Since X_s is the base change of X_{0,s_0} by the faithfully flat map $\kappa(s_0) \to \kappa(s)$ we see that $H^1(X_{0,s_0}, \mathcal{O}_{X_{0,s_0}}) = 0$ and hence $R^1 f_{0,*}\mathcal{O}_{X_{0,s_0}} = 0$. Similarly, as $\mathcal{O}_{Y_s} \to f_{s,*}\mathcal{O}_{X_s}$ is an isomorphism, so is $\mathcal{O}_{Y_{0,s_0}} \to f_{0,*}\mathcal{O}_{X_{0,s_0}}$. Since the dimensions of the fibres of $X_s \to Y_s$ are at most 1, the same is true for the morphism $X_{0,s_0} \to Y_{0,s_0}$. Finally, since $X \to S$ is flat, after increasing 0 we may assume X_0 is flat over S_0, see Limits, Lemma 8.7. Thus it suffices to prove the lemma for $X_0 \to Y_0 \to S_0$ and the point s_0.

Combining the reduction arguments above we reduce to the case where S and Y affine, S Noetherian, the fibres of f have dimension ≤ 1, and $R^1 f_*\mathcal{O}_X = 0$ universally. Let $y \in Y_s$ be a point. Claim:

$$\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$$

is an isomorphism. The claim implies the lemma. Namely, since $f_*\mathcal{O}_X$ is coherent (Cohomology of Schemes, Proposition 19.1) the claim means we can replace Y by an open neighbourhood of y and obtain an isomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Then we conclude that Y is flat over S by Lemma 72.6. Finally, the isomorphism $\mathcal{O}_Y \to$

$f_*\mathcal{O}_X$ remains an isomorphism after any base change $S' \to S$ by the final statement of Lemma 72.6.

Proof of the claim. We already know that $\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$ is surjective (Lemma 72.5) and that $(f_*\mathcal{O}_X)_y$ is $\mathcal{O}_{S,s}$-flat (Lemma 72.6) and that the induced map

$$\mathcal{O}_{Y_s,y} = \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y/\mathfrak{m}_s(f_*\mathcal{O}_X)_y \to (f_{s,*}\mathcal{O}_{X_s})_y$$

is injective by the assumption in the lemma. Then it follows from Algebra, Lemma 99.1 that $\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$ is injective as desired. $\square$

0E24 **Lemma 72.8.** *Let $f : X \to Y$ be a proper morphism of Noetherian schemes such that $f_*\mathcal{O}_X = \mathcal{O}_Y$, such that the fibres of f have dimension ≤ 1, and such that $H^1(X_y, \mathcal{O}_{X_y}) = 0$ for $y \in Y$. Then $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is a bijection onto the subgroup of $\mathcal{L} \in \mathrm{Pic}(X)$ with $\mathcal{L}|_{X_y} \cong \mathcal{O}_{X_y}$ for all $y \in Y$.*

Proof. By the projection formula (Cohomology, Lemma 54.2) we see that $f_*f^*\mathcal{N} \cong \mathcal{N}$ for $\mathcal{N} \in \mathrm{Pic}(Y)$. We claim that for $\mathcal{L} \in \mathrm{Pic}(X)$ with $\mathcal{L}|_{X_y} \cong \mathcal{O}_{X_y}$ for all $y \in Y$ we have $\mathcal{N} = f_*\mathcal{L}$ is invertible and $\mathcal{L} \cong f^*\mathcal{N}$. This will finish the proof.

The $\mathcal{O}_Y$-module $\mathcal{N} = f_*\mathcal{L}$ is coherent by Cohomology of Schemes, Proposition 19.1. Thus to see that it is an invertible $\mathcal{O}_Y$-module, it suffices to check on stalks (Algebra, Lemma 78.2). Since the map from a Noetherian local ring to its completion is faithfully flat, it suffices to check the completion $(f_*\mathcal{L})_y^\wedge$ is free (see Algebra, Section 97 and Lemma 78.6). For this we will use the theorem of formal functions as formulated in Cohomology of Schemes, Lemma 20.7. Since $f_*\mathcal{O}_X = \mathcal{O}_Y$ and hence $(f_*\mathcal{O}_X)_y^\wedge \cong \mathcal{O}_{Y,y}^\wedge$, it suffices to show that $\mathcal{L}|_{X_n} \cong \mathcal{O}_{X_n}$ for each n (compatibly for varying n. By Lemma 4.1 we have an exact sequence

$$H^1(X_y, \mathfrak{m}_y^n\mathcal{O}_X/\mathfrak{m}_y^{n+1}\mathcal{O}_X) \to \mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$$

with notation as in the theorem on formal functions. Observe that we have a surjection

$$\mathcal{O}_{X_y}^{\oplus r_n} \cong \mathfrak{m}_y^n/\mathfrak{m}_y^{n+1} \otimes_{\kappa(y)} \mathcal{O}_{X_y} \longrightarrow \mathfrak{m}_y^n\mathcal{O}_X/\mathfrak{m}_y^{n+1}\mathcal{O}_X$$

for some integers $r_n \geq 0$. Since $\dim(X_y) \leq 1$ this surjection induces a surjection on first cohomology groups (by the vanishing of cohomology in degrees ≥ 2 coming from Cohomology, Proposition 20.7). Hence the H^1 in the sequence is zero and the transition maps $\mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$ are injective as desired.

We still have to show that $f^*\mathcal{N} \cong \mathcal{L}$. This is proved by the same method and we omit the details. $\square$

73. Affine stratifications

0F2R This material is taken from [RV04]. Please read a little bit about stratifications in Topology, Section 28 before reading this section.

If X is a scheme, then a stratification of X usually means a stratification of the underlying topological space of X. The strata are locally closed subsets. We will view these strata as reduced locally closed subschemes of X using Schemes, Remark 12.6.

0F2S **Definition 73.1.** Let X be a scheme. An *affine stratification* is a locally finite stratification $X = \coprod_{i \in I} X_i$ whose strata X_i are affine and such that the inclusion morphisms $X_i \to X$ are affine.

The condition that a stratification $X = \coprod X_i$ is locally finite is, in the presence of the condition that the inclusion morphisms $X_i \to X$ are quasi-compact, equivalent to the condition that the strata are locally constructible subsets of X, see Properties, Lemma 2.7.

The condition that $X_i \to X$ is an affine morphism is independent on the scheme structure we put on the locally closed subset X_i, see Lemma 3.1. Moreover, if X is separated (or more generally has affine diagonal) and $X = \coprod X_i$ is a locally finite stratification with affine strata, then the morphisms $X_i \to X$ are affine. See Morphisms, Lemma 11.11. This allows us to disregard the condition of affineness of the inclusion morphisms $X_i \to X$ in most cases of interest.

We are often interested in the case where the partially ordered index set I of the stratification is finite. Recall that the *length* of a partially ordered set I is the supremum of the lengths p of chains $i_0 < i_1 < \ldots < i_p$ of elements of I.

0F2T **Lemma 73.2.** *Let X be a scheme. Let $X = \coprod_{i \in I} X_i$ be a finite affine stratification. There exists an affine stratification with index set $\{0, \ldots, n\}$ where n is the length of I.*

Proof. Recall that we have a partial ordering on I such that the closure of X_i is contained in $\bigcup_{j \leq i} X_j$ for all $i \in I$. Let $I' \subset I$ be the set of maximal indices of I. If $i \in I'$, then X_i is open in X because the union of the closures of the other strata is the complement of X_i. Let $U = \bigcup_{i \in I'} X_i$ viewed as an open subscheme of X so that $U_{red} = \coprod_{i \in I'} X_i$ as schemes. Then U is an affine scheme by Schemes, Lemma 6.8 and Lemma 2.3. The morphism $U \to X$ is affine as each $X_i \to X$, $i \in I'$ is affine by the same reasoning using Lemma 3.1. The complement $Z = X \setminus U$ endowed with the reduced induced scheme structure has the affine stratification $Z = \bigcup_{i \in I \setminus I'} X_i$. Here we use that a morphism of schemes $T \to Z$ is affine if and only if the composition $T \to X$ is affine; this follows from Morphisms, Lemmas 11.9, 11.7, and 11.11. Observe that the partially ordered set $I \setminus I'$ has length exactly one less than the length of I. Hence by induction we find that Z has an affine stratification $Z = Z_0 \amalg \ldots \amalg Z_{n-1}$ with index set $\{1, \ldots, n\}$. Setting $Z_n = U$ we obtain the desired stratification of X. $\square$

If a scheme X has a finite affine stratification, then of course X is quasi-compact. A bit less obvious is the fact that it forces X to be quasi-separated as well.

0F2U **Lemma 73.3.** *Let X be a scheme. The following are equivalent*

1. *X has a finite affine stratification, and*
2. *X is quasi-compact and quasi-separated.*

Proof. Let $X = \bigcup X_i$ be a finite affine stratification. Since each X_i is affine hence quasi-compact, we conclude that X is quasi-compact. Let $U, V \subset X$ be affine open. Then $U \cap X_i$ and $V \cap X_i$ are affine open in X_i since $X_i \to X$ is an affine morphism. Hence $U \cap V \cap X_i$ is an affine open of the affine scheme X_i (see Schemes, Lemma 21.7 for example). Therefore $U \cap V = \coprod U \cap V \cap X_i$ is quasi-compact as a finite union of affine strata. We conclude that X is quasi-separated by Schemes, Lemma 21.6.

Assume X is quasi-compact and quasi-separated. We may use the induction principle of Cohomology of Schemes, Lemma 4.1 to prove the assertion that X has a finite affine stratification. If X is empty, then it has an empty affine stratification.

If X is nonempty affine then it has an affine stratification with one stratum. Next, assume $X = U \cup V$ where U is quasi-compact open, V is affine open, and we have a finite affine stratifications $U = \bigcup_{i \in I} U_i$ and $U \cap V = \coprod_{j \in J} W_j$. Denote $Z = X \setminus V$ and $Z' = X \setminus U$. Note that Z is closed in U and Z' is closed in V. Observe that $U_i \cap Z$ and $U_i \cap W_j = U_i \times_U W_j$ are affine schemes affine over U. (Hints: use that $U_i \times_U W_j \to W_j$ is affine as a base change of $U_i \to U$, hence $U_i \cap W_j$ is affine, hence $U_i \cap W_j \to U_i$ is affine, hence $U_i \cap W_j \to U$ is affine.) It follows that

$$U = \coprod\nolimits_{i \in I} (U_i \cap Z) \amalg \coprod\nolimits_{(i,j) \in I \times J} (U_i \cap W_j)$$

is a finite affine stratification with partial ordering on $I \amalg I \times J$ given by $i' \leq (i,j) \Leftrightarrow i' \leq i$ and $(i', j') \leq (i,j) \Leftrightarrow i' \leq i$ and $j' \leq j$. Observe that $(U_i \cap Z) \times_X V = \emptyset$ and $(U_i \cap W_j) \times_X V = U_i \cap W_j$ are affine. Hence the morphisms $U_i \cap Z \to X$ and $U_i \cap W_j \to X$ are affine because we can check affineness of a morphism locally on the target (Morphisms, Lemma 11.3) and we have affineness over both U and V. To finish the proof we take the stratification above and we add one additional stratum, namely Z', whose index we add as a minimal element to the partially ordered set. $\square$

0F2V **Definition 73.4.** Let X be a nonempty quasi-compact and quasi-separated scheme. The *affine stratification number* is the smallest integer $n \geq 0$ such that the following equivalent conditions are satisfied

(1) there exists a finite affine stratification $X = \coprod_{i \in I} X_i$ where I has length n,
(2) there exists an affine stratification $X = X_0 \amalg X_1 \amalg \ldots \amalg X_n$ with index set $\{0, \ldots, n\}$.

