

QUOT AND HILBERT SPACES

05X4

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1. Introduction

05X5 As initially conceived, the purpose of this chapter was to write about Quot and Hilbert functors and to prove that these are algebraic spaces provided certain technical conditions are satisfied. This material, in the setting of schemes, is covered in Grothendieck’s lectures in the séminair Bourbaki, see [Gro95a], [Gro95b], [Gro95e], [Gro95f], [Gro95c], and [Gro95d]. For projective schemes the Quot and Hilbert schemes live inside Grassmannians of spaces of sections of suitable very ample invertible sheaves, and this provides a method of construction for these schemes. Our approach is different: we use Artin’s axioms to prove Quot and Hilb are algebraic spaces.

Upon further consideration, it turned out to be more convenient for the development of theory in the Stacks project, to start the discussion with the stack $\mathcal{Coh}_{X/B}$ of coherent sheaves (with proper support over the base) as introduced in [Lie06b]. For us $f : X \rightarrow B$ is a morphism of algebraic spaces satisfying suitable technical conditions, although this can be generalized (see below). Given modules $\mathcal{F}$ and $\mathcal{G}$ on X , under suitably hypotheses, the functor $T/B \mapsto \mathrm{Hom}_{X_T}(\mathcal{F}_T, \mathcal{G}_T)$ is an

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algebraic space $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ over B . See Section 3. The subfunctor $\mathrm{Isom}(\mathcal{F}, \mathcal{G})$ of isomorphisms is shown to be an algebraic space in Section 4. This is used in the next sections to show the diagonal of the stack $\mathcal{Coh}_{X/B}$ is representable. We prove $\mathcal{Coh}_{X/B}$ is an algebraic stack in Section 5 when $X \rightarrow B$ is flat and in Section 6 in general. Please see the introduction of this section for pointers to the literature.

Having proved this, it is rather straightforward to prove that $\mathrm{Quot}_{\mathcal{F}/X/B}$, $\mathrm{Hilb}_{X/B}$, and $\mathrm{Pic}_{X/B}$ are algebraic spaces and that $\mathcal{Pic}_{X/B}$ is an algebraic stack. See Sections 8, 9, 11, and 10.

In the usual manner we deduce that the functor $\mathrm{Mor}_B(Z, X)$ of relative morphisms is an algebraic space (under suitable hypotheses) in Section 12.

In Section 13 we prove that the stack in groupoids

$$\mathcal{Spaces}'_{fp, flat, proper}$$

parametrizing flat families of proper algebraic spaces satisfies all of Artin's axioms (including openness of versality) except for formal effectiveness. We've chosen the very awkward notation for this stack intentionally, because the reader should be careful in using its properties.

In Section 14 we prove that the stack *Polarized* parametrizing flat families of polarized proper algebraic spaces is an algebraic stack. Because of our work on flat families of proper algebraic spaces, this comes down to proving formal effectiveness for polarized schemes which is often known as Grothendieck's algebraization theorem.

In Section 15 we prove that the stack *Curves* parametrizing families of curves is algebraic.

In Section 16 we study moduli of complexes on a proper morphism and we obtain an algebraic stack $\mathcal{Complexes}_{X/B}$. The idea of the statement and the proof are taken from [Lie06a].

What is not in this chapter? There is almost no discussion of the properties the resulting moduli spaces and moduli stacks possess (beyond their algebraicity); to read about this we refer to Moduli Stacks, Section 1. In most of the results discussed, we can generalize the constructions by considering a morphism $\mathcal{X} \rightarrow \mathcal{B}$ of algebraic stacks instead of a morphism $X \rightarrow B$ of algebraic space. We will discuss this (insert future reference here). In the case of Hilbert spaces there is a more general notion of "Hilbert stacks" which we will discuss in a separate chapter, see (insert future reference here).

2. Conventions

05X6 We have intentionally placed this chapter, as well as the chapters "Examples of Stacks", "Sheaves on Algebraic Stacks", "Criteria for Representability", and "Artin's Axioms" before the general development of the theory of algebraic stacks. The reason for this is that starting with the next chapter (see Properties of Stacks, Section 2) we will no longer distinguish between a scheme and the algebraic stack it gives rise to. Thus our language will become more flexible and easier for a human to parse, but also less precise. These first few chapters, including the initial chapter "Algebraic Stacks", lay the groundwork that later allow us to ignore some of the very technical distinctions between different ways of thinking about algebraic

stacks. But especially in the chapters “Artin’s Axioms” and “Criteria of Representability” we need to be very precise about what objects exactly we are working with, as we are trying to show that certain constructions produce algebraic stacks or algebraic spaces.

Unfortunately, this means that some of the notation, conventions and terminology is awkward and may seem backwards to the more experienced reader. We hope the reader will forgive us!

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf} . And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S . In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. The Hom functor

08JS In this section we study the functor of homomorphisms defined below.

08JT **Situation 3.1.** Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . Let $\mathcal{F}, \mathcal{G}$ be quasi-coherent $\mathcal{O}_X$ -modules. For any scheme T over B we will denote $\mathcal{F}_T$ and $\mathcal{G}_T$ the base changes of $\mathcal{F}$ and $\mathcal{G}$ to T , in other words, the pullbacks via the projection morphism $X_T = X \times_B T \rightarrow X$. We consider the functor

$$08JU \quad (3.1.1) \quad \mathrm{Hom}(\mathcal{F}, \mathcal{G}) : (Sch/B)^{opp} \longrightarrow \mathrm{Sets}, \quad T \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T)$$

In Situation 3.1 we sometimes think of $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ as a functor $(Sch/S)^{opp} \rightarrow \mathrm{Sets}$ endowed with a morphism $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \rightarrow B$. Namely, if T is a scheme over S , then an element of $\mathrm{Hom}(\mathcal{F}, \mathcal{G})(T)$ consists of a pair (h, u) , where h is a morphism $h : T \rightarrow B$ and $u : \mathcal{F}_T \rightarrow \mathcal{G}_T$ is an $\mathcal{O}_{X_T}$ -module map where $X_T = T \times_{h, B} X$ and $\mathcal{F}_T$ and $\mathcal{G}_T$ are the pullbacks to X_T . In particular, when we say that $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \rightarrow \mathrm{Sets}$ is an algebraic space.

08JV **Lemma 3.2.** *In Situation 3.1 the functor $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for the fpqc topology.*

Proof. Let $\{T_i \rightarrow T\}_{i \in I}$ be an fpqc covering of schemes over B . Set $X_i = X_{T_i} = X \times_S T_i$ and $\mathcal{F}_i = u_{T_i}$ and $\mathcal{G}_i = \mathcal{G}_{T_i}$. Note that $\{X_i \rightarrow X_T\}_{i \in I}$ is an fpqc covering of X_T , see Topologies on Spaces, Lemma 9.3. Thus a family of maps $u_i : \mathcal{F}_i \rightarrow \mathcal{G}_i$ such that u_i and u_j restrict to the same map on $X_{T_i \times_T T_j}$ comes from a unique map $u : \mathcal{F}_T \rightarrow \mathcal{G}_T$ by descent (Descent on Spaces, Proposition 4.1). $\square$

Sanity check: Hom sheaf plays the same role among algebraic spaces over S .

0D3S **Lemma 3.3.** *In Situation 3.1. Let T be an algebraic space over S . We have*

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, \mathrm{Hom}(\mathcal{F}, \mathcal{G})) = \{(h, u) \mid h : T \rightarrow B, u : \mathcal{F}_T \rightarrow \mathcal{G}_T\}$$

where $\mathcal{F}_T, \mathcal{G}_T$ denote the pullbacks of $\mathcal{F}$ and $\mathcal{G}$ to the algebraic space $X \times_{B, h} T$.

Proof. Choose a scheme U and a surjective étale morphism $p : U \rightarrow T$. Let $R = U \times_T U$ with projections $t, s : R \rightarrow U$.

Let $v : T \rightarrow \text{Hom}(\mathcal{F}, \mathcal{G})$ be a natural transformation. Then $v(p)$ corresponds to a pair (h_U, u_U) over U . As v is a transformation of functors we see that the pullbacks of (h_U, u_U) by s and t agree. Since $T = U/R$ (Spaces, Lemma 9.1), we obtain a morphism $h : T \rightarrow B$ such that $h_U = h \circ p$. Then $\mathcal{F}_U$ is the pullback of $\mathcal{F}_T$ to X_U and similarly for $\mathcal{G}_U$. Hence u_U descends to a $\mathcal{O}_{X_T}$ -module map $u : \mathcal{F}_T \rightarrow \mathcal{G}_T$ by Descent on Spaces, Proposition 4.1.

Conversely, let (h, u) be a pair over T . Then we get a natural transformation $v : T \rightarrow \text{Hom}(\mathcal{F}, \mathcal{G})$ by sending a morphism $a : T' \rightarrow T$ where T' is a scheme to $(h \circ a, a^*u)$. We omit the verification that the construction of this and the previous paragraph are mutually inverse. $\square$

08JW **Remark 3.4.** In Situation 3.1 let $B' \rightarrow B$ be a morphism of algebraic spaces over S . Set $X' = X \times_B B'$ and denote $\mathcal{F}', \mathcal{G}'$ the pullback of $\mathcal{F}, \mathcal{G}$ to X' . Then we obtain a functor $\text{Hom}(\mathcal{F}', \mathcal{G}') : (\text{Sch}/B')^{\text{opp}} \rightarrow \text{Sets}$ associated to the base change $f' : X' \rightarrow B'$. For a scheme T over B' it is clear that we have

$$\text{Hom}(\mathcal{F}', \mathcal{G}')(T) = \text{Hom}(\mathcal{F}, \mathcal{G})(T)$$

where on the right hand side we think of T as a scheme over B via the composition $T \rightarrow B' \rightarrow B$. This trivial remark will occasionally be useful to change the base algebraic space.

08K3 **Lemma 3.5.** In Situation 3.1 let $\{X_i \rightarrow X\}_{i \in I}$ be an fppf covering and for each $i, j \in I$ let $\{X_{ijk} \rightarrow X_i \times_X X_j\}$ be an fppf covering. Denote $\mathcal{F}_i$, resp. $\mathcal{F}_{ijk}$ the pullback of $\mathcal{F}$ to X_i , resp. X_{ijk} . Similarly define $\mathcal{G}_i$ and $\mathcal{G}_{ijk}$. For every scheme T over B the diagram

$$\text{Hom}(\mathcal{F}, \mathcal{G})(T) \longrightarrow \prod_i \text{Hom}(\mathcal{F}_i, \mathcal{G}_i)(T) \xrightleftharpoons[\text{pr}_1^*]{\text{pr}_0^*} \prod_{i,j,k} \text{Hom}(\mathcal{F}_{ijk}, \mathcal{G}_{ijk})(T)$$

presents the first arrow as the equalizer of the other two.

Proof. Let $u_i : \mathcal{F}_{i,T} \rightarrow \mathcal{G}_{i,T}$ be an element in the equalizer of pr_0^* and pr_1^* . Since the base change of an fppf covering is an fppf covering (Topologies on Spaces, Lemma 7.3) we see that $\{X_{i,T} \rightarrow X_T\}_{i \in I}$ and $\{X_{ijk,T} \rightarrow X_{i,T} \times_{X_T} X_{j,T}\}$ are fppf coverings. Applying Descent on Spaces, Proposition 4.1 we first conclude that u_i and u_j restrict to the same morphism over $X_{i,T} \times_{X_T} X_{j,T}$, whereupon a second application shows that there is a unique morphism $u : \mathcal{F}_T \rightarrow \mathcal{G}_T$ restricting to u_i for each i . This finishes the proof. $\square$

08K4 **Lemma 3.6.** In Situation 3.1. If $\mathcal{F}$ is of finite presentation and f is quasi-compact and quasi-separated, then $\text{Hom}(\mathcal{F}, \mathcal{G})$ is limit preserving.