The equivalence of the conditions holds by Lemma 73.2. The existence of a finite affine stratification is proven in Lemma 73.3.

0F2W **Lemma 73.5.** *Let X be a separated scheme which has an open covering by $n+1$ affines. Then the affine stratification number of X is at most n.*

Proof. Say $X = U_0 \cup \ldots \cup U_n$ is an affine open covering. Set

$$X_i = (U_i \cup \ldots \cup U_n) \setminus (U_{i+1} \cup \ldots \cup U_n)$$

Then X_i is affine as a closed subscheme of U_i. The morphism $X_i \to X$ is affine by Morphisms, Lemma 11.11. Finally, we have $\overline{X_i} \subset X_i \cup X_{i-1} \cup \ldots X_0$. $\square$

0F2X **Lemma 73.6.** *Let X be a Noetherian scheme of dimension $\infty > d \geq 0$. Then the affine stratification number of X is at most d.*

Proof. By induction on d. If $d = 0$, then X is affine, see Properties, Lemma 10.5. Assume $d > 0$. Let $\eta_1, \ldots, \eta_n$ be the generic points of the irreducible components of X (Properties, Lemma 5.7). We can cover X by affine opens containing $\eta_1, \ldots, \eta_n$, see Properties, Lemma 29.4. Since X is quasi-compact we can find a finite affine open covering $X = \bigcup_{j=1,\ldots,m} U_j$ with $\eta_1, \ldots, \eta_n \in U_j$ for all $j = 1, \ldots, m$. Choose an affine open $U \subset U_1 \cap \ldots \cap U_m$ containing $\eta_1, \ldots, \eta_n$ (possible by the lemma already quoted). Then the morphism $U \to X$ is affine because $U \to U_j$ is affine for all j, see Morphisms, Lemma 11.3. Let $Z = X \setminus U$. By construction $\dim(Z) < \dim(X)$. By induction hypothesis we can find an affine stratification $Z = \bigcup_{i \in \{0, \ldots, n\}} Z_i$ of Z with $n \leq \dim(Z)$. Setting $U = X_{n+1}$ and $X_i = Z_i$ for $i \leq n$ we conclude. $\square$

0F2Y **Proposition 73.7.** *Let X be a nonempty quasi-compact and quasi-separated scheme with affine stratification number n. Then $H^p(X, \mathcal{F}) = 0$, $p > n$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof. We will prove this by induction on the affine stratification number n. If $n = 0$, then X is affine and the result is Cohomology of Schemes, Lemma 2.2. Assume $n > 0$. By Definition 73.4 there is an affine scheme U and an affine open immersion $j : U \to X$ such that the complement Z has affine stratification number $n-1$. As U and j are affine we have $H^p(X, j_*(\mathcal{F}|_U)) = 0$ for $p > 0$, see Cohomology of Schemes, Lemmas 2.4 and 2.3. Denote $\mathcal{K}$ and $\mathcal{Q}$ the kernel and cokernel of the map $\mathcal{F} \to j_*(\mathcal{F}|_U)$. Thus we obtain an exact sequence

$$0 \to \mathcal{K} \to \mathcal{F} \to j_*(\mathcal{F}|_U) \to \mathcal{Q} \to 0$$

of quasi-coherent $\mathcal{O}_X$-modules (see Schemes, Section 24). A standard argument, breaking our exact sequence into short exact sequences and using the long exact cohomology sequence, shows it suffices to prove $H^p(X, \mathcal{K}) = 0$ and $H^p(X, \mathcal{Q}) = 0$ for $p \geq n$. Since $\mathcal{F} \to j_*(\mathcal{F}|_U)$ restricts to an isomorphism over U, we see that $\mathcal{K}$ and $\mathcal{Q}$ are supported on Z. By Properties, Lemma 22.3 we can write these modules as the filtered colimits of their finite type quasi-coherent submodules. Using the fact that cohomology of sheaves on X commutes with filtered colimits, see Cohomology, Lemma 19.1, we conclude it suffices to show that if $\mathcal{G}$ is a finite type quasi-coherent module whose support is contained in Z, then $H^p(X, \mathcal{G}) = 0$ for $p \geq n$. Let $Z' \subset X$ be the scheme theoretic support of $\mathcal{G} \oplus \mathcal{O}_Z$; we may and do think of $\mathcal{G}$ as a quasi-coherent module on Z', see Morphisms, Section 5. Then Z' and Z have the same underlying topological space and hence the same affine stratification number, namely $n - 1$. Hence $H^p(X, \mathcal{G}) = H^p(Z', \mathcal{G})$ (equality by Cohomology of Schemes, Lemma 2.4) vanishes for $p \geq n$ by induction hypothesis. □

0F2Z **Example 73.8.** Let k be a field and let $X = \mathbf{P}^n_k$ be n-dimensional projective space over k. Lemma 73.5 applies to this by Constructions, Lemma 13.3. Hence the affine stratification number of $\mathbf{P}^n_k$ is at most n. On the other hand, we have nonzero cohomology in degree n for some quasi-coherent modules on $\mathbf{P}^n_k$, see Cohomology of Schemes, Lemma 8.1. Using Proposition 73.7 we conclude that the affine stratification number of $\mathbf{P}^n_k$ is equal to n.

74. Universally open morphisms

0F30 Some material on universally open morphisms.

0F31 **Lemma 74.1.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is universally open,*
(2) *for every morphism $S' \to S$ which is locally of finite presentation the base change $X_{S'} \to S'$ is open, and*
(3) *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is open.*

Proof. It is clear that (1) implies (2) and (2) implies (3). Let us prove that (3) implies (1). Suppose that the base change $X_T \to T$ is not open for some morphism of schemes $g : T \to S$. Then we can find some affine opens $V \subset S$, $U \subset X$, $W \subset T$ with $f(U) \subset V$ and $g(W) \subset V$ such that $U \times_V W \to W$ is not open. If we can show that this implies $\mathbf{A}^n \times U \to \mathbf{A}^n \times V$ is not open, then $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$

is not open and the proof is complete. This reduces us to the result proved in the next paragraph.

Let $A \to B$ be a ring map such that $A' \to B' = A' \otimes_A B$ does not induce an open map of spectra for some A-algebra A'. As the principal opens give a basis for the topology of $\text{Spec}(B')$ we conclude that the image of $D(g)$ in $\text{Spec}(A')$ is not open for some $g \in B'$. Write $g = \sum_{i=1,\ldots,n} a'_i \otimes b_i$ for some n, $a'_i \in A'$, and $b_i \in B$. Consider the element $h = \sum_{i=1,\ldots,n} x_i b_i$ in $B[x_1, \ldots, x_n]$. Assume that $D(h)$ maps to an open subset under the morphism

$$\text{Spec}(B[x_1, \ldots, x_n]) \longrightarrow \text{Spec}(A[x_1, \ldots, x_n])$$

in order to get a contradiction. Then $D(h)$ would map surjectively onto a quasi-compact open $U \subset \text{Spec}(A[x_1, \ldots, x_n])$. Let $A[x_1, \ldots, x_n] \to A'$ be the A-algebra homomorphism sending x_i to a'_i. This also induces a B-algebra homomorphism $B[x_1, \ldots, x_n] \to B'$ sending h to g. Since

$$\begin{tikzcd} \text{Spec}(B[x_1, \ldots, x_n]) \arrow[d] & \text{Spec}(B') \arrow[l] \arrow[d] \\ \text{Spec}(A[x_1, \ldots, x_n]) & \text{Spec}(A') \arrow[l] \end{tikzcd}$$

is cartesian the image of $D(g)$ in $\text{Spec}(A')$ is equal to the inverse image of U in $\text{Spec}(A')$ and hence open which is the desired contradiction. $\square$

0F32 **Lemma 74.2.** *Let $f : X \to Y$ be a morphism of schemes. If*

(1) *f is locally quasi-finite,*
(2) *Y is geometrically unibranch and locally Noetherian, and*
(3) *every irreducible component of X dominates an irreducible component of Y,*

then f is universally open.

Proof. For any n the scheme $\mathbf{A}^n \times Y$ is geometrically unibranch by Lemma 36.4 and Properties, Lemma 15.6. Hence the hypotheses of the lemma hold for the morphisms $\mathbf{A}^n \times X \to \mathbf{A}^n \times Y$ for all n. By Lemma 74.1 it suffices to prove f is open. By Morphisms, Lemma 23.2 it suffices to show that generalizations lift along f. Suppose that $y' \rightsquigarrow y$ is a specialization of points in Y and $x \in X$ is a point mapping to y. As in Lemma 41.1 choose a diagram

$$\begin{tikzcd} u \arrow[d] & U \arrow[r] \arrow[d] & X \arrow[d] \\ v & V \arrow[r] & Y \end{tikzcd}$$

where $(V, v) \to (Y, y)$ is an elementary étale neighbourhood, $U \to V$ is finite, u is the unique point of U mapping to v, $U \subset V \times_Y X$ is open, and $v \mapsto y$ and $u \mapsto x$. Let E be an irreducible component of U passing through u (there is at least one of these). Since $U \to X$ is étale, E maps to an irreducible component of X, which in turn dominates an irreducible component of Y (by assumption). Since $U \to V$ is finite hence closed, we conclude that the image $E' \subset V$ of E is an irreducible closed subset passing through v which dominates an irreducible component of Y. Since $V \to Y$ is étale E' must be an irreducible component of V passing through v. Since Y is geometrically unibranch we see that E' is the unique irreducible component of V passing through v (Lemma 36.2). Since V is locally Noetherian we may after shrinking V assume that $E' = V$ (equality of sets).

Since $V \to Y$ is étale we can find a specialization $v' \leadsto v$ whose image is $y' \leadsto y$. By the above we can find $u' \in U$ mapping to v'. Then $u' \leadsto u$ because u is the only point of U mapping to v and $U \to V$ is closed. Then finally the image $x' \in X$ of u' is a point specializing to x and mapping to y' and the proof is complete. □

0F33 **Lemma 74.3.** *Let $A \to B$ be a ring map. Say B is generated as an A-module by $b_1, \ldots, b_d \in B$. Set $h = \sum x_i b_i \in B[x_1, \ldots, x_d]$. Then $\text{Spec}(B) \to \text{Spec}(A)$ is universally open if and only if the image of $D(h)$ in $\text{Spec}(A[x_1, \ldots, x_d])$ is open.*

Proof. If $\text{Spec}(B) \to \text{Spec}(A)$ is universally open, then of course the image of $D(h)$ is open. Conversely, assume the image U of $D(h)$ is open. Let $A \to A'$ be a ring map. It suffices to show that the image of any principal open $D(g) \subset \text{Spec}(A' \otimes_A B)$ in $\text{Spec}(A')$ is open. We may write $g = \sum_{i=1,\ldots,d} a'_i \otimes b_i$ for some $a'_i \in A'$. Let $A[x_1, \ldots, x_n] \to A'$ be the A-algebra homomorphism sending x_i to a'_i. This also induces a B-algebra homomorphism $B[x_1, \ldots, x_n] \to A' \otimes_A B$ sending h to g. Since

$$\begin{tikzcd} \text{Spec}(B[x_1, \ldots, x_n]) \arrow[d] & \text{Spec}(B') \arrow[l] \arrow[d] \\ \text{Spec}(A[x_1, \ldots, x_n]) & \text{Spec}(A') \arrow[l] \end{tikzcd}$$

is cartesian the image of $D(g)$ in $\text{Spec}(A')$ is equal to the inverse image of U in $\text{Spec}(A')$ and hence open. □

0F34 **Lemma 74.4.** *Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms. Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. Assume S_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to S_i and let $f : X \to Y$ be the base change of f_0 to S. If*

1. *f is locally quasi-finite and universally open, and*
2. *f_0 is locally of finite presentation,*

then there exists an $i \geq 0$ such that f_i is locally quasi-finite and universally open.

Proof. By Limits, Lemma 18.2 after increasing 0 we may assume f_0 is locally quasi-finite. Let $x \in X$. By étale localization of quasi-finite morphisms we can find a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] \\ Y & V \arrow[l] \end{tikzcd}$$

where $V \to Y$ is étale, $U \subset X_V$ is open, $U \to V$ is finite, and x is in the image of $U \to X$, see Lemma 41.1. After shrinking V we may assume V and U are affine. Since X is quasi-compact, it follows, by taking a finite disjoint union of such V and U, that we can make a diagram as above such that $U \to X$ is surjective. By Limits, Lemmas 10.1, 4.11, 8.15, 8.3, 8.10, and 4.13 after possibly increasing 0 we may assume we have a diagram

$$\begin{tikzcd} X_0 & U_0 \arrow[l] \arrow[d] \\ Y_0 \arrow[u, leftarrow] & V_0 \arrow[l] \end{tikzcd}$$

where V_0 is affine, $V_0 \to Y_0$ is étale, $U_0 \subset (X_0)_{V_0}$ is open, $U_0 \to V_0$ is finite, and $U_0 \to X_0$ is surjective. Since $V_i \to Y_i$ is étale and hence universally open, follows that it suffices to prove that $U_i \to V_i$ is universally open for large enough i. This reduces us to the case discussed in the next paragraph.

Let $A = \mathrm{colim}\, A_i$ be a filtered colimit of rings. Let $A_0 \to B_0$ be a ring map. Set $B = A \otimes_{A_0} B_0$ and $B_i = A_i \otimes_{A_0} B_0$. Assume $A_0 \to B_0$ is finite, of finite presentation, and $A \to B$ is universally open. We have to show that $A_i \to B_i$ is universally open for i large enough. Pick $b_{0,1}, \ldots, b_{0,d} \in B_0$ which generate B_0 as an A_0-module. Set $h_0 = \sum_{j=1,\ldots,d} x_j b_{0,j}$ in $B_0[x_1, \ldots, x_d]$. Denote h, resp. h_i the image of h_0 in $B[x_1, \ldots, x_d]$, resp. $B_i[x_1, \ldots, x_d]$. The image U of $D(h)$ in $\mathrm{Spec}(A[x_1, \ldots, x_d])$ is open as $A \to B$ is universally open. Of course U is quasi-compact as the image of an affine scheme. For i large enough there is a quasi-compact open $U_i \subset \mathrm{Spec}(A_i[x_1, \ldots, x_d])$ whose inverse image in $\mathrm{Spec}(A[x_1, \ldots, x_d])$ is U, see Limits, Lemma 4.11. After increasing i we may assume that $D(h_i)$ maps into U_i; this follows from the same lemma by considering the pullback of U_i in $D(h_i)$. Finally, for i even larger the morphism of schemes $D(h_i) \to U_i$ will be surjective by an application of the already used Limits, Lemma 8.15. We conclude $A_i \to B_i$ is universally open by Lemma 74.3. $\square$

0F35 **Lemma 74.5.** *Let $f : X \to Y$ be a locally quasi-finite morphism. Then*

(1) *the functions $n_{X/Y}$ of Lemmas 27.3 and 28.3 agree,*
(2) *if X is quasi-compact, then $n_{X/Y}$ attains a maximum $d < \infty$.*

Proof. Agreement of the functions is immediate from the fact that the (geometric) fibres of a locally quasi-finite morphism are discrete, see Morphisms, Lemma 20.8. Boundedness follows from Morphisms, Lemmas 57.2 and 57.9. $\square$

0F36 **Lemma 74.6.** *Let $f : X \to Y$ be a separated, locally quasi-finite, and universally open morphism of schemes. Let $n_{X/Y}$ be as in Lemma 74.5. If $n_{X/Y}(y) \geq d$ for some $y \in Y$ and $d \geq 0$, then $n_{X/Y} \geq d$ in an open neighbourhood of y.*

Proof. The question is local on Y hence we may assume Y affine. Let K be an algebraic closure of the residue field $\kappa(y)$. Our assumption is that $(X_y)_K$ has $\geq d$ connected components. Then for a suitable quasi-compact open $X' \subset X$ the scheme $(X'_y)_K$ has $\geq d$ connected components; details omitted. After replacing X by X' we may assume X is quasi-compact. Then f is quasi-finite. Let $x_1, \ldots, x_n$ be the points of X lying over y. Apply Lemma 41.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

as in locus citatus. Observe that $n_{X/Y}(y) = \sum_i m_i$ in this situation; some details omitted. Since f is universally open, we see that $V_{i,j} \to U$ is open for all i, j. Hence after shrinking U we may assume $V_{i,j} \to U$ is surjective for all i, j. This proves that

$n_{U \times_Y X/U} \geq \sum_i m_i = n_{X/Y}(y) \geq d$. Since the construction of $n_{X/Y}$ is compatible with base change the proof is complete. □

0F37 **Lemma 74.7.** *Let $f : X \to Y$ be a separated, locally quasi-finite, and universally open morphism of schemes. Let $n_{X/Y}$ be as in Lemma 74.5. If $n_{X/Y}$ attains a maximum $d < \infty$, then the set*

$$Y_d = \{y \in Y \mid n_{X/Y}(y) = d\}$$

is open in Y and the morphism $f^{-1}(Y_d) \to Y_d$ is finite.

Proof. The openness of Y_d is immediate from Lemma 74.6. To prove finiteness over Y_d we redo the argument of the proof of that lemma. Namely, let $y \in Y_d$. Then there are at most d points of X lying over y. Say $x_1, \ldots, x_n$ are the points of X lying over y. Apply Lemma 41.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

as in locus citatus. Observe that $d = n_{X/Y}(y) = \sum_i m_i$ in this situation; some details omitted. Since f is universally open, we see that $V_{i,j} \to U$ is open for all i, j. Hence after shrinking U we may assume $V_{i,j} \to U$ is surjective for all i, j and we may assume U maps into W. This proves that $n_{U \times_Y X/U} \geq \sum_i m_i = d$. Since the construction of $n_{X/Y}$ is compatible with base change we know that $n_{U \times_Y X/U} = d$. This means that W has to be empty and we conclude that $U \times_Y X \to U$ is finite. By Descent, Lemma 23.23 this implies that $X \to Y$ is finite over the image of the open morphism $U \to Y$. In other words, we see that f is finite over an open neighbourhood of y as desired. □

75. Weightings

0F38 The material in this section is taken from [AGV71, Exposee XVII, 6.2.4].

Let $\pi : U \to V$ be a locally quasi-finite morphism of schemes with finite fibres. Given a function $w : U \to \mathbf{Z}$ we define a function

$$\int_\pi w : V \longrightarrow \mathbf{Z}, \quad v \longmapsto \sum\nolimits_{u \in U,\ \pi(u) = v} w(u)[\kappa(u) : \kappa(v)]_s$$

Note that the field extensions are finite (Morphisms, Lemma 20.5), $[\kappa' : \kappa]_s$ is the separable degree (Fields, Definition 14.7), and the sum is finite as the fibres of π are assumed finite. Another way to compute the value of $\int_\pi w$ at a point $v \in V$ is as follows. Choose an algebraically closed field k and a morphism $\overline{v} : \mathrm{Spec}(k) \to V$ whose image is v. Then we have

$$(\int\nolimits_\pi w)(v) = \sum\nolimits_{\overline{u} \in U_{\overline{v}}} w(\overline{u})$$

where of course $w(\overline{u})$ denotes the value of w at the image u of the point $\overline{u}$ under the morphism $U_{\overline{v}} \to U$. Note that we may view $\overline{u} \in U_{\overline{v}}$ as morphisms $\overline{u} : \mathrm{Spec}(k) \to U$ such that $\pi \circ \overline{u} = \overline{v}$. Namely, since $U \to V$ is locally quasi-finite with finite fibres, the scheme $U_{\overline{v}}$ is the spectrum of a finite dimension algebra over k and all of whose prime ideals are maximal ideals with residue field k. To see that the equality holds, note that the number of morphisms $\overline{u}$ lying over a given u is equal to $[\kappa(u) : \kappa(v)]_s$ by Fields, Lemma 14.8.

0F39 **Lemma 75.1.** *Given a cartesian square*

$$\begin{tikzcd} U & U' \arrow[l, "h"'] \arrow[d, "\pi'"] \\ V \arrow[from=1-1, "\pi"'] & V' \arrow[l, "g"] \end{tikzcd}$$

with π locally quasi-finite with finite fibres and a function $w : U \to \mathbf{Z}$ we have $(\int_\pi w) \circ g = \int_{\pi'}(w \circ h)$.

Proof. This follows immediately from the second description of $\int_\pi w$ given above. To prove it from the definition, you use that if E/F is a finite extension of fields and F'/F is another field extension, then writing $(E \otimes_F F')_{red} = \prod E'_i$ as a product of fields finite over F', we have

$$[E : F]_s = \sum [E'_i : F']_s$$

To prove this equality pick an algebraically closed field extension Ω/F' and observe that

$$\begin{aligned} [E : F]_s &= |\operatorname{Mor}_F(E, \Omega)| \\ &= |\operatorname{Mor}_{F'}(E \otimes_F F', \Omega)| \\ &= |\operatorname{Mor}_{F'}((E \otimes_F F')_{red}, \Omega)| \\ &= \sum |\operatorname{Mor}_{F'}(E'_i, \Omega)| \\ &= \sum [E'_i : F']_s \end{aligned}$$

where we have used Fields, Lemma 14.8. □

0F3A **Definition 75.2.** Let $f : X \to Y$ be a locally quasi-finite morphism. A *weighting* or a *pondération* of f is a map $w : X \to \mathbf{Z}$ such that for any diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & U \arrow[l, "h"'] \arrow[d, "\pi"] \\ Y & V \arrow[l, "g"] \end{tikzcd}$$

where $V \to Y$ is étale, $U \subset X_V$ is open, and $U \to V$ finite, the function $\int_\pi (w \circ h)$ is locally constant.