Proof. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B -schemes. We have to show that

$$\text{Hom}(\mathcal{F}, \mathcal{G})(T) = \text{colim } \text{Hom}(\mathcal{F}, \mathcal{G})(T_i)$$

Pick $0 \in I$. We may replace B by T_0 , X by X_{T_0} , $\mathcal{F}$ by $\mathcal{F}_{T_0}$, $\mathcal{G}$ by $\mathcal{G}_{T_0}$, and I by $\{i \in I \mid i \geq 0\}$. See Remark 3.4. Thus we may assume $B = \text{Spec}(R)$ is affine.

When B is affine, then X is quasi-compact and quasi-separated. Choose a surjective étale morphism $U \rightarrow X$ where U is an affine scheme (Properties of Spaces, Lemma

6.3). Since X is quasi-separated, the scheme $U \times_X U$ is quasi-compact and we may choose a surjective étale morphism $V \rightarrow U \times_X U$ where V is an affine scheme. Applying Lemma 3.5 we see that $\text{Hom}(\mathcal{F}, \mathcal{G})$ is the equalizer of two maps between

$$\text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) \quad \text{and} \quad \text{Hom}(\mathcal{F}|_V, \mathcal{G}|_V)$$

This reduces us to the case that X is affine.

In the affine case the statement of the lemma reduces to the following problem: Given a ring map $R \rightarrow A$, two A -modules M, N and a directed system of R -algebras $C = \text{colim } C_i$. When is it true that the map

$$\text{colim } \text{Hom}_{A \otimes_R C_i}(M \otimes_R C_i, N \otimes_R C_i) \longrightarrow \text{Hom}_{A \otimes_R C}(M \otimes_R C, N \otimes_R C)$$

is bijective? By Algebra, Lemma 127.5 this holds if $M \otimes_R C$ is of finite presentation over $A \otimes_R C$, i.e., when M is of finite presentation over A . $\square$

08K5 **Lemma 3.7.** *Let S be a scheme. Let B be an algebraic space over S . Let $i : X' \rightarrow X$ be a closed immersion of algebraic spaces over B . Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ -module and let $\mathcal{G}'$ be a quasi-coherent $\mathcal{O}_{X'}$ -module. Then*

$$\text{Hom}(\mathcal{F}, i_* \mathcal{G}') = \text{Hom}(i^* \mathcal{F}, \mathcal{G}')$$

as functors on (Sch/B) .

Proof. Let $g : T \rightarrow B$ be a morphism where T is a scheme. Denote $i_T : X'_T \rightarrow X_T$ the base change of i . Denote $h : X_T \rightarrow X$ and $h' : X'_T \rightarrow X'$ the projections. Observe that $(h')^* i^* \mathcal{F} = i_T^* h^* \mathcal{F}$. As a closed immersion is affine (Morphisms of Spaces, Lemma 20.6) we have $h^* i_* \mathcal{G} = i_{T,*} (h')^* \mathcal{G}$ by Cohomology of Spaces, Lemma 11.1. Thus we have

$$\begin{aligned} \text{Hom}(\mathcal{F}, i_* \mathcal{G}')(T) &= \text{Hom}_{\mathcal{O}_{X_T}}(h^* \mathcal{F}, h^* i_* \mathcal{G}') \\ &= \text{Hom}_{\mathcal{O}_{X_T}}(h^* \mathcal{F}, i_{T,*} (h')^* \mathcal{G}) \\ &= \text{Hom}_{\mathcal{O}_{X'_T}}(i_T^* h^* \mathcal{F}, (h')^* \mathcal{G}) \\ &= \text{Hom}_{\mathcal{O}_{X'_T}}((h')^* i^* \mathcal{F}, (h')^* \mathcal{G}) \\ &= \text{Hom}(i^* \mathcal{F}, \mathcal{G}')(T) \end{aligned}$$

as desired. The middle equality follows from the adjointness of the functors $i_{T,*}$ and i_T^* . $\square$

08JX **Lemma 3.8.** *Let S be a scheme. Let B be an algebraic space over S . Let K be a pseudo-coherent object of $D(\mathcal{O}_B)$.*

- (1) *If for all $g : T \rightarrow B$ in (Sch/B) the cohomology sheaf $H^{-1}(Lg^* K)$ is zero, then the functor*

$$(\text{Sch}/B)^{\text{opp}} \longrightarrow \text{Sets}, \quad (g : T \rightarrow B) \longmapsto H^0(T, H^0(Lg^* K))$$

is an algebraic space affine and of finite presentation over B .

- (2) *If for all $g : T \rightarrow B$ in (Sch/B) the cohomology sheaves $H^i(Lg^* K)$ are zero for $i < 0$, then K is perfect, K locally has tor amplitude in $[0, b]$, and the functor*

$$(\text{Sch}/B)^{\text{opp}} \longrightarrow \text{Sets}, \quad (g : T \rightarrow B) \longmapsto H^0(T, Lg^* K)$$

is an algebraic space affine and of finite presentation over B .

Proof. Under the assumptions of (2) we have $H^0(T, Lg^*K) = H^0(T, H^0(Lg^*K))$. Let us prove that the rule $T \mapsto H^0(T, H^0(Lg^*K))$ satisfies the sheaf property for the fppf topology. To do this assume we have an fppf covering $\{h_i : T_i \rightarrow T\}$ of a scheme $g : T \rightarrow B$ over B . Set $g_i = g \circ h_i$. Note that since h_i is flat, we have $Lh_i^* = h_i^*$ and h_i^* commutes with taking cohomology. Hence

$$H^0(T_i, H^0(Lg_i^*K)) = H^0(T_i, H^0(h_i^*Lg^*K)) = H^0(T, h_i^*H^0(Lg^*K))$$

Similarly for the pullback to $T_i \times_T T_j$. Since Lg^*K is a pseudo-coherent complex on T (Cohomology on Sites, Lemma 45.3) the cohomology sheaf $\mathcal{F} = H^0(Lg^*K)$ is quasi-coherent (Derived Categories of Spaces, Lemma 13.6). Hence by Descent on Spaces, Proposition 4.1 we see that

$$H^0(T, \mathcal{F}) = \text{Ker}(\prod H^0(T_i, h_i^*\mathcal{F}) \rightarrow \prod H^0(T_i \times_T T_j, (T_i \times_T T_j \rightarrow T)^*\mathcal{F}))$$

In this way we see that the rules in (1) and (2) satisfy the sheaf property for fppf coverings. This means we may apply Bootstrap, Lemma 11.2 to see it suffices to prove the representability étale locally on B . Moreover, we may check whether the end result is affine and of finite presentation étale locally on B , see Morphisms of Spaces, Lemmas 20.3 and 28.4. Hence we may assume that B is an affine scheme.

Assume $B = \text{Spec}(A)$ is an affine scheme. By the results of Derived Categories of Spaces, Lemmas 13.6, 4.2, and 13.2 we deduce that in the rest of the proof we may think of K as a perfect object of the derived category of complexes of modules on B in the Zariski topology. By Derived Categories of Schemes, Lemmas 10.1, 3.5, and 10.2 we can find a pseudo-coherent complex $M^\bullet$ of A -modules such that K is the corresponding object of $D(\mathcal{O}_B)$. Our assumption on pullbacks implies that $M^\bullet \otimes_A^L \kappa(\mathfrak{p})$ has vanishing H^{-1} for all primes $\mathfrak{p} \subset A$. By More on Algebra, Lemma 76.4 we can write

$$M^\bullet = \tau_{\geq 0}M^\bullet \oplus \tau_{\leq -1}M^\bullet$$

with $\tau_{\geq 0}M^\bullet$ perfect with Tor amplitude in $[0, b]$ for some $b \geq 0$ (here we also have used More on Algebra, Lemmas 74.12 and 66.16). Note that in case (2) we also see that $\tau_{\leq -1}M^\bullet = 0$ in $D(A)$ whence $M^\bullet$ and K are perfect with tor amplitude in $[0, b]$. For any B -scheme $g : T \rightarrow B$ we have

$$H^0(T, H^0(Lg^*K)) = H^0(T, H^0(Lg^*\tau_{\geq 0}K))$$

(by the dual of Derived Categories, Lemma 16.1) hence we may replace K by $\tau_{\geq 0}K$ and correspondingly $M^\bullet$ by $\tau_{\geq 0}M^\bullet$. In other words, we may assume $M^\bullet$ has tor amplitude in $[0, b]$.

Assume $M^\bullet$ has tor amplitude in $[0, b]$. We may assume $M^\bullet$ is a bounded above complex of finite free A -modules (by our definition of pseudo-coherent complexes, see More on Algebra, Definition 64.1 and the discussion following the definition). By More on Algebra, Lemma 66.2 we see that $M = \text{Coker}(M^{-1} \rightarrow M^0)$ is flat. By Algebra, Lemma 78.2 we see that M is finite locally free. Hence $M^\bullet$ is quasi-isomorphic to

$$M \rightarrow M^1 \rightarrow M^2 \rightarrow \dots \rightarrow M^d \rightarrow 0 \dots$$

Note that this is a K-flat complex (Cohomology, Lemma 26.9), hence derived pull-back of K via a morphism $T \rightarrow B$ is computed by the complex

$$g^*\widetilde{M} \rightarrow g^*\widetilde{M}^1 \rightarrow \dots$$

Thus it suffices to show that the functor

$$(g : T \rightarrow B) \mapsto \text{Ker}(\Gamma(T, g^* \widetilde{M}) \rightarrow \Gamma(T, g^*(\widetilde{M^1}))$$

is representable by an affine scheme of finite presentation over B .

We may still replace B by the members of an affine open covering in order to prove this last statement. Hence we may assume that M is finite free (recall that M^1 is finite free to begin with). Write $M = A^{\oplus n}$ and $M^1 = A^{\oplus m}$. Let the map $M \rightarrow M^1$ be given by the $m \times n$ matrix (a_{ij}) with coefficients in A . Then $\widetilde{M} = \mathcal{O}_B^{\oplus n}$ and $\widetilde{M^1} = \mathcal{O}_B^{\oplus m}$. Thus the functor above is equal to the functor

$$(g : T \rightarrow B) \mapsto \{(f_1, \dots, f_n) \in \Gamma(T, \mathcal{O}_T) \mid \sum g^{\sharp}(a_{ij})f_i = 0, j = 1, \dots, m\}$$

Clearly this is representable by the affine scheme

$$\text{Spec} \left(A[x_1, \dots, x_n] / \left(\sum a_{ij}x_i; j = 1, \dots, m \right) \right)$$

and the lemma has been proved. $\square$

The functor $\text{Hom}(\mathcal{F}, \mathcal{G})$ is representable in a number of situations. All of our results will be based on the following basic case. The proof of this lemma as given below is in some sense the natural generalization to the proof of [DG67, III, Cor 7.7.8].

08JY **Lemma 3.9.** *In Situation 3.1 assume that*

- (1) B is a Noetherian algebraic space,
- (2) f is locally of finite type and quasi-separated,
- (3) $\mathcal{F}$ is a finite type $\mathcal{O}_X$ -module, and
- (4) $\mathcal{G}$ is a finite type $\mathcal{O}_X$ -module, flat over B , with support proper over B .