Of course taking $w = 0$ we obtain a weighting of any locally quasi-finite morphism f, albeit not a very interesting one. It will turn out that *positive* weightings, i.e., $w : X \to \mathbf{Z}_{>0}$ are the most interesting ones for various purposes.

0F3B **Lemma 75.3.** *Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}$ be a weighting. Let $f' : X' \to Y'$ be the base change of f by a morphism $Y' \to Y$. Then the composition $w' : X' \to \mathbf{Z}$ of w and the projection $X' \to X$ is a weighting of f'.*

Proof. Consider a diagram

$$\begin{tikzcd} X' \arrow[d, "f'"'] & U' \arrow[l, "h'"'] \arrow[d, "\pi'"] \\ Y' & V' \arrow[l, "g'"] \end{tikzcd}$$

as in Definition 75.2 for the morphism f'. For any $v' \in V'$ we have to show that $\int_{\pi'}(w' \circ h')$ is constant in an open neighbourhood of v'. By Lemma 75.1 (and the fact that étale morphisms are open) we may replace V' by any étale neighbourhood of v'. After replacing V' by an étale neighbourhood of v' we may assume that $U' = U'_1 \amalg \ldots \amalg U'_n$ where each U'_i has a unique point u'_i lying over v' such that $\kappa(u'_i)/\kappa(v')$ is purely inseparable, see Lemma 41.5. Clearly, it suffices to prove that $\int_{U'_i \to V'} w'|_{U'_i}$ is constant in a neighbourhood of v'. This reduces us to the case discussed in the next paragraph.

We have $v' \in V'$ and there is a unique point u' of U' lying over v' with $\kappa(u')/\kappa(v')$ purely inseparable. Denote $x \in X$ and $y \in Y$ the image of u' and v'. We can find an étale neighbourhood $(V, v) \to (Y, y)$ and an open $U \subset X_V$ such that $\pi : U \to V$ is finite and such that there is a unique point $u \in U$ lying over v which maps to $x \in X$ via the projection $h : U \to X$ such that moreover $\kappa(u)/\kappa(v)$ is purely inseparable. This is possible by the lemma used above. Consider the morphism

$$U'' = U \times_X U' \longrightarrow V \times_Y V' = V''$$

Since u and u' both map to $x \in X$ there is a point $u'' \in U''$ mapping to (u, u'). Denote $v'' \in V''$ the image of u''. After replacing V', v' by V'', v'' we may assume that the composition $V' \to Y' \to Y$ factors through a map of étale neighbourhoods $(V', v') \to (V, v)$ such that the induced morphism $X'_{V'} = X_{V'} \to X_V$ sends u' to u. Inside the base change $X'_{V'} = X_{V'}$ we have two open subschemes, namely U' and the inverse image $U_{V'}$ of $U \subset X_V$. By construction both contain a unique point lying over v', namely u' for both of them. Thus after shrinking V' we may assume these open subsets are the same; namely, $U' \setminus (U' \cap U_{V'})$ and $U_{V'} \setminus (U' \cap U_{V'})$ have a closed image in V' and these images do not contain v'. Thus $U' = U_{V'}$ and we find a cartesian diagram as in Lemma 75.1. Since $\int_\pi (w \circ h)$ is locally constant by assumption we conclude. $\square$

0GK8 **Lemma 75.4.** *Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}$ be a weighting of f. If $X' \subset X$ is open, then $w|_{X'}$ is a weighting of $f|_{X'} : X' \to Y$.*

Proof. Immediate from the definition. $\square$

0GK9 **Lemma 75.5.** *Let $f : X \to Y$ and $g : Y \to Z$ be locally quasi-finite morphisms. Let $w_f : X \to \mathbf{Z}$ be a weighting of f and let $w_g : Y \to \mathbf{Z}$ be a weighting of g. Then the function*

$$X \longrightarrow \mathbf{Z}, \quad x \longmapsto w_f(x) w_g(f(x))$$

is a weighting of $g \circ f$.

Proof. Let us set $w_{g \circ f}(x) = w_f(x) w_g(f(x))$ for $x \in X$. Consider a diagram

$$\begin{tikzcd} X \arrow[d, "g \circ f"'] & U \arrow[l] \arrow[d, "\pi"] \\ Z & W \arrow[l] \end{tikzcd}$$

where $W \to Z$ is étale, $U \subset X_W$ is open, and $U \to W$ finite. We have to show that $\int_\pi w_{g \circ f}|_U$ is locally constant. Choose a point $w \in W$. By Lemma 75.1 (and the fact that étale morphisms are open) it suffices to show that $\int_\pi w_{g \circ f}|_U$ is constant after replacing (W, w) by an étale neighbourhood. After replacing (W, w) by an étale neighbourhood we may assume $U = U_1 \amalg \ldots \amalg U_n$ where each U_i has a unique point

u_i lying over w such that $\kappa(u_i)/\kappa(w)$ is purely inseparable, see Lemma 41.5. Clearly, it suffices to show that $\int_{U_i \to W} w_{g\circ f}|_{U_i}$ is constant in an étale neighbourhood of w. This reduces us to the case discussed in the next paragraph.

We have $w \in W$ and there is a unique point $u \in U$ lying over w with $\kappa(u)/\kappa(w)$ purely inseparable. Consider the point $v = f(u) \in Y$. After replacing (W, w) by an elementary étale neighbourhood we may assume there is an open neighbourhood $V \subset Y_W$ of v such that $V \to W$ is finite, see Lemma 41.1. Then $f_W^{-1}(V) \cap U$ is an open neighbourhood of u where $f_W : X_W \to Y_W$ is the base change of f to W. Hence after Zariski shrinking W, we may assume $f_W(U) \subset V$. Thus we obtain morphisms

$$U \xrightarrow{a} V \xrightarrow{b} W$$

and $U \to V$ is finite as $V \to W$ is separated (because finite). Since w_f and w_g are weightings of f and g we see that $\int_a w_f|_U$ is locally constant on V and $\int_b w_g|_V$ is locally constant on W. Thus after shrinking W one more time we may assume these functions are constant say with values n and m. It follows immediately that $\int_\pi w_{g\circ f}|_U = \int_{b\circ a} w_{g\circ f}|_U$ is constant with value nm as desired. □

0F3C **Lemma 75.6.** *Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}$ be a weighting. If $w(x) > 0$ for all $x \in X$, then f is universally open.*

Proof. Since the property is preserved by base change, see Lemma 75.3, it suffices to prove that f is open. Since we may also replace X by any open of X, it suffices to prove that $f(X)$ is open. Let $y \in f(X)$. Choose $x \in X$ with $f(x) = y$. It suffices to prove that $f(X)$ contains an open neighbourhood of y and it suffices to do so after replacing Y by an étale neighbourhood of y. By étale localization of quasi-finite morphisms, see Section 41, we may assume there is an open neighbourhood $U \subset X$ of x such that $\pi = f|_U : U \to Y$ is finite. Then $\int_\pi w|_U$ is locally constant and has positive value at y. Hence $\pi(U)$ contains an open neighbourhood of y and the proof is complete. □

0F3D **Lemma 75.7.** *Let $f : X \to Y$ be a morphism of schemes. Assume f is locally quasi-finite, locally of finite presentation, and flat. Then there is a positive weighting $w : X \to \mathbf{Z}_{>0}$ of f given by the rule that sends $x \in X$ lying over $y \in Y$ to*

$$w(x) = \mathit{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})[\kappa(x) : \kappa(y)]_i$$

where $[\kappa' : \kappa]_i$ is the inseparable degree (Fields, Definition 14.7).

Proof. Consider a diagram as in Definition 75.2. Let $u \in U$ with images x, y, v in X, Y, V. Then we claim that

$$\text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}) = \text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})$$

and

$$[\kappa(x) : \kappa(y)]_i = [\kappa(u) : \kappa(v)]_i$$

The first equality follows as $\mathcal{O}_{X,x} \to \mathcal{O}_{U,u}$ is a flat local homomorphism such that $\mathfrak{m}_y\mathcal{O}_{U,u} = \mathfrak{m}_v\mathcal{O}_{U,u}$ and $\mathfrak{m}_x\mathcal{O}_{U,u} = \mathfrak{m}_u$ (because $\mathcal{O}_{Y,y} \to \mathcal{O}_{V,v}$ and $\mathcal{O}_{X,x} \to \mathcal{O}_{U,u}$ are unramified) and hence the equality by Algebra, Lemma 52.13. The second equality follows because $\kappa(v)/\kappa(y)$ is a finite separable extension and $\kappa(u)$ is a factor of $\kappa(x) \otimes_{\kappa(y)} \kappa(v)$ and hence the inseparable degree is unchanged. Having said this, we see that formation of the function in the lemma commutes with étale base change. This reduces the problem to the discussion of the next paragraph.

Assume that f is a finite, flat morphism of finite presentation. We have to show that $\int_f w$ is locally constant on Y. In fact, f is finite locally free (Morphisms, Lemma 48.2) and we will show that $\int_f w$ is equal to the degree of f (which is a locally constant function on Y). Namely, for $y \in Y$ we see that

$$
\begin{aligned}
(\textstyle\int_f w)(y) &= \sum\nolimits_{f(x)=y} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})[\kappa(x):\kappa(y)]_i[\kappa(x):\kappa(y)]_s \\
&= \sum\nolimits_{f(x)=y} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})[\kappa(x):\kappa(y)] \\
&= \text{length}_{\mathcal{O}_{Y,y}}((f_*\mathcal{O}_X)_y/\mathfrak{m}_y(f_*\mathcal{O}_X)_y)
\end{aligned}
$$

Last equality by Algebra, Lemma 52.12. The final number is the rank of $f_*\mathcal{O}_X$ at y as desired. $\square$

0F3E **Lemma 75.8.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

1. *f is locally quasi-finite, and*
2. *Y is geometrically unibranch and locally Noetherian.*

Then there is a weighting $w : X \to \mathbf{Z}_{\geq 0}$ given by the rule that sends $x \in X$ lying over $y \in Y$ to the "generic separable degree" of $\mathcal{O}^{sh}_{X,x}$ over $\mathcal{O}^{sh}_{Y,y}$.

Proof. It follows from Algebra, Lemma 156.3 that $\mathcal{O}^{sh}_{Y,y} \to \mathcal{O}^{sh}_{X,x}$ is finite. Since Y is geometrically unibranch there is a unique minimal prime $\mathfrak{p}$ in $\mathcal{O}^{sh}_{Y,y}$, see More on Algebra, Lemma 106.5. Write

$$(\kappa(\mathfrak{p}) \otimes_{\mathcal{O}^{sh}_{Y,y}} \mathcal{O}^{sh}_{X,x})_{red} = \prod K_i$$

as a finite product of fields. We set $w(x) = \sum [K_i : \kappa(\mathfrak{p})]_s$.

Since this definition is clearly insensitive to étale localization, in order to show that w is a weighting we reduce to showing that if f is a finite morphism, then $\int_f w$ is locally constant. Observe that the value of $\int_f w$ in a generic point η of Y is just the number of points of the geometric fibre $X_{\overline{\eta}}$ of $X \to Y$ over η. Moreover, since Y is unibranch a point y of Y is the specialization of a unique generic point η. Hence it suffices to show that $(\int_f w)(y)$ is equal to the number of points of $X_{\overline{\eta}}$. After passing to an affine neighbourhood of y we may assume $X \to Y$ is given by a finite ring map $A \to B$. Suppose $\mathcal{O}^{sh}_{Y,y}$ is constructed using a map $\kappa(y) \to k$ into an algebraically closed field k. Then

$$\mathcal{O}^{sh}_{Y,y} \otimes_A B = \prod\nolimits_{f(x)=y} \prod\nolimits_{\varphi \in \text{Mor}_{\kappa(y)}(\kappa(x),k)} \mathcal{O}^{sh}_{X,x}$$

by Algebra, Lemma 153.4 and the lemma used above. Observe that the minimal prime $\mathfrak{p}$ of $\mathcal{O}^{sh}_{Y,y}$ maps to the prime of A corresponding to η. Hence we see that the desired equality holds because the number of points of a geometric fibre is unchanged by a field extension. $\square$

76. More on weightings

0F3F We prove a few more basic properties of weightings. Although at first it appears that weightings can be very wild, it actually turns out the condition imposed in Definition 75.2 is rather strong.

0F3G **Lemma 76.1.** *Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}$ be a weighting of f. Then the level sets of the function w are locally constructible in X.*

Proof. In the proof below we will use Lemmas 75.4 and 75.3 without further mention. We will also use elementary properties of constructible subsets of schemes and topological spaces, see Topology, Section 15 and Properties, Section 2. Using this the reader sees question is local on X and Y; details omitted. Hence we may assume X and Y are affine. If we can find a surjective morphism $Y' \to Y$ of finite presentation such that the level sets of w pull back to locally constructible subsets of $X' = Y' \times_Y X$, then we conclude by Morphisms, Theorem 22.3.

Assume X and Y affine. We may choose an immersion $X \to T$ where $T \to Y$ is finite, see Lemma 43.3. By Morphisms, Lemma 48.6 after replacing Y by Y' surjective finite locally free over Y, replacing X by $Y' \times_Y X$ and T by a scheme finite locally free over Y' containing $Y' \times_Y T$ as a closed subscheme, we may assume T is finite locally free over Y, contains closed subschemes T_i mapping isomorphically to Y such that $T = \bigcup_{i=1,\ldots,n} T_i$ (set theoretically). Since $T_i \subset T$ is a constructible closed subset (as the image of a finitely presented morphism $Y \to T$ of schemes), we see that for $I \subset \{1, \ldots, n\}$ the intersection $\bigcap_{i \in I} T_i$ is a constructible closed subset of T and hence maps to a constructible closed subset of Y.

For a disjoint union decomposition $\{1, \ldots, n\} = I_1 \amalg \ldots \amalg I_r$ with nonempty parts consider the subset $Y_{I_1,\ldots,I_r} \subset Y$ consisting of points $y \in Y$ such that $T_y = \{x_1, \ldots, x_r\}$ consists of exactly r points with $x_j \in T_i \Leftrightarrow i \in I_j$. By our remarks above this is a constructible partition of Y. There exists an affine scheme Y' of finite presentation over Y such that the image of $Y' \to Y$ is exactly $Y_{I_1,\ldots,I_r}$, see Algebra, Lemma 29.4. Hence we may assume that $Y = Y_{I_1,\ldots,I_r}$ for some disjoint union decomposition $\{1, \ldots, n\} = I_1 \amalg \ldots \amalg I_r$. In this case $T = T(1) \amalg \ldots \amalg T(r)$ with $T(j) = \bigcap_{i \in I_j} T_i$ is a decomposition of T into disjoint closed (and hence open) subsets. Intersecting with the locally closed subscheme X we obtain an analogous decomposition $X = X(1) \amalg \ldots \amalg X(r)$ into open and closed parts. The morphism $X(j) \to Y$ an immersion. Since w is a weighting, it follows that $w|_{X(j)}$ is locally constant[16] and we conclude. □

0F3H **Lemma 76.2.** *Let $f : X \to Y$ be a locally quasi-finite morphism of finite presentation. Let $w : X \to \mathbf{Z}$ be a weighting of f. Then the level sets of the function $\int_f w$ are locally constructible in Y.*

Proof. By Lemma 75.1 formation of the function $\int_f w$ commutes with arbitrary base change and by Lemma 75.3 after base change we still have a weighthing. This means that if we can find $Y' \to Y$ surjective and of finite presentation, then it suffices to prove the result after base change to Y', see Morphisms, Theorem 22.3.

The question is local on Y hence we may assume Y is affine. Then X is quasi-compact and quasi-separated (as f is of finite presentation). Suppose that $X = U \cup V$ are quasi-compact open. Then we have

$$\int_f w = \int_{f|_U} w|_U + \int_{f|_V} w|_V - \int_{f|_{U \cap V}} w|_{U \cap V}$$

[16] In fact, if $f : X \to Y$ is an immersion and w is a weighting of f, then f restricts to an open map on the locus where w is nonzero.

Thus if we know the result for $w|_U$, $w|_V$, $w|_{U\cap V}$ then we know the result for w. By the induction principle (Cohomology of Schemes, Lemma 4.1) it suffices to prove the lemma when X is affine.

Assume X and Y are affine. We may choose an open immersion $X \to T$ where $T \to Y$ is finite, see Lemma 43.3. Because we may still base change with a suitable $Y' \to Y$ we can use Morphisms, Lemma 48.6 to reduce to the case where all residue field extensions induced by the morphism $T \to Y$ (and a foriori induced by $X \to Y$) are trivial. In this situation $\int_f w$ is just taking the sums of the values of w in fibres. The level sets of w are locally constructible in X (Lemma 76.1). The function w only takes a finite number of values by Properties, Lemma 2.7. Hence we conclude by Morphisms, Theorem 22.3 and some elementary arguments on sums of integers. $\square$

0F3I **Lemma 76.3.** *Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Then w is upper semi-continuous.*

Proof. Let $x \in X$ with image $y \in Y$. Choose an étale neighbourhood $(V, v) \to (Y, y)$ and an open $U \subset X_V$ such that $\pi : U \to V$ is finite and there is a unique point $u \in U$ mapping to v with $\kappa(u)/\kappa(v)$ purely inseparable. See Lemma 41.3. Then $(\int_\pi w|_U)(v) = w(u)$. It follows from Definition 75.2 that after replacing V by a neighbourhood of v we we have $w|_U(u') \leq w|_U(u) = w(x)$ for all $u' \in U$. Namely, $w|_U(u')$ occurs as a summand in the expression for $(\int_\pi w|_U)(\pi(u'))$. This proves the lemma because the étale morphism $U \to X$ is open. $\square$

0F3J **Lemma 76.4.** *Let $f : X \to Y$ be a separated, locally quasi-finite morphism with finite fibres. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Then $\int_f w$ is lower semi-continuous.*

Proof. Let $y \in Y$. Let $x_1, \ldots, x_r \in X$ be the points lying over y. Apply Lemma 41.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

as in locus citatus. Observe that $(\int_f w)(y) = \sum w(v_{i,j})$ where $w(v_{i,j}) = w(x_i)$. Since $\int_{V_{i,j}\to U} w|_{V_{i,j}}$ is locally constant by definition, we may after shrinking U assume these functions are constant with value $w(v_{i,j})$. We conclude that

$$\int\nolimits_{U\times_Y X\to U} w|_{U\times_Y X} = \int\nolimits_{W\to U} w|_W + \sum \int\nolimits_{V_{i,j}\to U} w|_{V_{i,j}} = \int\nolimits_{W\to U} w|_W + (\int\nolimits_f w)(y)$$

This is $\geq (\int_f w)(y)$ and we conclude because $U \to Y$ is open and formation of the integral commutes with base change (Lemma 75.1). $\square$

0F3K **Lemma 76.5.** *Let $f : X \to Y$ be a locally quasi-finite morphism with X quasi-compact. Let $w : X \to \mathbf{Z}$ be a weighting of f. Then $\int_f w$ attains its maximum.*

Proof. It follows from Lemma 76.1 and Properties, Lemma 2.7 that w only takes a finite number of values on X. It follows from Morphisms, Lemma 57.9 that $X \to Y$ has bounded geometric fibres. This shows that $\int_f w$ is bounded. $\square$

0F3L **Lemma 76.6.** *Let $f : X \to Y$ be a separated, locally quasi-finite morphism. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Assume $\int_w f$ attains its maximum d and let $Y_d \subset Y$ be the open set of points y with $(\int_f w)(y) = d$. Then the morphism $f^{-1}(Y_d) \to Y_d$ is finite.*

Proof. Observe that Y_d is open by Lemma 76.4. Let $y \in Y_d$. Say $x_1, \ldots, x_n$ are the points of X lying over y. Apply Lemma 41.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

as in locus citatus. Observe that $d = \sum w(v_{i,j})$ where $w(v_{i,j}) = w(x_i)$. Since $\int_{V_{i,j} \to U} w|_{V_{i,j}}$ is locally constant by definition, we may after shrinking U assume these functions are constant with value $w(v_{i,j})$. We conclude that

$$\int_{U \times_Y X \to U} w|_{U \times_Y X} = \int_{W \to U} w|_W + \sum \int_{V_{i,j} \to U} w|_{V_{i,j}} = \int_{W \to U} w|_W + (\int_f w)(y)$$

This is $\geq (\int_f w)(y) = d$ and we conclude that W must be the emptyset. Thus $U \times_Y X \to U$ is finite. By Descent, Lemma 23.23 this implies that $X \to Y$ is finite over the image of the open morphism $U \to Y$. In other words, we see that f is finite over an open neighbourhood of y as desired. $\square$

0F3M **Lemma 76.7.** *Let $A \to B$ be a ring map which is finite and of finite presentation. There exists a finitely presented ring map $A \to A_{univ}$ and an idempotent $e_{univ} \in B \otimes_A A_{univ}$ such that for any ring map $A \to A'$ and idempotent $e \in B \otimes_A A'$ there is a ring map $A_{univ} \to A'$ mapping e_{univ} to e.*