Then the functor $\text{Hom}(\mathcal{F}, \mathcal{G})$ is an algebraic space affine and of finite presentation over B .

Proof. We may replace X by a quasi-compact open neighbourhood of the support of $\mathcal{G}$, hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \rightarrow \mathcal{F}$ by a perfect complex P of the triple $(X, \mathcal{F}, -1)$, see Derived Categories of Spaces, Definition 14.1 and Theorem 14.7). Then the induced map

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Hom}_{D(\mathcal{O}_X)}(P, \mathcal{G})$$

is an isomorphism because $P \rightarrow \mathcal{F}$ induces an isomorphism $H^0(P) \rightarrow \mathcal{F}$ and $H^i(P) = 0$ for $i > 0$. Moreover, for any morphism $g : T \rightarrow B$ denote $h : X_T = T \times_B X \rightarrow X$ the projection and set $P_T = Lh^*P$. Then it is equally true that

$$\text{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T) \longrightarrow \text{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T)$$

is an isomorphism, as $P_T = Lh^*P \rightarrow Lh^*\mathcal{F} \rightarrow \mathcal{F}_T$ induces an isomorphism $H^0(P_T) \rightarrow \mathcal{F}_T$ (because h^* is right exact and $H^i(P) = 0$ for $i > 0$). Thus it suffices to prove the result for the functor

$$T \mapsto \text{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T).$$

By the Leray spectral sequence (see Cohomology on Sites, Remark 14.4) we have

$$\text{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T) = H^0(X_T, R\mathcal{H}om(P_T, \mathcal{G}_T)) = H^0(T, Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T))$$

where $f_T : X_T \rightarrow T$ is the base change of f . By Derived Categories of Spaces, Lemma 21.5 we have

$$Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T) = Lg^*Rf_*R\mathcal{H}om(P, \mathcal{G}).$$

By Derived Categories of Spaces, Lemma 22.3 the object $K = Rf_*R\mathcal{H}om(P, \mathcal{G})$ of $D(\mathcal{O}_B)$ is perfect. This means we can apply Lemma 3.8 as long as we can prove that the cohomology sheaf $H^i(Lg^*K)$ is 0 for all $i < 0$ and $g : T \rightarrow B$ as above. This is clear from the last displayed formula as the cohomology sheaves of $Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T)$ are zero in negative degrees due to the fact that $R\mathcal{H}om(P_T, \mathcal{G}_T)$ has vanishing cohomology sheaves in negative degrees as P_T is perfect with vanishing cohomology sheaves in positive degrees. $\square$

Here is a cheap consequence of Lemma 3.9.

08K6 **Proposition 3.10.** *In Situation 3.1 assume that*

- (1) *f is of finite presentation, and*
- (2) *$\mathcal{G}$ is a finitely presented $\mathcal{O}_X$ -module, flat over B , with support proper over B .*

Then the functor $\text{Hom}(\mathcal{F}, \mathcal{G})$ is an algebraic space affine over B . If $\mathcal{F}$ is of finite presentation, then $\text{Hom}(\mathcal{F}, \mathcal{G})$ is of finite presentation over B .

Proof. By Lemma 3.2 the functor $\text{Hom}(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for fppf coverings. This mean we may¹ apply Bootstrap, Lemma 11.1 to check the representability étale locally on B . Moreover, we may check whether the end result is affine or of finite presentation étale locally on B , see Morphisms of Spaces, Lemmas 20.3 and 28.4. Hence we may assume that B is an affine scheme.

Assume B is an affine scheme. As f is of finite presentation, it follows X is quasi-compact and quasi-separated. Thus we can write $\mathcal{F} = \text{colim } \mathcal{F}_i$ as a filtered colimit of $\mathcal{O}_X$ -modules of finite presentation (Limits of Spaces, Lemma 9.1). It is clear that

$$\text{Hom}(\mathcal{F}, \mathcal{G}) = \lim \text{Hom}(\mathcal{F}_i, \mathcal{G})$$

Hence if we can show that each $\text{Hom}(\mathcal{F}_i, \mathcal{G})$ is representable by an affine scheme, then we see that the same thing holds for $\text{Hom}(\mathcal{F}, \mathcal{G})$. Use the material in Limits, Section 2 and Limits of Spaces, Section 4. Thus we may assume that $\mathcal{F}$ is of finite presentation.

Say $B = \text{Spec}(R)$. Write $R = \text{colim } R_i$ with each R_i a finite type $\mathbf{Z}$ -algebra. Set $B_i = \text{Spec}(R_i)$. By the results of Limits of Spaces, Lemmas 7.1 and 7.2 we can find an i , a morphism of algebraic spaces $X_i \rightarrow B_i$, and finitely presented $\mathcal{O}_{X_i}$ -modules $\mathcal{F}_i$ and $\mathcal{G}_i$ such that the base change of $(X_i, \mathcal{F}_i, \mathcal{G}_i)$ to B recovers $(X, \mathcal{F}, \mathcal{G})$. By Limits of Spaces, Lemma 6.12 we may, after increasing i , assume that $\mathcal{G}_i$ is flat over B_i . By Limits of Spaces, Lemma 12.3 we may similarly assume the scheme theoretic support of $\mathcal{G}_i$ is proper over B_i . At this point we can apply Lemma 3.9 to see that $H_i = \text{Hom}(\mathcal{F}_i, \mathcal{G}_i)$ is an algebraic space affine of finite presentation over B_i . Pulling back to B (using Remark 3.4) we see that $H_i \times_{B_i} B = \text{Hom}(\mathcal{F}, \mathcal{G})$ and we win. $\square$

¹We omit the verification of the set theoretical condition (3) of the referenced lemma.

4. The Isom functor

08K7 In Situation 3.1 we can consider the subfunctor

$$\text{Isom}(\mathcal{F}, \mathcal{G}) \subset \text{Hom}(\mathcal{F}, \mathcal{G})$$

whose value on a scheme T over B is the set of *invertible* $\mathcal{O}_{X_T}$ -homomorphisms $u : \mathcal{F}_T \rightarrow \mathcal{G}_T$.

We sometimes think of $\text{Isom}(\mathcal{F}, \mathcal{G})$ as a functor $(\text{Sch}/S)^{\text{opp}} \rightarrow \text{Sets}$ endowed with a morphism $\text{Isom}(\mathcal{F}, \mathcal{G}) \rightarrow B$. Namely, if T is a scheme over S , then an element of $\text{Isom}(\mathcal{F}, \mathcal{G})(T)$ consists of a pair (h, u) , where h is a morphism $h : T \rightarrow B$ and $u : \mathcal{F}_T \rightarrow \mathcal{G}_T$ is an $\mathcal{O}_{X_T}$ -module isomorphism where $X_T = T \times_{h, B} X$ and $\mathcal{F}_T$ and $\mathcal{G}_T$ are the pullbacks to X_T . In particular, when we say that $\text{Isom}(\mathcal{F}, \mathcal{G})$ is an algebraic space, we mean that the corresponding functor $(\text{Sch}/S)^{\text{opp}} \rightarrow \text{Sets}$ is an algebraic space.

08K8 **Lemma 4.1.** *In Situation 3.1 the functor $\text{Isom}(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for the fpqc topology.*

Proof. We have already seen that $\text{Hom}(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property. Hence it remains to show the following: Given an fpqc covering $\{T_i \rightarrow T\}_{i \in I}$ of schemes over B and an $\mathcal{O}_{X_T}$ -linear map $u : \mathcal{F}_T \rightarrow \mathcal{G}_T$ such that u_{T_i} is an isomorphism for all i , then u is an isomorphism. Since $\{X_i \rightarrow X_T\}_{i \in I}$ is an fpqc covering of X_T , see Topologies on Spaces, Lemma 9.3, this follows from Descent on Spaces, Proposition 4.1. $\square$

Sanity check: Isom sheaf plays the same role among algebraic spaces over S .

0D3T **Lemma 4.2.** *In Situation 3.1. Let T be an algebraic space over S . We have*

$$\text{Mor}_{\text{Sh}((\text{Sch}/S)_{\text{fpf}})}(T, \text{Isom}(\mathcal{F}, \mathcal{G})) = \{(h, u) \mid h : T \rightarrow B, u : \mathcal{F}_T \rightarrow \mathcal{G}_T \text{ isomorphism}\}$$

where $\mathcal{F}_T, \mathcal{G}_T$ denote the pullbacks of $\mathcal{F}$ and $\mathcal{G}$ to the algebraic space $X \times_{B, h} T$.

Proof. Observe that the left and right hand side of the equality are subsets of the left and right hand side of the equality in Lemma 3.3. We omit the verification that these subsets correspond under the identification given in the proof of that lemma. $\square$

08K9 **Proposition 4.3.** *In Situation 3.1 assume that*

- (1) f is of finite presentation, and
- (2) $\mathcal{F}$ and $\mathcal{G}$ are finitely presented $\mathcal{O}_X$ -modules, flat over B , with support proper over B .

Then the functor $\text{Isom}(\mathcal{F}, \mathcal{G})$ is an algebraic space affine of finite presentation over B .

Proof. We will use the abbreviations $H = \text{Hom}(\mathcal{F}, \mathcal{G})$, $I = \text{Hom}(\mathcal{F}, \mathcal{F})$, $H' = \text{Hom}(\mathcal{G}, \mathcal{F})$, and $I' = \text{Hom}(\mathcal{G}, \mathcal{G})$. By Proposition 3.10 the functors H, I, H', I' are algebraic spaces and the morphisms $H \rightarrow B, I \rightarrow B, H' \rightarrow B$, and $I' \rightarrow B$ are affine and of finite presentation. The composition of maps gives a morphism

$$c : H' \times_B H \longrightarrow I \times_B I', \quad (u', u) \longmapsto (u \circ u', u' \circ u)$$

of algebraic spaces over B . Since $I \times_B I' \rightarrow B$ is separated, the section $\sigma : B \rightarrow I \times_B I'$ corresponding to $(\text{id}_{\mathcal{F}}, \text{id}_{\mathcal{G}})$ is a closed immersion (Morphisms of Spaces,

Lemma 4.7). Moreover, σ is of finite presentation (Morphisms of Spaces, Lemma 28.9). Hence

$$\text{Isom}(\mathcal{F}, \mathcal{G}) = (H' \times_B H) \times_{c, I \times_B I', \sigma} B$$

is an algebraic space affine of finite presentation over B as well. Some details omitted. $\square$

5. The stack of coherent sheaves

08KA In this section we prove that the stack of coherent sheaves on X/B is algebraic under suitable hypotheses. This is a special case of [Lie06b, Theorem 2.1.1] which treats the case of the stack of coherent sheaves on an Artin stack over a base.

08KB **Situation 5.1.** Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . Assume that f is of finite presentation. We denote $\text{Coh}_{X/B}$ the category whose objects are triples $(T, g, \mathcal{F})$ where

- (1) T is a scheme over S ,
- (2) $g : T \rightarrow B$ is a morphism over S , and setting $X_T = T \times_{g, B} X$
- (3) $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{X_T}$ -module of finite presentation, flat over T , with support proper over T .

A morphism $(T, g, \mathcal{F}) \rightarrow (T', g', \mathcal{F}')$ is given by a pair (h, φ) where

- (1) $h : T \rightarrow T'$ is a morphism of schemes over B (i.e., $g' \circ h = g$), and
- (2) $\varphi : (h')^* \mathcal{F}' \rightarrow \mathcal{F}$ is an isomorphism of $\mathcal{O}_{X_T}$ -modules where $h' : X_T \rightarrow X_{T'}$ is the base change of h .