Proof. Choose $b_1, \ldots, b_n \in B$ generating B as an A-module. For each i choose a monic $P_i \in A[x]$ such that $P_i(b_i) = 0$ in B, see Algebra, Lemma 36.3. Thus B is a quotient of the finite free A-algebra $B' = A[x_1, \ldots, x_n]/(P_1(x_1), \ldots, P_n(x_n))$. Let $J \subset B'$ be the kernel of the surjection $B' \to B$. Then $J = (f_1, \ldots, f_m)$ is finitely generated as B is a finitely generated A-algebra, see Algebra, Lemma 6.2. Choose an A-basis $b'_1, \ldots, b'_N$ of B'. Consider the algebra

$$A_{univ} = A[z_1, \ldots, z_N, y_1, \ldots, y_m]/I$$

where I is the ideal generated by the coefficients in $A[z_1, \ldots, z_n, y_1, \ldots, y_m]$ of the basis elements $b'_1, \ldots, b'_N$ of the expression

$$(\sum z_j b'_j)^2 - \sum z_j b'_j + \sum y_k f_k$$

in $B'[z_1, \ldots, z_N, y_1, \ldots, y_m]$. By construction the element $\sum z_j b'_j$ maps to an idempotent e_{univ} in the algebra $B \otimes_A A_{univ}$. Moreover, if $e \in B \otimes_A A'$ is an idempotent, then we can lift e to an element of the form $\sum b'_j \otimes a'_j$ in $B' \otimes_A A'$ and we can find $a''_k \in A'$ such that

$$(\sum b'_j \otimes a'_j)^2 - \sum b'_j \otimes a'_j + \sum f_k \otimes a''_k$$

is zero in $B' \otimes_A A'$. Hence we get an A-algebra map $A_{univ} \to A$ sending z_j to a'_j and y_k to a''_k mapping e_{univ} to e. This finishes the proof. $\square$

0F3N **Lemma 76.8.** *Let $X \to Y$ be a morphism of affine schemes which is quasi-finite and of finite presentation. There exists a morphism $Y_{univ} \to Y$ of finite presentation and an open subscheme $U_{univ} \subset Y_{univ} \times_Y X$ such that $U_{univ} \to Y_{univ}$ is finite with the following property: given any morphism $Y' \to Y$ of affine schemes and an open subscheme $U' \subset Y' \times_Y X$ such that $U' \to Y'$ is finite, there exists a morphism $Y' \to Y_{univ}$ such that the inverse image of U_{univ} is U'.*

Proof. Recall that a finite type morphism is quasi-finite if and only if it has relative dimension 0, see Morphisms, Lemma 29.5. By Lemma 34.9 applied with $d = 0$ we reduce to the case where X and Y are Noetherian. We may choose an open immersion $X \to X'$ such that $X' \to Y$ is finite, see Algebra, Lemma 123.14. Note that if we have $Y' \to Y$ and U' as in (2), then

$$U' \to Y' \times_Y X \to Y' \times_Y X'$$

is open immersion between schemes finite over Y' and hence is closed as well. We conclude that U' corresponds to an idempotent in

$$\Gamma(Y', \mathcal{O}_{Y'}) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(X', \mathcal{O}_{X'})$$

whose corresponding open and closed subset is contained in the open $Y' \times_Y X$. Let $Y'_{univ} \to Y$ and idempotent

$$e'_{univ} \in \Gamma(Y_{univ}, \mathcal{O}_{Y_{univ}}) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(X', \mathcal{O}_{X'})$$

be the pair constructed in Lemma 76.7 for the ring map $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X', \mathcal{O}_{X'})$ (here we use that Y is Noetherian to see that X' is of finite presentation over Y). Let $U'_{univ} \subset Y'_{univ} \times_Y X'$ be the corresponding open and closed subscheme. Then we see that

$$U'_{univ} \setminus Y'_{univ} \times_Y X$$

is a closed subset of U'_{univ} and hence has closed image $T \subset Y'_{univ}$. If we set $Y_{univ} = Y'_{univ} \setminus T$ and U_{univ} the restriction of U'_{univ} to $Y_{univ} \times_Y X$, then we see that the lemma is true. $\square$

0F3P **Lemma 76.9.** *Let $Y = \lim Y_i$ be a directed limit of affine schemes. Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of affine schemes which is quasi-finite and of finite presentation. Let $f : X \to Y$ and $f_i : X_i \to Y_i$ for $i \geq 0$ be the base changes of f_0. If $w : X \to \mathbf{Z}$ is a weighting of f, then for sufficiently large i there exists a weighting $w_i : X_i \to \mathbf{Z}$ of f_i whose pullback to X is w.*

Proof. By Lemma 76.1 the level sets of w are constructible subsets E_k of X. This implies the function w only takes a finite number of values by Properties, Lemma 2.7. Thus there exists an i such that E_k descends to a construcible subset $E_{i,k}$ in X_i for all k; moreover, we may assume $X_i = \coprod E_{i,k}$. This follows as the topological space of X is the limit in the category of topological spaces of the spectral spaces X_i along a directed system with spectral transition maps. See Limits, Section 4 and Topology, Section 24. We define $w_i : X_i \to \mathbf{Z}$ such that its level sets are the constructible sets $E_{i,k}$.

Choose $Y_{i,univ} \to Y_i$ and $U_{i,univ} \subset Y_{i,univ} \times_{Y_i} X_i$ as in Lemma 76.8. By the universal property of the construction, in order to show that w_i is a weighting, it would suffice to show that

$$\tau_i = \int_{U_{i,univ} \to Y_{i,univ}} w_i|_{U_{i,univ}}$$

is locally constant on $Y_{i,univ}$. By Lemma 76.2 this function has constructible level sets but it may not (yet) be locally constant. Set $Y_{univ} = Y_{i,univ} \times_{Y_i} Y$ and let $U_{univ} \subset Y_{univ} \times_Y X$ be the inverse image of $U_{i,univ}$. Then, since the pullback of w to $Y_{univ} \times_Y X$ is a weighting for $Y_{univ} \times_Y X \to Y_{univ}$ (Lemma 75.3) we do have that

$$\tau = \int_{U_{univ} \to Y_{univ}} w_i|_{U_{univ}}$$

is locally constant on Y_{univ}. Thus the level sets of τ are open and closed. Finally, we have $Y_{univ} = \lim_{i' \geq i} Y_{i',univ}$ and the level sets of τ are the inverse limits of the level sets of $\tau_{i'}$ (similarly defined). Hence the references above imply that for sufficiently large i' the level sets of $\tau_{i'}$ are open as well. For such an index i' we conclude that $w_{i'}$ is a weighting of $f_{i'}$ as desired. $\square$

77. Weightings and affine stratification numbers

0F3Q In this section we give a bound for the affine stratification number of a scheme which has a certain kind of cover by an affine scheme.

0F3R **Lemma 77.1.** *Let $f : X \to Y$ be a morphism of affine schemes which is quasi-finite and of finite presentation. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Let $d < \infty$ be the maximum value of $\int_f w$. The open*

$$Y_d = \{y \in Y \mid (\textstyle\int_f w)(y) = d\}$$

of Y is affine.

Proof. Observe that $\int_f w$ attains its maximum by Lemma 76.5. The set Y_d is open by Lemma 76.4. Thus the statement of the lemma makes sense.

Reduction to the Noetherian case; please skip this paragraph. Recall that a finite type morphism is quasi-finite if and only if it has relative dimension 0, see Morphisms, Lemma 29.5. By Lemma 34.9 applied with $d = 0$ we can find a quasi-finite morphism $f_0 : X_0 \to Y_0$ of affine Noetherian schemes and a morphism $Y \to Y_0$ such that f is the base change of f_0. Then we can write $Y = \lim Y_i$ as a directed limit of affine schemes of finite type over Y_0, see Algebra, Lemma 127.2. By Lemma 76.9 we can find an i such that our weighting w descends to a weighting w_i of the base change $f_i : X_i \to Y_i$ of f_0. Now if the lemma holds for f_i, w_i, then it implies the lemma for f as formation of $\int_f w$ commutes with base change, see Lemma 75.1.

Assume X and Y Noetherian. Let $X' \to Y'$ be the base change of f by a morphism $g : Y' \to Y$. The formation of $\int_f w$ and hence the open Y_d commute with base change. If g is finite and surjective, then $Y'_d \to Y_d$ is finite and surjective. In this case proving that Y_d is affine is equivalent to showing that Y'_d is affine, see Cohomology of Schemes, Lemma 13.3.

We may choose an immersion $X \to T$ with T finite over Y, see Lemma 43.3. We are going to apply Morphisms, Lemma 48.6 to the finite morphism $T \to Y$. This lemma tells us that there is a finite surjective morphism $Y' \to Y$ such that $Y' \times_Y T$ is a closed subscheme of a scheme T' finite over Y' which has a special form. By the discussion in the first paragraph, we may replace Y by Y', T by T', and X by $Y' \times_Y X$. Thus we may assume there is an immersion $X \to T$ (not necessarily open or closed) and closed subschemes $T_i \subset T$, $i = 1, \ldots, n$ where

1. $T \to Y$ is finite (and locally free),
2. $T_i \to Y$ is an isomorphism, and
3. $T = \bigcup_{i=1,\ldots,n} T_i$ set theoretically.

Let $Y' = \coprod Y_k$ be the disjoint union of the irreducible components of Y (viewed as integral closed subschemes of Y). Then we may base change once more by $Y' \to Y$; here we are using that Y is Noetherian. Thus we may in addition assume Y is integral and Noetherian.

We also may and do assume that $T_i \neq T_j$ if $i \neq j$ by removing repeats. Since Y and hence all T_i are integral, this means that if T_i and T_j intersect, then they intersect in a closed subset which maps to a proper closed subset of Y.

Observe that $V_i = X \cap T_i$ is a locally closed subset which is in addition a closed subscheme of X hence affine. Let $\eta \in Y$ and $\eta_i \in T_i$ be the generic points. If $\eta \notin Y_d$, then $Y_d = \emptyset$ and we're done. Assume $\eta \in Y_d$. Denote $I \in \{1, \ldots, n\}$ the subset of indices i such that $\eta_i \in V_i$. For $i \in I$ the locally closed subset $V_i \subset T_i$ contains the generic point of the irreducible space T_i and hence is open. On the other hand, since f is open (Lemma 75.6), for any $x \in X$ we can find an $i \in I$ and a specialization $\eta_i \leadsto x$. It follows that $x \in T_i$ and hence $x \in V_i$. In other words, we see that $X = \bigcup_{i \in I} V_i$ set theoretically. We claim that $Y_d = \bigcap_{i \in I} \text{Im}(V_i \to Y)$; this will finish the proof as the intersection of affine opens $\text{Im}(V_i \to Y)$ of Y is affine.

For $y \in Y$ let $f^{-1}(\{y\}) = \{x_1, \ldots, x_r\}$ in X. For each $i \in I$ there is at most one $j(i) \in \{1, \ldots, x_r\}$ such that $\eta_i \leadsto x_{j(i)}$. In fact, $j(i)$ exists and is equal to j if and only if $x_j \in V_i$. If $i \in I$ is such that $j = j(i)$ exists, then $V_i \to Y$ is an isomorphism in a neighbourhood of $x_j \mapsto y$. Hence $\bigcup_{i \in I,\ j(i) = j} V_i \to Y$ is finite after replacing source and target by neighbourhoods of $x_j \mapsto y$. Thus the definition of a weighting tells us that $w(x_j) = \sum_{i \in I,\ j(i) = j} w(\eta_i)$. Thus we see that

$$(\textstyle\int_f w)(\eta) = \sum_{i \in I} w(\eta_i) \geq \sum_{j(i) \text{ exists}} w(\eta_i) = \sum_j w(x_j) = (\int_f w)(y)$$

Thus equality holds if and only if y is contained in $\bigcap_{i \in I} \text{Im}(V_i \to Y)$ which is what we wanted to show. $\square$

0F3S **Proposition 77.2.** *Let $f : X \to Y$ be a surjective quasi-finite morphism of schemes. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Assume X affine and Y separated*[17]*. Then the affine stratification number of Y is at most the number of distinct values of $\int_f w$.*

Proof. Note that since Y is separated, the morphism $X \to Y$ is affine (Morphisms, Lemma 11.11). The function $\int_f w$ attains its maximum d by Lemma 76.5. We will use induction on d. Consider the open subscheme $Y_d = \{y \in Y \mid (\int_f w)(y) = d\}$ of Y and recall that $f^{-1}(Y_d) \to Y_d$ is finite, see Lemma 76.6. By Lemma 77.1 for every affine open $W \subset Y$ we have that $Y_d \cap W$ is affine (this uses that $W \times_Y X$ is affine, being affine over X). Hence $Y_d \to Y$ is an affine morphism of schemes. We conclude that $f^{-1}(Y_d) = Y_d \times_Y X$ is an affine scheme being affine over X. Then $f^{-1}(Y_d) \to Y_d$ is surjective and hence Y_d is affine by Limits, Lemma 11.1. Set $X' = X \setminus f^{-1}(Y_d)$ and $Y' = Y \setminus Y_d$ viewed as closed subschemes of X and Y. Since X' is closed in X it is affine. Since Y' is closed in Y it is separated. The morphism $f' : X' \to Y'$ is surjective and w induces a weighting w' of f', see Lemma 75.3. By induction Y' has an affine stratification of length $\leq$ the number of distinct values of $\int_{f'} w'$ and the proof is complete. $\square$

78. Completely decomposed morphisms

0GTH Nishnevich studied the notion of a completely decomposed family of étale morphisms, in order to define what is now called the Nishnevich topology, see for example [Nis89].

[17]It suffices if the diagonal of Y is affine.

0GTI **Definition 78.1.** A morphism $f : X \to Y$ of schemes is said to be *completely decomposed*[18] if for all points $y \in Y$ there is a point $x \in X$ with $f(x) = y$ such that the field extension $\kappa(x)/\kappa(y)$ is trivial. A family of morphisms $\{f_i : X_i \to Y\}_{i \in I}$ of schemes with fixed target is said to be *completely decomposed* if $\coprod f_i : \coprod Y_i \to X$ is completely decomposed.

We start with some basic lemmas.

0GTJ **Lemma 78.2.** *The composition of two completely decomposed morphisms of schemes is completely decomposed. If $\{f_i : X_i \to Y\}_{i \in I}$ is completely decomposed and for each i we have a family $\{X_{ij} \to X_i\}_{j \in J_i}$ which is completely decomposed, then the family $\{X_{ij} \to Y\}_{i \in I, j \in J_i}$ is completely decomposed.*

Proof. Omitted. □

0GTK **Lemma 78.3.** *The base change of a completely decomposed morphism of schemes is completely decomposed. If $\{f_i : X_i \to Y\}_{i \in I}$ is completely decomposed and $Y' \to Y$ is a morphism of schemes, then $\{X_i \times_Y Y' \to Y'\}_{i \in I}$ is completely decomposed.*

Proof. Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of schemes. Let $y' \in Y'$ be a point with image $y = g(y')$ in Y. If $x \in X$ is a point such that $f(x) = y$ and $\kappa(x) = \kappa(y)$, then there exists a unique point $x' \in X' = X \times_Y Y'$ which maps to y' in Y' and to x in X and moreover $\kappa(x') = \kappa(y')$, see Schemes, Lemma 17.5. From this fact the lemma follows easily; we omit the details. □

0GTL **Lemma 78.4.** *Let $f : X \to Y$ be a morphism of schemes. Assume f is completely decomposed, f is locally of finite presentation, and Y is quasi-compact and quasi-separated. Then there exist $n \geq 0$ and morphisms $Z_i \to Y$, $i = 1, \ldots, n$ with the following properties* [EHIK21, Lemma 2.1.2]

1. *$\coprod Z_i \to Y$ is surjective,*
2. *$Z_i \to Y$ is an immersion for all i,*
3. *$Z_i \to Y$ is of finite presentation for all i, and*
4. *the base change $X \times_Y Z_i \to Z_i$ has a section for all i.*

Proof. Let $y \in Y$. By assumption there is a morphism $\sigma : \mathrm{Spec}(\kappa(y)) \to X$ over Y. We can write $\mathrm{Spec}(\kappa(y))$ as a directed limit of affine schemes Z over Y such that $Z \to Y$ is an immersion of finite presentation. Namely, choose an affine open $y \in \mathrm{Spec}(A) \subset Y$ and say y corresponds to the prime ideal $\mathfrak{p}$ of A. Then $\kappa(\mathfrak{p})$ is the filtered colimit of the rings $(A/I)_f$ where $I \subset \mathfrak{p}$ is a finitely generated ideal and $f \in A$, $f \notin \mathfrak{p}$. The morphisms $Z = \mathrm{Spec}((A/I)_f) \to Y$ are immersions of finite presentation; quasi-compactness of $Z \to Y$ follows as Y is quasi-separated, see Schemes, Lemma 21.14. By Limits, Proposition 6.1 for some such Z there is a morphism $\sigma' : Z \to X$ over Y agreeing with σ on the spectrum of $\kappa(\mathfrak{p})$. Since σ' is a morphism over Y, we obtain a section of the projection $X \times_Y Z \to Z$

We conclude that Y is the union of the images of immersions $Z \to Y$ of finite presentation such that $X \times_Y Z \to Z$ has a section. Since the image of $Z \to Y$ is constructible (Morphisms, Lemma 22.2) and since Y is compact in the constructible topology (Properties, Lemma 2.4 and Topology, Lemma 23.2), we see that a finite number of these suffice. □

[18] This may be nonstandard terminology.

0GTM **Lemma 78.5.** *Let $S = \lim_{\lambda \in \Lambda} S_\lambda$ be a limit of a directed system of schemes with affine transition morphisms. Let $0 \in \Lambda$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. For $\lambda \geq 0$ let $f_\lambda : X_\lambda \to Y_\lambda$ be the base change of f_0 to S_λ and let $f : X \to Y$ be the base change of f_0 to S. If*

(1) *f is completely decomposed,*
(2) *Y_0 is quasi-compact and quasi-separated, and*
(3) *f_0 is locally of finite presentation,*

then there exists an $\lambda \geq 0$ such that f_λ is completely decomposed.

Proof. Since Y_0 is quasi-compact and quasi-separated, the scheme Y, which is affine over Y_0, is quasi-compact and quasi-separated. Choose $n \geq 0$ and $Z_i \to Y$, $i = 1, \ldots, n$ as in Lemma 78.4. Denote $\sigma_i : Z_i \to X$ morphisms over Y which exist by our choice of Z_i. After increasing $0 \in \Lambda$ we may assume there exist morphisms $Z_{i,0} \to Y_0$ of finite presentation whose base changes to S are the morphisms $Z_i \to Y$, see Limits, Lemma 10.1. By Limits, Lemma 8.13 we may assume, after possibly increasing 0, that $Z_{i,0} \to Y_0$ is an immersion. Since $\coprod Z_i \to Y$ is surjective, we may assume, after possibly increasing 0, that $\coprod Z_{i,0} \to Y_0$ is surjective, see Limits, Lemma 8.15. Observe that $Z_i = \lim_{\lambda \geq 0} Z_{i,\lambda}$ where $Z_{i,\lambda} = Y_\lambda \times_{Y_0} Z_{i,0}$. Let us view the compositions

$$Z_i \xrightarrow{\sigma_i} X \to X_0$$

as morphisms over Y_0. Since f_0 is locally of finite presentation by Limits, Proposition 6.1 we can find a $\lambda \geq 0$ such that there exist morphisms $\sigma'_{i,\lambda} : Z_{i,\lambda} \to X_0$ over Y_0 whose precomposition with $Z_i \to Z_{i,\lambda}$ are the displayed arrows. Of course, then $\sigma'_{i,\lambda}$ determines a morphism $\sigma_{i,\lambda} : Z_{i,\lambda} \to X_\lambda = X_0 \times_{Y_0} Y_\lambda$ over Y_λ. Since $\coprod Z_{i,\lambda} \to Y_\lambda$ is surjective we conclude that $X_\lambda \to Y_\lambda$ is completely decomposed. $\square$

79. Families of ample invertible modules

0GTN We continue the discussion from Morphisms, Section 12.

0GTP **Lemma 79.1.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *Y has an ample family of invertible modules,*
(2) *there exists an f-ample invertible module on X.*

Then X has an ample family of invertible modules.

Proof. Let $\mathcal{L}$ be an f-ample invertible module on X. This in particular implies that f is quasi-compact, see Morphisms, Definition 37.1. Since Y is quasi-compact by Morphisms, Definition 12.1 we see that X is quasi-compact (and hence X itself satisfies the first condition of Morphisms, Definition 12.1). Let $x \in X$ with image $y \in Y$. By assumption (2) we can find an invertible $\mathcal{O}_Y$-module $\mathcal{N}$ and a section $t \in \Gamma(Y, \mathcal{N})$ such that the locus Y_t where t does not vanish is affine. Then $\mathcal{L}$ is ample over $f^{-1}(Y_t) = X_{f^*t}$ and hence we can find a section $s \in \Gamma(X_{f^*t}, \mathcal{L})$ such that $(X_{f^*t})_s$ is affine and contains x. By Properties, Lemma 17.2 for some $n \geq 0$ the product $(f^*t)^n s$ extends to a section $s' \in \Gamma(X, f^*\mathcal{N}^{\otimes n} \otimes \mathcal{L})$. Then finally the section $s'' = f^*ts'$ of $f^*\mathcal{N}^{\otimes n+1} \otimes \mathcal{L}$ vanishes at every point of $X \setminus X_{f^*t}$ hence we see that $X_{s''} = (X_{f^*t})_s$ is affine as desired. $\square$

0GTQ **Lemma 79.2.** *Let $f : X \to Y$ be an affine or quasi-affine morphism of schemes. If Y has an ample family of invertible modules, so does X.*

Proof. By Morphisms, Lemma 37.6 this is a special case of Lemma 79.1. $\square$

80. Blowing up and ample families of invertible modules

0GTR We prove a result from [Gro10].