Thus $\text{Coh}_{X/B}$ is a category and the rule

$$p : \text{Coh}_{X/B} \longrightarrow (\text{Sch}/S)_{fppf}, \quad (T, g, \mathcal{F}) \longmapsto T$$

is a functor. For a scheme T over S we denote $\text{Coh}_{X/B, T}$ the fibre category of p over T . These fibre categories are groupoids.

08W5 **Lemma 5.2.** *In Situation 5.1 the functor $p : \text{Coh}_{X/B} \longrightarrow (\text{Sch}/S)_{fppf}$ is fibred in groupoids.*

Proof. We show that p is fibred in groupoids by checking conditions (1) and (2) of Categories, Definition 35.1. Given an object $(T', g', \mathcal{F}')$ of $\text{Coh}_{X/B}$ and a morphism $h : T \rightarrow T'$ of schemes over S we can set $g = h \circ g'$ and $\mathcal{F} = (h')^* \mathcal{F}'$ where $h' : X_T \rightarrow X_{T'}$ is the base change of h . Then it is clear that we obtain a morphism $(T, g, \mathcal{F}) \rightarrow (T', g', \mathcal{F}')$ of $\text{Coh}_{X/B}$ lying over h . This proves (1). For (2) suppose we are given morphisms

$$(h_1, \varphi_1) : (T_1, g_1, \mathcal{F}_1) \rightarrow (T, g, \mathcal{F}) \quad \text{and} \quad (h_2, \varphi_2) : (T_2, g_2, \mathcal{F}_2) \rightarrow (T, g, \mathcal{F})$$

of $\text{Coh}_{X/B}$ and a morphism $h : T_1 \rightarrow T_2$ such that $h_2 \circ h = h_1$. Then we can let φ be the composition

$$(h')^* \mathcal{F}_2 \xrightarrow{(h')^* \varphi_2^{-1}} (h')^* (h_2)^* \mathcal{F} = (h_1)^* \mathcal{F} \xrightarrow{\varphi_1} \mathcal{F}_1$$

to obtain the morphism $(h, \varphi) : (T_1, g_1, \mathcal{F}_1) \rightarrow (T_2, g_2, \mathcal{F}_2)$ that witnesses the truth of condition (2). $\square$

08W6 **Lemma 5.3.** *In Situation 5.1. Denote $\mathcal{X} = \text{Coh}_{X/B}$. Then $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.*

Proof. Consider two objects $x = (T, g, \mathcal{F})$ and $y = (T, h, \mathcal{G})$ of $\mathcal{X}$ over a scheme T . We have to show that $\text{Isom}_{\mathcal{X}}(x, y)$ is an algebraic space over T , see Algebraic Stacks, Lemma 10.11. If for $a : T' \rightarrow T$ the restrictions $x|_{T'}$ and $y|_{T'}$ are isomorphic in the fibre category $\mathcal{X}_{T'}$, then $g \circ a = h \circ a$. Hence there is a transformation of presheaves

$$\text{Isom}_{\mathcal{X}}(x, y) \longrightarrow \text{Equalizer}(g, h)$$

Since the diagonal of B is representable (by schemes) this equalizer is a scheme. Thus we may replace T by this equalizer and the sheaves $\mathcal{F}$ and $\mathcal{G}$ by their pullbacks. Thus we may assume $g = h$. In this case we have $\text{Isom}_{\mathcal{X}}(x, y) = \text{Isom}(\mathcal{F}, \mathcal{G})$ and the result follows from Proposition 4.3. $\square$

08KC **Lemma 5.4.** *In Situation 5.1 the functor $p : \text{Coh}_{X/B} \rightarrow (\text{Sch}/S)_{\text{fppf}}$ is a stack in groupoids.*

Proof. To prove that $\text{Coh}_{X/B}$ is a stack in groupoids, we have to show that the presheaves Isom are sheaves and that descent data are effective. The statement on Isom follows from Lemma 5.3, see Algebraic Stacks, Lemma 10.11. Let us prove the statement on descent data. Suppose that $\{a_i : T_i \rightarrow T\}$ is an fppf covering of schemes over S . Let (ξ_i, φ_{ij}) be a descent datum for $\{T_i \rightarrow T\}$ with values in $\text{Coh}_{X/B}$. For each i we can write $\xi_i = (T_i, g_i, \mathcal{F}_i)$. Denote $\text{pr}_0 : T_i \times_T T_j \rightarrow T_i$ and $\text{pr}_1 : T_i \times_T T_j \rightarrow T_j$ the projections. The condition that $\xi_i|_{T_i \times_T T_j} = \xi_j|_{T_i \times_T T_j}$ implies in particular that $g_i \circ \text{pr}_0 = g_j \circ \text{pr}_1$. Thus there exists a unique morphism $g : T \rightarrow B$ such that $g_i = g \circ a_i$, see Descent on Spaces, Lemma 7.2. Denote $X_T = T \times_{g,B} X$. Set $X_i = X_{T_i} = T_i \times_{g_i,B} X = T_i \times_{a_i,T} X_T$ and

$$X_{ij} = X_{T_i} \times_{X_T} X_{T_j} = X_i \times_{X_T} X_j$$

with projections pr_i and pr_j to X_i and X_j . Observe that the pullback of $(T_i, g_i, \mathcal{F}_i)$ by $\text{pr}_0 : T_i \times_T T_j \rightarrow T_i$ is given by $(T_i \times_T T_j, g_i \circ \text{pr}_0, \text{pr}_i^* \mathcal{F}_i)$. Hence a descent datum for $\{T_i \rightarrow T\}$ in $\text{Coh}_{X/B}$ is given by the objects $(T_i, g \circ a_i, \mathcal{F}_i)$ and for each pair i, j an isomorphism of $\mathcal{O}_{X_{ij}}$ -modules

$$\varphi_{ij} : \text{pr}_i^* \mathcal{F}_i \longrightarrow \text{pr}_j^* \mathcal{F}_j$$

satisfying the cocycle condition over (the pullback of X to) $T_i \times_T T_j \times_T T_k$. Ok, and now we simply use that $\{X_i \rightarrow X_T\}$ is an fppf covering so that we can view $(\mathcal{F}_i, \varphi_{ij})$ as a descent datum for this covering. By Descent on Spaces, Proposition 4.1 this descent datum is effective and we obtain a quasi-coherent sheaf $\mathcal{F}$ over X_T restricting to $\mathcal{F}_i$ on X_i . By Morphisms of Spaces, Lemma 31.5 we see that $\mathcal{F}$ is flat over T and Descent on Spaces, Lemma 6.2 guarantees that $\mathcal{Q}$ is of finite presentation as an $\mathcal{O}_{X_T}$ -module. Finally, by Descent on Spaces, Lemma 11.19 we see that the scheme theoretic support of $\mathcal{F}$ is proper over T as we've assumed the scheme theoretic support of $\mathcal{F}_i$ is proper over T_i (note that taking scheme theoretic support commutes with flat base change by Morphisms of Spaces, Lemma 30.10). In this way we obtain our desired object over T . $\square$

08LP **Remark 5.5.** In Situation 5.1 the rule $(T, g, \mathcal{F}) \mapsto (T, g)$ defines a 1-morphism

$$\text{Coh}_{X/B} \longrightarrow \mathcal{S}_B$$

of stacks in groupoids (see Lemma 5.4, Algebraic Stacks, Section 7, and Examples of Stacks, Section 10). Let $B' \rightarrow B$ be a morphism of algebraic spaces over S . Let $\mathcal{S}_{B'} \rightarrow \mathcal{S}_B$ be the associated 1-morphism of stacks fibred in sets. Set $X' = X \times_B B'$.

We obtain a stack in groupoids $\mathcal{Coh}_{X'/B'} \rightarrow (\mathcal{Sch}/S)_{fppf}$ associated to the base change $f' : X' \rightarrow B'$. In this situation the diagram

$$\begin{array}{ccc} \mathcal{Coh}_{X'/B'} & \longrightarrow & \mathcal{Coh}_{X/B} \\ \downarrow & & \downarrow \\ \mathcal{S}_{B'} & \longrightarrow & \mathcal{S}_B \end{array} \quad \begin{array}{c} \text{or in} \\ \text{another} \\ \text{notation} \end{array} \quad \begin{array}{ccc} \mathcal{Coh}_{X'/B'} & \longrightarrow & \mathcal{Coh}_{X/B} \\ \downarrow & & \downarrow \\ \mathcal{Sch}/B' & \longrightarrow & \mathcal{Sch}/B \end{array}$$

is 2-fibre product square. This trivial remark will occasionally be useful to change the base algebraic space.

08KD **Lemma 5.6.** *In Situation 5.1 assume that $B \rightarrow S$ is locally of finite presentation. Then $p : \mathcal{Coh}_{X/B} \rightarrow (\mathcal{Sch}/S)_{fppf}$ is limit preserving (Artin's Axioms, Definition 11.1).*

Proof. Write $B(T)$ for the discrete category whose objects are the S -morphisms $T \rightarrow B$. Let $T = \lim T_i$ be a filtered limit of affine schemes over S . Assigning to an object $(T, h, \mathcal{F})$ of $\mathcal{Coh}_{X/B, T}$ the object h of $B(T)$ gives us a commutative diagram of fibre categories

$$\begin{array}{ccc} \operatorname{colim} \mathcal{Coh}_{X/B, T_i} & \longrightarrow & \mathcal{Coh}_{X/B, T} \\ \downarrow & & \downarrow \\ \operatorname{colim} B(T_i) & \longrightarrow & B(T) \end{array}$$

We have to show the top horizontal arrow is an equivalence. Since we have assumed that B is locally of finite presentation over S we see from Limits of Spaces, Remark 3.11 that the bottom horizontal arrow is an equivalence. This means that we may assume $T = \lim T_i$ be a filtered limit of affine schemes over B . Denote $g_i : T_i \rightarrow B$ and $g : T \rightarrow B$ the corresponding morphisms. Set $X_i = T_i \times_{g_i, B} X$ and $X_T = T \times_{g, B} X$. Observe that $X_T = \operatorname{colim} X_i$ and that the algebraic spaces X_i and X_T are quasi-separated and quasi-compact (as they are of finite presentation over the affines T_i and T). By Limits of Spaces, Lemma 7.2 we see that

$$\operatorname{colim} FP(X_i) = FP(X_T).$$

where $FP(W)$ is short hand for the category of finitely presented $\mathcal{O}_W$ -modules. The results of Limits of Spaces, Lemmas 6.12 and 12.3 tell us the same thing is true if we replace $FP(X_i)$ and $FP(X_T)$ by the full subcategory of objects flat over T_i and T with scheme theoretic support proper over T_i and T . This proves the lemma. $\square$

08LQ **Lemma 5.7.** *In Situation 5.1. Let*

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout in the category of schemes over S where $Z \rightarrow Z'$ is a thickening and $Z \rightarrow Y$ is affine, see More on Morphisms, Lemma 14.3. Then the functor on fibre categories

$$\mathcal{Coh}_{X/B, Y'} \longrightarrow \mathcal{Coh}_{X/B, Y} \times_{\mathcal{Coh}_{X/B, Z}} \mathcal{Coh}_{X/B, Z'}$$

is an equivalence.

Proof. Observe that the corresponding map

$$B(Y') \longrightarrow B(Y) \times_{B(Z)} B(Z')$$

is a bijection, see Pushouts of Spaces, Lemma 6.1. Thus using the commutative diagram

$$\begin{array}{ccc} \mathrm{Coh}_{X/B, Y'} & \longrightarrow & \mathrm{Coh}_{X/B, Y} \times_{\mathrm{Coh}_{X/B, Z}} \mathrm{Coh}_{X/B, Z'} \\ \downarrow & & \downarrow \\ B(Y') & \longrightarrow & B(Y) \times_{B(Z)} B(Z') \end{array}$$

we see that we may assume that Y' is a scheme over B' . By Remark 5.5 we may replace B by Y' and X by $X \times_B Y'$. Thus we may assume $B = Y'$. In this case the statement follows from Pushouts of Spaces, Lemma 6.6. $\square$

08W7 **Lemma 5.8.** *Let*

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ \downarrow & & \downarrow \\ T & \longrightarrow & T' \end{array}$$

be a cartesian square of algebraic spaces where $T \rightarrow T'$ is a first order thickening. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$ -module flat over T' . Set $\mathcal{F} = i^ \mathcal{F}'$. The following are equivalent*

- (1) $\mathcal{F}'$ is a quasi-coherent $\mathcal{O}_{X'}$ -module of finite presentation,
- (2) $\mathcal{F}'$ is an $\mathcal{O}_{X'}$ -module of finite presentation,
- (3) $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$ -module of finite presentation,
- (4) $\mathcal{F}$ is an $\mathcal{O}_X$ -module of finite presentation,

Proof. Recall that a finitely presented module is quasi-coherent hence the equivalence of (1) and (2) and (3) and (4). The equivalence of (2) and (4) is a special case of Deformation Theory, Lemma 11.3. $\square$

08W8 **Lemma 5.9.** *In Situation 5.1 assume that S is a locally Noetherian scheme and $B \rightarrow S$ is locally of finite presentation. Let k be a finite type field over S and let $x_0 = (\mathrm{Spec}(k), g_0, \mathcal{G}_0)$ be an object of $\mathcal{X} = \mathrm{Coh}_{X/B}$ over k . Then the spaces $T\mathcal{F}_{\mathcal{X}, k, x_0}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X}, k, x_0})$ (Artin's Axioms, Section 8) are finite dimensional.*

Proof. Observe that by Lemma 5.7 our stack in groupoids $\mathcal{X}$ satisfies property (RS*) defined in Artin's Axioms, Section 21. In particular $\mathcal{X}$ satisfies (RS). Hence all associated predeformation categories are deformation categories (Artin's Axioms, Lemma 6.1) and the statement makes sense.

In this paragraph we show that we can reduce to the case $B = \mathrm{Spec}(k)$. Set $X_0 = \mathrm{Spec}(k) \times_{g_0, B} X$ and denote $\mathcal{X}_0 = \mathrm{Coh}_{X_0/k}$. In Remark 5.5 we have seen that $\mathcal{X}_0$ is the 2-fibre product of $\mathcal{X}$ with $\mathrm{Spec}(k)$ over B as categories fibred in groupoids over $(\mathrm{Sch}/S)_{fppf}$. Thus by Artin's Axioms, Lemma 8.2 we reduce to proving that B , $\mathrm{Spec}(k)$, and $\mathcal{X}_0$ have finite dimensional tangent spaces and infinitesimal automorphism spaces. The tangent space of B and $\mathrm{Spec}(k)$ are finite dimensional by Artin's Axioms, Lemma 8.1 and of course these have vanishing Inf . Thus it suffices to deal with $\mathcal{X}_0$.

Let $k[\epsilon]$ be the dual numbers over k . Let $\mathrm{Spec}(k[\epsilon]) \rightarrow B$ be the composition of $g_0 : \mathrm{Spec}(k) \rightarrow B$ and the morphism $\mathrm{Spec}(k[\epsilon]) \rightarrow \mathrm{Spec}(k)$ coming from the inclusion

$k \rightarrow k[\epsilon]$. Set $X_0 = \operatorname{Spec}(k) \times_B X$ and $X_\epsilon = \operatorname{Spec}(k[\epsilon]) \times_B X$. Observe that X_ϵ is a first order thickening of X_0 flat over the first order thickening $\operatorname{Spec}(k) \rightarrow \operatorname{Spec}(k[\epsilon])$. Unwinding the definitions and using Lemma 5.8 we see that $T\mathcal{F}_{\mathcal{X}_0, k, x_0}$ is the set of lifts of $\mathcal{G}_0$ to a flat module on X_ϵ . By Deformation Theory, Lemma 12.1 we conclude that

$$T\mathcal{F}_{\mathcal{X}_0, k, x_0} = \operatorname{Ext}_{\mathcal{O}_{X_0}}^1(\mathcal{G}_0, \mathcal{G}_0)$$

Here we have used the identification $\epsilon k[\epsilon] \cong k$ of $k[\epsilon]$ -modules. Using Deformation Theory, Lemma 12.1 once more we see that

$$\operatorname{Inf}(\mathcal{F}_{\mathcal{X}, k, x_0}) = \operatorname{Ext}_{\mathcal{O}_{X_0}}^0(\mathcal{G}_0, \mathcal{G}_0)$$

These spaces are finite dimensional over k as $\mathcal{G}_0$ has support proper over $\operatorname{Spec}(k)$. Namely, X_0 is of finite presentation over $\operatorname{Spec}(k)$, hence Noetherian. Since $\mathcal{G}_0$ is of finite presentation it is a coherent $\mathcal{O}_{X_0}$ -module. Thus we may apply Derived Categories of Spaces, Lemma 8.4 to conclude the desired finiteness. $\square$

08W9 **Lemma 5.10.** *In Situation 5.1 assume that S is a locally Noetherian scheme and that $f : X \rightarrow B$ is separated. Let $\mathcal{X} = \operatorname{Coh}_{X/B}$. Then the functor Artin's Axioms, Equation (9.3.1) is an equivalence.*

Proof. Let A be an S -algebra which is a complete local Noetherian ring with maximal ideal $\mathfrak{m}$ whose residue field k is of finite type over S . We have to show that the category of objects over A is equivalent to the category of formal objects over A . Since we know this holds for the category $\mathcal{S}_B$ fibred in sets associated to B by Artin's Axioms, Lemma 9.5, it suffices to prove this for those objects lying over a given morphism $\operatorname{Spec}(A) \rightarrow B$.

Set $X_A = \operatorname{Spec}(A) \times_B X$ and $X_n = \operatorname{Spec}(A/\mathfrak{m}^n) \times_B X$. By Grothendieck's existence theorem (More on Morphisms of Spaces, Theorem 42.11) we see that the category of coherent modules $\mathcal{F}$ on X_A with support proper over $\operatorname{Spec}(A)$ is equivalent to the category of systems $(\mathcal{F}_n)$ of coherent modules $\mathcal{F}_n$ on X_n with support proper over $\operatorname{Spec}(A/\mathfrak{m}^n)$. The equivalence sends $\mathcal{F}$ to the system $(\mathcal{F} \otimes_A A/\mathfrak{m}^n)$. See discussion in More on Morphisms of Spaces, Remark 42.12. To finish the proof of the lemma, it suffices to show that $\mathcal{F}$ is flat over A if and only if all $\mathcal{F} \otimes_A A/\mathfrak{m}^n$ are flat over $A/\mathfrak{m}^n$. This follows from More on Morphisms of Spaces, Lemma 24.3. $\square$

08WA **Lemma 5.11.** *In Situation 5.1 assume that S is a locally Noetherian scheme, $S = B$, and $f : X \rightarrow B$ is flat. Let $\mathcal{X} = \operatorname{Coh}_{X/B}$. Then we have openness of versality for $\mathcal{X}$ (see Artin's Axioms, Definition 13.1).*

First proof. This proof is based on the criterion of Artin's Axioms, Lemma 24.4. Let $U \rightarrow S$ be of finite type morphism of schemes, x an object of $\mathcal{X}$ over U and $u_0 \in U$ a finite type point such that x is versal at u_0 . After shrinking U we may assume that u_0 is a closed point (Morphisms, Lemma 16.1) and $U = \operatorname{Spec}(A)$ with $U \rightarrow S$ mapping into an affine open $\operatorname{Spec}(\Lambda)$ of S . Let $\mathcal{F}$ be the coherent module on $X_A = \operatorname{Spec}(A) \times_S X$ flat over A corresponding to the given object x .

According to Deformation Theory, Lemma 12.1 we have an isomorphism of functors

$$T_x(M) = \operatorname{Ext}_{X_A}^1(\mathcal{F}, \mathcal{F} \otimes_A M)$$

and given any surjection $A' \rightarrow A$ of Λ -algebras with square zero kernel I we have an obstruction class

$$\xi_{A'} \in \operatorname{Ext}_{X_A}^2(\mathcal{F}, \mathcal{F} \otimes_A I)$$

This uses that for any $A' \rightarrow A$ as above the base change $X_{A'} = \text{Spec}(A') \times_B X$ is flat over A' . Moreover, the construction of the obstruction class is functorial in the surjection $A' \rightarrow A$ (for fixed A) by Deformation Theory, Lemma 12.3. Apply Derived Categories of Spaces, Lemma 23.3 to the computation of the Ext groups $\text{Ext}_{X_A}^i(\mathcal{F}, \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes_A^{\mathbf{L}} M) \longrightarrow \text{Ext}_{X_A}^i(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for $i \leq m$ compatible with boundary maps. This object K , together with the displayed identifications above gives us a datum as in Artin's Axioms, Situation 24.2. Finally, condition (iv) of Artin's Axioms, Lemma 24.3 holds by Deformation Theory, Lemma 12.5. Thus Artin's Axioms, Lemma 24.4 does indeed apply and the lemma is proved. $\square$

Second proof. This proof is based on Artin's Axioms, Lemma 22.2. Conditions (1), (2), and (3) of that lemma correspond to Lemmas 5.3, 5.7, and 5.6.

We have constructed an obstruction theory in the chapter on deformation theory. Namely, given an S -algebra A and an object x of $\text{Coh}_{X/B}$ over $\text{Spec}(A)$ given by $\mathcal{F}$ on X_A we set $\mathcal{O}_x(M) = \text{Ext}_{X_A}^2(\mathcal{F}, \mathcal{F} \otimes_A M)$ and if $A' \rightarrow A$ is a surjection with kernel I , then as obstruction element we take the element

$$o_x(A') = o(\mathcal{F}, \mathcal{F} \otimes_A I, 1) \in \mathcal{O}_x(I) = \text{Ext}_{X_A}^2(\mathcal{F}, \mathcal{F} \otimes_A I)$$

of Deformation Theory, Lemma 12.1. All properties of an obstruction theory as defined in Artin's Axioms, Definition 22.1 follow from this lemma except for functoriality of obstruction classes as formulated in condition (ii) of the definition. But as stated in the footnote to assumption (4) of Artin's Axioms, Lemma 22.2 it suffices to check functoriality of obstruction classes for a fixed A which follows from Deformation Theory, Lemma 12.3. Deformation Theory, Lemma 12.1 also tells us that $T_x(M) = \text{Ext}_{X_A}^1(\mathcal{F}, \mathcal{F} \otimes_A M)$ for any A -module M .

To finish the proof it suffices to show that $T_x(\prod M_n) = \prod T_x(M_n)$ and $\mathcal{O}_x(\prod M_n) = \prod \mathcal{O}_x(M_n)$. Apply Derived Categories of Spaces, Lemma 23.3 to the computation of the Ext groups $\text{Ext}_{X_A}^i(\mathcal{F}, \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes_A^{\mathbf{L}} M) \longrightarrow \text{Ext}_{X_A}^i(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for $i = 1, 2$. A straightforward argument shows that

$$H^i(K \otimes_A^{\mathbf{L}} \prod M_n) = \prod H^i(K \otimes_A^{\mathbf{L}} M_n)$$

whenever K is a pseudo-coherent object of $D(A)$. In fact, this property (for all i) characterizes pseudo-coherent complexes, see More on Algebra, Lemma 65.5. $\square$

08WC **Theorem 5.12** (Algebraicity of the stack of coherent sheaves; flat case). *Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . Assume that f is of finite presentation, separated, and flat². Then $\text{Coh}_{X/B}$ is an algebraic stack over S .*

²This assumption is not necessary. See Section 6.