0GTS **Lemma 80.1.** *Let X be a scheme. Suppose given effective Cartier divisors $D_1, \ldots, D_m$ on X and invertible modules $\mathcal{L}_1, \ldots, \mathcal{L}_m$ such that $\bigcap D_i = \emptyset$ and $\mathcal{L}_i|_{X \setminus D_i}$ is ample. Then X has an ample family of invertible modules.*

Proof. Let $x \in X$. Choose an index $i \in \{1, \ldots, m\}$ such that $x \not\in D_i$. Set $U_i = X \setminus D_i$. Since $\mathcal{L}_i|_{U_i}$ we can find an $n \geq 1$ and a section $s \in \Gamma(U_i, \mathcal{L}_i^{\otimes n})$ such that the locus $(U_i)_s$ where s doesn't vanish is affine (Properties, Definition 26.1). Since U_i is the locus where the canonical section $1 \in \mathcal{O}_X(D_i)$ doesn't vanish, we see from Properties, Lemma 17.2 there exists an $N \geq 0$ such that s extends to a section

$$s' \in \Gamma(X, \mathcal{L}_i^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{O}_X(ND_i))$$

After replacing N by $N+1$ we see that s' vanishes at every point of D_i and hence that $X_{s'} = (U_i)_s$ is affine. This proves that X has an ample family of invertible modules, see Morphisms, Definition 12.1. $\square$

0GTT **Lemma 80.2.** *Let X be a quasi-compact and quasi-separated scheme with finitely many irreducible components. There exists a quasi-compact dense open $U \subset X$ and a U-admissible blowing up $X' \to X$ such that the scheme X' has an ample family of invertible modules.* [Gro10, Proposition 1.3.1]

Proof. Let $\eta_1, \ldots, \eta_n \in X$ be the generic points of the irreducible components of X. By Properties, Lemma 29.4 and the fact that X is quasi-compact we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ such that each U_i contains $\eta_1, \ldots, \eta_n$. In particular the quasi-compact open subset $U = U_1 \cap \ldots \cap U_m$ is dense in X. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be a finite type quasi-coherent ideal sheaf such that $U_i = X \setminus Z_i$ where $Z_i = V(\mathcal{I}_i)$, see Properties, Lemma 24.1. Let

$$f : X' \longrightarrow X$$

be the blowing up of X in the ideal sheaf $\mathcal{I} = \mathcal{I}_1 \cdots \mathcal{I}_m$. Note that f is a U-admissible blowing up as $V(\mathcal{I})$ is (set theoretically) the union of the Z_i which are disjoint from U. Also, f is a projective morphism and $\mathcal{O}_{X'}(1)$ is f-relatively ample, see Divisors, Lemma 32.13. By Divisors, Lemma 32.12 for each i the morphism f' factors as $X' \to X'_i \to X$ where $X'_i \to X$ is the blowing up in $\mathcal{I}_i$ and $X' \to X'_i$ is another blowing up (namely in the pullback of the products of the ideals $\mathcal{I}_j$ omitting $\mathcal{I}_i$). It follows from this that $D_i = f^{-1}(Z_i) \subset X'$ is an effective Cartier divisor, see Divisors, Lemmas 32.11 and 32.4. We have $X' \setminus D_i = f^{-1}(U_i)$. As $\mathcal{O}_{X'}(1)$ is f-ample, the restriction of $\mathcal{O}_{X'}(1)$ to $X' \setminus D_i$ is ample. It follows from Lemma 80.1 that X' has an ample family of invertible modules. $\square$

0GTU **Proposition 80.3.** *Let X be a quasi-compact and quasi-separated scheme. There exists a morphism $f : Y \to X$ which is of finite presentation, proper, and completely decomposed (Definition 78.1) such that the scheme Y has an ample family of invertible modules.*

Proof. By Limits, Proposition 5.4 there exists an affine morphism $X \to X_0$ where X_0 is a scheme of finite type over $\mathbf{Z}$. Below we produce a morphism $Y_0 \to X_0$ with all the desired properties. Then setting $Y = X \times_{X_0} Y_0$ and f equal to the projection $f : Y \to X$ we conclude. To see this observe that f is of finite presentation

(Morphisms, Lemma 21.4), f is proper (Morphisms, Lemma 41.5), f is completely decomposed (Lemma 78.3). Finally, since $Y \to Y_0$ is affine (as the base change of $X \to X_0$) we see that Y has an ample family of invertible modules by Lemma 79.2. This reduces us to the case discussed in the next paragraph.

Assume X is of finite type over $\mathbf{Z}$. In particular $\dim(X) < \infty$. We will argue by induction on $\dim(X)$. If $\dim(X) = 0$, then X is affine and has the resolution property. In general, there exists a dense open $U \subset X$ and a U-admissible blowing up $X' \to X$ such that X' has an ample family of invertible modules, see Lemma 80.2. Since $f : X' \to X$ is an isomorphism over U we see that every point of U lifts to a point of X' with the same residue field. Let $Z = X \setminus U$ with the reduced induced scheme structure. Then $\dim(Z) < \dim(X)$ as U is dense in X (see above). By induction we find a proper, completely decomposed morphism $W \to Z$ such that W has an ample family of invertible modules. Then it follows that $Y = X' \amalg W \to X$ is the desired morphism. $\square$

81. The extensive criterion for closed immersions

0H2N In this section, we give a criterion for a morphism of schemes to be a closed immersion.

0H2P **Lemma 81.1.** *A morphism $f : X \to Y$ of affine schemes is a closed immersion if and only if for every injective ring map $A \to B$ and commutative square*

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

there exists a lift $\mathrm{Spec}(A) \to X$ *making the two triangles commute.*

Proof. Let the morphism f be given by the ring map $\phi : R \to S$. Then f is a closed immersion if and only if ϕ is surjective.

First, we assume that ϕ is surjective. Let $\psi : A \to B$ be an injective ring map, and suppose we are given a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "\phi"] & A \arrow[d, "\psi"] \\ S \arrow[r, "\beta"] \arrow[ur, dotted] & B \end{tikzcd}$$

Then we define a lift $S \to A$ by $s \mapsto \alpha(r)$, where $r \in R$ is such that $\phi(r) = s$. This is well-defined because ψ is injective and the square commutes. Since taking the ring spectrum defines an anti-equivalence between commutative rings and affine schemes, the desired lifting property for f holds.

Next, we assume that ϕ has lifts against all injective ring maps $\psi : A \to B$. Note that $\phi(R)$ is a subring of S, so we obtain a commutative square

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\phi"] & \phi(R) \arrow[d] \\ S \arrow[r, equal] \arrow[ur, dotted] & S \end{tikzcd}$$

in which a lift $S \to \phi(R)$ exists. Hence, the inclusion $\phi(R) \to S$ must be an isomorphism, which shows that ϕ is surjective, and we win. $\square$

0H2Q **Lemma 81.2.** *Let X be a scheme. If the canonical morphism $X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ of Schemes, Lemma 6.4 has a retraction, then X is an affine scheme.*

Proof. Write $S = \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ and $f : X \to S$ the morphism given in the lemma. Let $s : S \to X$ be a retraction; so $\mathrm{id}_X = sf$. Then $fsf = \mathrm{id}_S f$. Since f induces an isomorphism $\Gamma(S, \mathcal{O}_S) \to \Gamma(X, \mathcal{O}_X)$ this means that fs and id_S induce the same map on $\Gamma(S, \mathcal{O}_S)$. Whence $fs = \mathrm{id}_S$ as S is affine. Hence f is an isomorphism and X is an affine scheme, as was to be shown. $\square$

0H2R **Lemma 81.3.** *Let X be a scheme. Let $f : X \to S = \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ be the canonical morphism of Schemes, Lemma 6.4. The largest quasi-coherent $\mathcal{O}_S$-module contained in the kernel of $f^\sharp : \mathcal{O}_S \to f_*\mathcal{O}_X$ is zero. If X is quasi-compact, then $f^\sharp$ is injective. In particular, if X is quasi-compact, then f is a dominant morphism.*

Proof. Let $M \subset \Gamma(S, \mathcal{O}_S)$ be the submodule corresponding to the largest quasi-coherent $\mathcal{O}_S$-module contained in the kernel of $f^\sharp$. Then any element $a \in M$ is mapped to zero by $f^\sharp$. However, $f^\sharp(a)$ is the element of

$$\Gamma(S, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = \Gamma(S, \mathcal{O}_S)$$

corresponding to a itself! Thus $a = 0$. Hence $M = 0$ which proves the first assertion. Note that this is equivalent to the morphism $f : X \to S$ being scheme-theoretically surjective.

If X is quasi-compact, then $\mathrm{Ker}(f^\sharp)$ is quasi-coherent by Morphisms, Lemma 6.3. Hence $\mathrm{Ker}(f^\sharp) = 0$ and $f^\sharp$ is injective. In this case, f is a dominant morphism by part (4) of Morphisms, Lemma 6.3. $\square$

0H2S **Lemma 81.4.** *Let $f : X \to Y$ be a quasi-compact morphism of schemes. Then f is a closed immersion if and only if for every injective ring map $A \to B$ and commutative square*

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

there exists a lift $\mathrm{Spec}\, A \to X$ making the diagram commute.

Proof. Assume that f is a closed immersion. Let $A \to B$ be an injective ring map and consider a commutative square

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Then $\mathrm{Spec}(A) \times_Y X \to \mathrm{Spec}(A)$ is a closed immersion and hence we get an ideal $I \subset A$ and a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(B) \arrow[r] \arrow[d] & \mathrm{Spec}(A/I) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r] \arrow[ru, dotted] & \mathrm{Spec}(A) \arrow[r] & Y
\end{tikzcd}$$

We obtain a lift by Lemma 81.1.

Assume that f has the lifting property stated in the lemma. To prove that f is a closed immersion is local on Y, hence we may and do assume Y is affine. In particular, Y is quasi-compact and therefore X is quasi-compact. Hence there exists a finite affine open covering $X = U_1 \cup \ldots \cup U_n$. The source of the morphism

$$\pi : U = \coprod U_i \longrightarrow X$$

is affine and the induced ring map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_U)$ is injective. By assumption, there exists a lift in the diagram

$$\begin{tikzcd}
U \arrow[rr, "\pi"] \arrow[d] & & X \arrow[d, "f"] \\
\mathrm{Spec}(\Gamma(X, \mathcal{O}_X)) \arrow[rr, "f'"] \arrow[rru, dotted, "h"] & & Y
\end{tikzcd}$$

where f' is the morphism of affine schemes corresponding to the ring map $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$. It follows from the fact that π is an epimorphism that the morphism h is a retraction of the canonical morphism $X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$; details omitted. Hence X is affine by Lemma 81.2. By Lemma 81.1 we conclude that f is a closed immersion. $\square$

82. Other chapters

Preliminaries

Schemes

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