Proof. Set $\mathcal{X} = \text{Coh}_{X/B}$. We have seen that $\mathcal{X}$ is a stack in groupoids over $(\text{Sch}/S)_{fppf}$ with diagonal representable by algebraic spaces (Lemmas 5.4 and 5.3). Hence it suffices to find a scheme W and a surjective and smooth morphism $W \rightarrow \mathcal{X}$.

Let B' be a scheme and let $B' \rightarrow B$ be a surjective étale morphism. Set $X' = B' \times_B X$ and denote $f' : X' \rightarrow B'$ the projection. Then $\mathcal{X}' = \text{Coh}_{X'/B'}$ is equal to the 2-fibre product of $\mathcal{X}$ with the category fibred in sets associated to B' over the category fibred in sets associated to B (Remark 5.5). By the material in Algebraic Stacks, Section 10 the morphism $\mathcal{X}' \rightarrow \mathcal{X}$ is surjective and étale. Hence it suffices to prove the result for $\mathcal{X}'$. In other words, we may assume B is a scheme.

Assume B is a scheme. In this case we may replace S by B , see Algebraic Stacks, Section 19. Thus we may assume $S = B$.

Assume $S = B$. Choose an affine open covering $S = \bigcup U_i$. Denote $\mathcal{X}_i$ the restriction of $\mathcal{X}$ to $(\text{Sch}/U_i)_{fppf}$. If we can find schemes W_i over U_i and surjective smooth morphisms $W_i \rightarrow \mathcal{X}_i$, then we set $W = \coprod W_i$ and we obtain a surjective smooth morphism $W \rightarrow \mathcal{X}$. Thus we may assume $S = B$ is affine.

Assume $S = B$ is affine, say $S = \text{Spec}(\Lambda)$. Write $\Lambda = \text{colim } \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \rightarrow \text{Spec}(\Lambda_i)$ which is of finite presentation, separated, and flat and whose base change to Λ is X . See Limits of Spaces, Lemmas 7.1, 6.9, and 6.12. If we show that $\text{Coh}_{X_i/\text{Spec}(\Lambda_i)}$ is an algebraic stack, then it follows by base change (Remark 5.5 and Algebraic Stacks, Section 19) that $\mathcal{X}$ is an algebraic stack. Thus we may assume that Λ is a finite type $\mathbf{Z}$ -algebra.

Assume $S = B = \text{Spec}(\Lambda)$ is affine of finite type over $\mathbf{Z}$. In this case we will verify conditions (1), (2), (3), (4), and (5) of Artin's Axioms, Lemma 17.1 to conclude that $\mathcal{X}$ is an algebraic stack. Note that Λ is a G-ring, see More on Algebra, Proposition 50.12. Hence all local rings of S are G-rings. Thus (5) holds. By Lemma 5.11 we have that $\mathcal{X}$ satisfies openness of versality, hence (4) holds. To check (2) we have to verify axioms [-1], [0], [1], [2], and [3] of Artin's Axioms, Section 14. We omit the verification of [-1] and axioms [0], [1], [2], [3] correspond respectively to Lemmas 5.4, 5.6, 5.7, 5.9. Condition (3) follows from Lemma 5.10. Finally, condition (1) is Lemma 5.3. This finishes the proof of the theorem. $\square$

6. The stack of coherent sheaves in the non-flat case

08WB In Theorem 5.12 the assumption that $f : X \rightarrow B$ is flat is not necessary. In this section we give a different proof which avoids the flatness assumption and avoids checking openness of versality by using the results in Flatness on Spaces, Section 12 and Artin's Axioms, Section 20.

For a different approach to this problem the reader may wish to consult [Art69] and follow the method discussed in the papers [OS03], [Lie06b], [Ols05], [HR13], [HR10], [Ryd11]. Some of these papers deal with the more general case of the stack of coherent sheaves on an algebraic stack over an algebraic stack and others deal with similar problems in the case of Hilbert stacks or Quot functors. Our strategy will be to show algebraicity of some cases of Hilbert stacks and Quot functors as a consequence of the algebraicity of the stack of coherent sheaves.

09DS **Theorem 6.1** (Algebraicity of the stack of coherent sheaves; general case). *Let S be a scheme. Let $f : X \rightarrow B$ be morphism of algebraic spaces over S . Assume that*

f is of finite presentation and separated. Then $\text{Coh}_{X/B}$ is an algebraic stack over S .

Proof. Only the last step of the proof is different from the proof in the flat case, but we repeat all the arguments here to make sure everything works.

Set $\mathcal{X} = \text{Coh}_{X/B}$. We have seen that $\mathcal{X}$ is a stack in groupoids over $(\text{Sch}/S)_{fppf}$ with diagonal representable by algebraic spaces (Lemmas 5.4 and 5.3). Hence it suffices to find a scheme W and a surjective and smooth morphism $W \rightarrow \mathcal{X}$.

Let B' be a scheme and let $B' \rightarrow B$ be a surjective étale morphism. Set $X' = B' \times_B X$ and denote $f' : X' \rightarrow B'$ the projection. Then $\mathcal{X}' = \text{Coh}_{X'/B'}$ is equal to the 2-fibre product of $\mathcal{X}$ with the category fibred in sets associated to B' over the category fibred in sets associated to B (Remark 5.5). By the material in Algebraic Stacks, Section 10 the morphism $\mathcal{X}' \rightarrow \mathcal{X}$ is surjective and étale. Hence it suffices to prove the result for $\mathcal{X}'$. In other words, we may assume B is a scheme.

Assume B is a scheme. In this case we may replace S by B , see Algebraic Stacks, Section 19. Thus we may assume $S = B$.

Assume $S = B$. Choose an affine open covering $S = \bigcup U_i$. Denote $\mathcal{X}_i$ the restriction of $\mathcal{X}$ to $(\text{Sch}/U_i)_{fppf}$. If we can find schemes W_i over U_i and surjective smooth morphisms $W_i \rightarrow \mathcal{X}_i$, then we set $W = \coprod W_i$ and we obtain a surjective smooth morphism $W \rightarrow \mathcal{X}$. Thus we may assume $S = B$ is affine.

Assume $S = B$ is affine, say $S = \text{Spec}(\Lambda)$. Write $\Lambda = \text{colim } \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \rightarrow \text{Spec}(\Lambda_i)$ which is separated and of finite presentation and whose base change to Λ is X . See Limits of Spaces, Lemmas 7.1 and 6.9. If we show that $\text{Coh}_{X_i/\text{Spec}(\Lambda_i)}$ is an algebraic stack, then it follows by base change (Remark 5.5 and Algebraic Stacks, Section 19) that $\mathcal{X}$ is an algebraic stack. Thus we may assume that Λ is a finite type $\mathbf{Z}$ -algebra.

Assume $S = B = \text{Spec}(\Lambda)$ is affine of finite type over $\mathbf{Z}$. In this case we will verify conditions (1), (2), (3), (4), and (5) of Artin's Axioms, Lemma 17.1 to conclude that $\mathcal{X}$ is an algebraic stack. Note that Λ is a G-ring, see More on Algebra, Proposition 50.12. Hence all local rings of S are G-rings. Thus (5) holds. To check (2) we have to verify axioms [-1], [0], [1], [2], and [3] of Artin's Axioms, Section 14. We omit the verification of [-1] and axioms [0], [1], [2], [3] correspond respectively to Lemmas 5.4, 5.6, 5.7, 5.9. Condition (3) is Lemma 5.10. Condition (1) is Lemma 5.3.

It remains to show condition (4) which is openness of versality. To see this we will use Artin's Axioms, Lemma 20.3. We have already seen that $\mathcal{X}$ has diagonal representable by algebraic spaces, has (RS*), and is limit preserving (see lemmas used above). Hence we only need to see that $\mathcal{X}$ satisfies the strong formal effectiveness formulated in Artin's Axioms, Lemma 20.3. This is Flatness on Spaces, Theorem 12.8 and the proof is complete. $\square$

7. The functor of quotients

082L In this section we discuss some generalities regarding the functor $Q_{\mathcal{F}/X/B}$ defined below. The notation $\text{Quot}_{\mathcal{F}/X/B}$ is reserved for a subfunctor of $Q_{\mathcal{F}/X/B}$. We urge the reader to skip this section on a first reading.

082M **Situation 7.1.** Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ -module. For any scheme T over B we will denote X_T the base change of X to T and $\mathcal{F}_T$ the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_B T \rightarrow X$. Given such a T we set

$$Q_{\mathcal{F}/X/B}(T) = \left\{ \begin{array}{l} \text{quotients } \mathcal{F}_T \rightarrow \mathcal{Q} \text{ where } \mathcal{Q} \text{ is a} \\ \text{quasi-coherent } \mathcal{O}_{X_T}\text{-module flat over } T \end{array} \right\}$$

We identify quotients if they have the same kernel. Suppose that $T' \rightarrow T$ is a morphism of schemes over B and $\mathcal{F}_T \rightarrow \mathcal{Q}$ is an element of $Q_{\mathcal{F}/X/B}(T)$. Then the pullback $\mathcal{Q}' = (X_{T'} \rightarrow X_T)^* \mathcal{Q}$ is a quasi-coherent $\mathcal{O}_{X_{T'}}$ -module flat over T' by Morphisms of Spaces, Lemma 31.3. Thus we obtain a functor

082N (7.1.1) $Q_{\mathcal{F}/X/B} : (Sch/B)^{opp} \longrightarrow Sets$

This is the *functor of quotients of $\mathcal{F}/X/B$* . We define a subfunctor

0CZL (7.1.2) $Q_{\mathcal{F}/X/B}^{fp} : (Sch/B)^{opp} \longrightarrow Sets$

which assigns to T the subset of $Q_{\mathcal{F}/X/B}(T)$ consisting of those quotients $\mathcal{F}_T \rightarrow \mathcal{Q}$ such that $\mathcal{Q}$ is of finite presentation as an $\mathcal{O}_{X_T}$ -module. This is a subfunctor by Properties of Spaces, Section 30.

In Situation 7.1 we sometimes think of $Q_{\mathcal{F}/X/B}$ as a functor $(Sch/S)^{opp} \rightarrow Sets$ endowed with a morphism $Q_{\mathcal{F}/X/S} \rightarrow B$. Namely, if T is a scheme over S , then an element of $Q_{\mathcal{F}/X/B}(T)$ is a pair $(h, \mathcal{Q})$ where h a morphism $h : T \rightarrow B$ and $\mathcal{Q}$ is a T -flat quotient $\mathcal{F}_T \rightarrow \mathcal{Q}$ of finite presentation on $X_T = X \times_{B,h} T$. In particular, when we say that $Q_{\mathcal{F}/X/S}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \rightarrow Sets$ is an algebraic space. Similar remarks apply to $Q_{\mathcal{F}/X/B}^{fp}$.

08IT **Remark 7.2.** In Situation 7.1 let $B' \rightarrow B$ be a morphism of algebraic spaces over S . Set $X' = X \times_B B'$ and denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to X' . Thus we have the functor $Q_{\mathcal{F}'/X'/B'}$ on the category of schemes over B' . For a scheme T over B' it is clear that we have

$$Q_{\mathcal{F}'/X'/B'}(T) = Q_{\mathcal{F}/X/B}(T)$$

where on the right hand side we think of T as a scheme over B via the composition $T \rightarrow B' \rightarrow B$. Similar remarks apply to $Q_{\mathcal{F}/X/B}^{fp}$. These trivial remarks will occasionally be useful to change the base algebraic space.

08IU **Remark 7.3.** Let S be a scheme, X an algebraic space over S , and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$ -module. Suppose that $\{f_i : X_i \rightarrow X\}_{i \in I}$ is an fpqc covering and for each $i, j \in I$ we are given an fpqc covering $\{X_{ijk} \rightarrow X_i \times_X X_j\}$. In this situation we have a bijection

$$\left\{ \begin{array}{l} \text{quotients } \mathcal{F} \rightarrow \mathcal{Q} \text{ where} \\ \mathcal{Q} \text{ is a quasi-coherent} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{families of quotients } f_i^* \mathcal{F} \rightarrow \mathcal{Q}_i \text{ where} \\ \mathcal{Q}_i \text{ is quasi-coherent and } \mathcal{Q}_i \text{ and } \mathcal{Q}_j \\ \text{restrict to the same quotient on } X_{ijk} \end{array} \right\}$$

Namely, let $(f_i^* \mathcal{F} \rightarrow \mathcal{Q}_i)_{i \in I}$ be an element of the right hand side. Then since $\{X_{ijk} \rightarrow X_i \times_X X_j\}$ is an fpqc covering we see that the pullbacks of $\mathcal{Q}_i$ and $\mathcal{Q}_j$ restrict to the same quotient of the pullback of $\mathcal{F}$ to $X_i \times_X X_j$ (by fully faithfulness in Descent on Spaces, Proposition 4.1). Hence we obtain a descent datum for quasi-coherent modules with respect to $\{X_i \rightarrow X\}_{i \in I}$. By Descent on Spaces, Proposition 4.1 we find a map of quasi-coherent $\mathcal{O}_X$ -modules $\mathcal{F} \rightarrow \mathcal{Q}$ whose restriction to X_i

recovers the given maps $f_i^* \mathcal{F} \rightarrow \mathcal{Q}_i$. Since the family of morphisms $\{X_i \rightarrow X\}$ is jointly surjective and flat, for every point $x \in |X|$ there exists an i and a point $x_i \in |X_i|$ mapping to x . Note that the induced map on local rings $\mathcal{O}_{X, \bar{x}} \rightarrow \mathcal{O}_{X_i, \bar{x}_i}$ is faithfully flat, see Morphisms of Spaces, Section 30. Thus we see that $\mathcal{F} \rightarrow \mathcal{Q}$ is surjective.

082P **Lemma 7.4.** *In Situation 7.1. The functors $Q_{\mathcal{F}/X/B}$ and $Q_{\mathcal{F}/X/B}^{fp}$ satisfy the sheaf property for the fpqc topology.*

Proof. Let $\{T_i \rightarrow T\}_{i \in I}$ be an fpqc covering of schemes over S . Set $X_i = X_{T_i} = X \times_S T_i$ and $\mathcal{F}_i = \mathcal{F}_{T_i}$. Note that $\{X_i \rightarrow X_T\}_{i \in I}$ is an fpqc covering of X_T (Topologies on Spaces, Lemma 9.3) and that $X_{T_i \times_T T_{i'}} = X_i \times_{X_T} X_{i'}$. Suppose that $\mathcal{F}_i \rightarrow \mathcal{Q}_i$ is a collection of elements of $Q_{\mathcal{F}/X/B}(T_i)$ such that $\mathcal{Q}_i$ and $\mathcal{Q}_{i'}$ restrict to the same element of $Q_{\mathcal{F}/X/B}(T_i \times_T T_{i'})$. By Remark 7.3 we obtain a surjective map of quasi-coherent $\mathcal{O}_{X_T}$ -modules $\mathcal{F}_T \rightarrow \mathcal{Q}$ whose restriction to X_i recovers the given quotients. By Morphisms of Spaces, Lemma 31.5 we see that $\mathcal{Q}$ is flat over T . Finally, in the case of $Q_{\mathcal{F}/X/B}^{fp}$, i.e., if $\mathcal{Q}_i$ are of finite presentation, then Descent on Spaces, Lemma 6.2 guarantees that $\mathcal{Q}$ is of finite presentation as an $\mathcal{O}_{X_T}$ -module. $\square$

Sanity check: $Q_{\mathcal{F}/X/B}$, $Q_{\mathcal{F}/X/B}^{fp}$ play the same role among algebraic spaces over S .

0D3U **Lemma 7.5.** *In Situation 7.1. Let T be an algebraic space over S . We have*

$$\mathrm{Mor}_{\mathrm{Sh}((\mathrm{Sch}/S)_{\mathrm{fpf}})}(T, Q_{\mathcal{F}/X/B}) = \left\{ (h, \mathcal{F}_T \rightarrow \mathcal{Q}) \text{ where } h : T \rightarrow B \text{ and } \mathcal{Q} \text{ is quasi-coherent and flat over } T \right\}$$

where $\mathcal{F}_T$ denotes the pullback of $\mathcal{F}$ to the algebraic space $X \times_{B,h} T$. Similarly, we have

$$\mathrm{Mor}_{\mathrm{Sh}((\mathrm{Sch}/S)_{\mathrm{fpf}})}(T, Q_{\mathcal{F}/X/B}^{fp}) = \left\{ (h, \mathcal{F}_T \rightarrow \mathcal{Q}) \text{ where } h : T \rightarrow B \text{ and } \mathcal{Q} \text{ is of finite presentation and flat over } T \right\}$$

Proof. Choose a scheme U and a surjective étale morphism $p : U \rightarrow T$. Let $R = U \times_T U$ with projections $t, s : R \rightarrow U$.

Let $v : T \rightarrow Q_{\mathcal{F}/X/B}$ be a natural transformation. Then $v(p)$ corresponds to a pair $(h_U, \mathcal{F}_U \rightarrow \mathcal{Q}_U)$ over U . As v is a transformation of functors we see that the pullbacks of $(h_U, \mathcal{F}_U \rightarrow \mathcal{Q}_U)$ by s and t agree. Since $T = U/R$ (Spaces, Lemma 9.1), we obtain a morphism $h : T \rightarrow B$ such that $h_U = h \circ p$. By Descent on Spaces, Proposition 4.1 the quotient $\mathcal{Q}_U$ descends to a quotient $\mathcal{F}_T \rightarrow \mathcal{Q}$ over X_T . Since $U \rightarrow T$ is surjective and flat, it follows from Morphisms of Spaces, Lemma 31.5 that $\mathcal{Q}$ is flat over T .

Conversely, let $(h, \mathcal{F}_T \rightarrow \mathcal{Q})$ be a pair over T . Then we get a natural transformation $v : T \rightarrow Q_{\mathcal{F}/X/B}$ by sending a morphism $a : T' \rightarrow T$ where T' is a scheme to $(h \circ a, \mathcal{F}_{T'} \rightarrow a^* \mathcal{Q})$. We omit the verification that the construction of this and the previous paragraph are mutually inverse.

In the case of $Q_{\mathcal{F}/X/B}^{fp}$ we add: given a morphism $h : T \rightarrow B$, a quasi-coherent sheaf on X_T is of finite presentation as an $\mathcal{O}_{X_T}$ -module if and only if the pullback to X_U is of finite presentation as an $\mathcal{O}_{X_U}$ -module. This follows from the fact that $X_U \rightarrow X_T$ is surjective and étale and Descent on Spaces, Lemma 6.2. $\square$

08IV **Lemma 7.6.** *In Situation 7.1 let $\{X_i \rightarrow X\}_{i \in I}$ be an fpqc covering and for each $i, j \in I$ let $\{X_{ijk} \rightarrow X_i \times_X X_j\}$ be an fpqc covering. Denote $\mathcal{F}_i$, resp. $\mathcal{F}_{ijk}$ the pullback of $\mathcal{F}$ to X_i , resp. X_{ijk} . For every scheme T over B the diagram*

$$Q_{\mathcal{F}/X/B}(T) \longrightarrow \prod_i Q_{\mathcal{F}_i/X_i/B}(T) \begin{array}{c} \xrightarrow{pr_0^*} \\ \xrightarrow{pr_1^*} \end{array} \prod_{i,j,k} Q_{\mathcal{F}_{ijk}/X_{ijk}/B}(T)$$

presents the first arrow as the equalizer of the other two. The same is true for the functor $Q_{\mathcal{F}/X/B}^{fp}$.

Proof. Let $\mathcal{F}_{i,T} \rightarrow \mathcal{Q}_i$ be an element in the equalizer of pr_0^* and pr_1^* . By Remark 7.3 we obtain a surjection $\mathcal{F}_T \rightarrow \mathcal{Q}$ of quasi-coherent $\mathcal{O}_{X_T}$ -modules whose restriction to $X_{i,T}$ recovers $\mathcal{F}_i \rightarrow \mathcal{Q}_i$. By Morphisms of Spaces, Lemma 31.5 we see that $\mathcal{Q}$ is flat over T as desired. In the case of the functor $Q_{\mathcal{F}/X/B}^{fp}$, i.e., if $\mathcal{Q}_i$ is of finite presentation, then $\mathcal{Q}$ is of finite presentation too by Descent on Spaces, Lemma 6.2. $\square$

082Q **Lemma 7.7.** *In Situation 7.1 assume also that (a) f is quasi-compact and quasi-separated and (b) $\mathcal{F}$ is of finite presentation. Then the functor $Q_{\mathcal{F}/X/B}^{fp}$ is limit preserving in the following sense: If $T = \lim T_i$ is a directed limit of affine schemes over B , then $Q_{\mathcal{F}/X/B}^{fp}(T) = \text{colim } Q_{\mathcal{F}/X/B}^{fp}(T_i)$.*

Proof. Let $T = \lim T_i$ be as in the statement of the lemma. Choose $i_0 \in I$ and replace I by $\{i \in I \mid i \geq i_0\}$. We may set $B = S = T_{i_0}$ and we may replace X by X_{T_0} and $\mathcal{F}$ by the pullback to X_{T_0} . Then $X_T = \lim X_{T_i}$, see Limits of Spaces, Lemma 4.1. Let $\mathcal{F}_T \rightarrow \mathcal{Q}$ be an element of $Q_{\mathcal{F}/X/B}^{fp}(T)$. By Limits of Spaces, Lemma 7.2 there exists an i and a map $\mathcal{F}_{T_i} \rightarrow \mathcal{Q}_i$ of $\mathcal{O}_{X_{T_i}}$ -modules of finite presentation whose pullback to X_T is the given quotient map.

We still have to check that, after possibly increasing i , the map $\mathcal{F}_{T_i} \rightarrow \mathcal{Q}_i$ is surjective and $\mathcal{Q}_i$ is flat over T_i . To do this, choose an affine scheme U and a surjective étale morphism $U \rightarrow X$ (see Properties of Spaces, Lemma 6.3). We may check surjectivity and flatness over T_i after pulling back to the étale cover $U_{T_i} \rightarrow X_{T_i}$ (by definition). This reduces us to the case where $X = \text{Spec}(B_0)$ is an affine scheme of finite presentation over $B = S = T_0 = \text{Spec}(A_0)$. Writing $T_i = \text{Spec}(A_i)$, then $T = \text{Spec}(A)$ with $A = \text{colim } A_i$ we have reached the following algebra problem. Let $M_i \rightarrow N_i$ be a map of finitely presented $B_0 \otimes_{A_0} A_i$ -modules such that $M_i \otimes_{A_i} A \rightarrow N_i \otimes_{A_i} A$ is surjective and $N_i \otimes_{A_i} A$ is flat over A . Show that for some $i' \geq i$ $M_{i'} \otimes_{A_{i'}} A \rightarrow N_{i'} \otimes_{A_{i'}} A$ is surjective and $N_{i'} \otimes_{A_{i'}} A$ is flat over A . The first follows from Algebra, Lemma 127.5 and the second from Algebra, Lemma 168.1. $\square$

08IW **Lemma 7.8.** *In Situation 7.1. Let*

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout in the category of schemes over B where $Z \rightarrow Z'$ is a thickening and $Z \rightarrow Y$ is affine, see *More on Morphisms*, Lemma 14.3. Then the natural map

$$Q_{\mathcal{F}/X/B}(Y') \longrightarrow Q_{\mathcal{F}/X/B}(Y) \times_{Q_{\mathcal{F}/X/B}(Z)} Q_{\mathcal{F}/X/B}(Z')$$

is bijective. If $X \rightarrow B$ is locally of finite presentation, then the same thing is true for $Q_{\mathcal{F}/X/B}^{fp}$.

Proof. Let us construct an inverse map. Namely, suppose we have $\mathcal{F}_Y \rightarrow \mathcal{A}$, $\mathcal{F}_{Z'} \rightarrow \mathcal{B}'$, and an isomorphism $\mathcal{A}|_{X_Z} \rightarrow \mathcal{B}'|_{X_Z}$ compatible with the given surjections. Then we apply Pushouts of Spaces, Lemma 6.6 to get a quasi-coherent module $\mathcal{A}'$ on $X_{Y'}$ flat over Y' . Since this sheaf is constructed as a fibre product (see proof of cited lemma) there is a canonical map $\mathcal{F}_{Y'} \rightarrow \mathcal{A}'$. That this map is surjective can be seen because it factors as

$$\begin{array}{c} \mathcal{F}_{Y'} \\ \downarrow \\ (X_Y \rightarrow X_{Y'})_* \mathcal{F}_Y \times_{(X_Z \rightarrow X_{Y'})_* \mathcal{F}_Z} (X_{Z'} \rightarrow X_{Y'})_* \mathcal{F}_{Z'} \\ \downarrow \\ \mathcal{A}' = (X_Y \rightarrow X_{Y'})_* \mathcal{A} \times_{(X_Z \rightarrow X_{Y'})_* \mathcal{A}|_{X_Z}} (X_{Z'} \rightarrow X_{Y'})_* \mathcal{B}' \end{array}$$

and the first arrow is surjective by *More on Algebra*, Lemma 6.5 and the second by *More on Algebra*, Lemma 6.6.

In the case of $Q_{\mathcal{F}/X/B}^{fp}$ all we have to show is that the construction above produces a finitely presented module. This is explained in *More on Algebra*, Remark 7.8 in the commutative algebra setting. The current case of modules over algebraic spaces follows from this by étale localization. $\square$

0CZU Remark 7.9 (Obstructions for quotients). In Situation 7.1 **assume** that $\mathcal{F}$ is flat over B . Let $T \subset T'$ be a first order thickening of schemes over B with ideal sheaf $\mathcal{J}$. Then $X_T \subset X_{T'}$ is a first order thickening of algebraic spaces whose ideal sheaf $\mathcal{I}$ is a quotient of $f_T^* \mathcal{J}$. We will think of sheaves on $X_{T'}$, resp. T' as sheaves on X_T , resp. T using the fundamental equivalence described in *More on Morphisms of Spaces*, Section 9. Let

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{F}_T \rightarrow \mathcal{Q} \rightarrow 0$$

define an element x of $Q_{\mathcal{F}/X/B}(T)$. Since $\mathcal{F}_{T'}$ is flat over T' we have a short exact sequence

$$0 \rightarrow f_T^* \mathcal{J} \otimes_{\mathcal{O}_{X_T}} \mathcal{F}_T \xrightarrow{i} \mathcal{F}_{T'} \xrightarrow{\pi} \mathcal{F}_T \rightarrow 0$$

and we have $f_T^* \mathcal{J} \otimes_{\mathcal{O}_{X_T}} \mathcal{F}_T = \mathcal{I} \otimes_{\mathcal{O}_{X_T}} \mathcal{F}_T$, see *Deformation Theory*, Lemma 11.2. Let us use the abbreviation $f_T^* \mathcal{J} \otimes_{\mathcal{O}_{X_T}} \mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_T} \mathcal{J}$ for an $\mathcal{O}_{X_T}$ -module $\mathcal{G}$. Since $\mathcal{Q}$ is flat over T , we obtain a short exact sequence

$$0 \rightarrow \mathcal{K} \otimes_{\mathcal{O}_T} \mathcal{J} \rightarrow \mathcal{F}_T \otimes_{\mathcal{O}_T} \mathcal{J} \rightarrow \mathcal{Q} \otimes_{\mathcal{O}_T} \mathcal{J} \rightarrow 0$$

Combining the above we obtain an canonical extension

$$0 \rightarrow \mathcal{Q} \otimes_{\mathcal{O}_T} \mathcal{J} \rightarrow \pi^{-1}(\mathcal{K})/i(\mathcal{K} \otimes_{\mathcal{O}_T} \mathcal{J}) \rightarrow \mathcal{K} \rightarrow 0$$

of $\mathcal{O}_{X_T}$ -modules. This defines a canonical class

$$o_x(T') \in \text{Ext}_{\mathcal{O}_{X_T}}^1(\mathcal{K}, \mathcal{Q} \otimes_{\mathcal{O}_T} \mathcal{J})$$

If $o_x(T')$ is zero, then we obtain a splitting of the short exact sequence defining it, in other words, we obtain a $\mathcal{O}_{X_{T'}}$ -submodule $\mathcal{K}' \subset \pi^{-1}(\mathcal{K})$ sitting in a short exact sequence $0 \rightarrow \mathcal{K} \otimes_{\mathcal{O}_T} \mathcal{J} \rightarrow \mathcal{K}' \rightarrow \mathcal{K} \rightarrow 0$. Then it follows from the lemma reference

above that $\mathcal{Q}' = \mathcal{F}_{T'}/\mathcal{K}'$ is a lift of x to an element of $Q_{\mathcal{F}/X/B}(T')$. Conversely, the reader sees that the existence of a lift implies that $o_x(T')$ is zero. Moreover, if $x \in Q_{\mathcal{F}/X/B}^{fp}(T)$, then automatically $x' \in Q_{\mathcal{F}/X/B}^{fp}(T')$ by Deformation Theory, Lemma 11.3. If we ever need this remark we will turn this remark into a lemma, precisely formulate the result and give a detailed proof (in fact, all of the above works in the setting of arbitrary ringed topoi).

0CZV **Remark 7.10** (Deformations of quotients). In Situation 7.1 **assume** that $\mathcal{F}$ is flat over B . We continue the discussion of Remark 7.9. Assume $o_x(T') = 0$. Then we claim that the set of lifts $x' \in Q_{\mathcal{F}/X/B}(T')$ is a principal homogeneous space under the group

$$\mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{K}, \mathcal{Q} \otimes_{\mathcal{O}_T} \mathcal{J})$$

Namely, given any $\mathcal{F}_{T'} \rightarrow \mathcal{Q}'$ flat over T' lifting the quotient $\mathcal{Q}$ we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{K} \otimes \mathcal{J} & \longrightarrow & \mathcal{F}_T \otimes \mathcal{J} & \longrightarrow & \mathcal{Q} \otimes \mathcal{J} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{F}_{T'} & \longrightarrow & \mathcal{Q}' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}_T & \longrightarrow & \mathcal{Q} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

(to see this use the observations made in the previous remark). Given a map $\varphi : \mathcal{K} \rightarrow \mathcal{Q} \otimes \mathcal{J}$ we can consider the subsheaf $\mathcal{K}'_\varphi \subset \mathcal{F}_{T'}$ consisting of those local sections s whose image in $\mathcal{F}_T$ is a local section k of $\mathcal{K}$ and whose image in $\mathcal{Q}'$ is the local section $\varphi(k)$ of $\mathcal{Q} \otimes \mathcal{J}$. Then set $\mathcal{Q}'_\varphi = \mathcal{F}_{T'}/\mathcal{K}'_\varphi$. Conversely, any second lift of x corresponds to one of the quotients constructed in this manner. If we ever need this remark we will turn this remark into a lemma, precisely formulate the result and give a detailed proof (in fact, all of the above works in the setting of arbitrary ringed topoi).

8. The Quot functor

09TQ In this section we prove the Quot functor is an algebraic space.

09TR **Situation 8.1.** Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . Assume that f is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ -module. For any scheme T over B we will denote X_T the base change of X to T and $\mathcal{F}_T$ the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \rightarrow X$. Given such a T we set

$$\mathrm{Quot}_{\mathcal{F}/X/B}(T) = \left\{ \begin{array}{l} \text{quotients } \mathcal{F}_T \rightarrow \mathcal{Q} \text{ where } \mathcal{Q} \text{ is a quasi-coherent} \\ \mathcal{O}_{X_T}\text{-module of finite presentation, flat over } T \\ \text{with support proper over } T \end{array} \right\}$$

By Derived Categories of Spaces, Lemma 7.8 this is a subfunctor of the functor $Q_{\mathcal{F}/X/B}^{fp}$ we discussed in Section 7. Thus we obtain a functor

$$09TS \quad (8.1.1) \quad \text{Quot}_{\mathcal{F}/X/B} : (Sch/B)^{opp} \longrightarrow Sets$$

This is the *Quot functor* associated to $\mathcal{F}/X/B$.

In Situation 8.1 we sometimes think of $\text{Quot}_{\mathcal{F}/X/B}$ as a functor $(Sch/S)^{opp} \rightarrow Sets$ endowed with a morphism $\text{Quot}_{\mathcal{F}/X/B} \rightarrow B$. Namely, if T is a scheme over S , then an element of $\text{Quot}_{\mathcal{F}/X/B}(T)$ is a pair $(h, \mathcal{Q})$ where h is a morphism $h : T \rightarrow B$ and $\mathcal{Q}$ is a finitely presented, T -flat quotient $\mathcal{F}_T \rightarrow \mathcal{Q}$ on $X_T = X \times_{B,h} T$ with support proper over T . In particular, when we say that $\text{Quot}_{\mathcal{F}/X/B}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \rightarrow Sets$ is an algebraic space.

09TT **Lemma 8.2.** *In Situation 8.1. The functor $\text{Quot}_{\mathcal{F}/X/B}$ satisfies the sheaf property for the fpqc topology.*

Proof. In Lemma 7.4 we have seen that the functor $Q_{\mathcal{F}/X/S}^{fp}$ is a sheaf. Recall that for a scheme T over S the subset $\text{Quot}_{\mathcal{F}/X/S}(T) \subset Q_{\mathcal{F}/X/S}(T)$ picks out those quotients whose support is proper over T . This defines a subsheaf by the result of Descent on Spaces, Lemma 11.19 combined with Morphisms of Spaces, Lemma 30.10 which shows that taking scheme theoretic support commutes with flat base change. $\square$

Sanity check: $\text{Quot}_{\mathcal{F}/X/B}$ plays the same role among algebraic spaces over S .

0D3V **Lemma 8.3.** *In Situation 8.1. Let T be an algebraic space over S . We have*

$$\text{Mor}_{Sh((Sch/S)_{fppf})}(T, \text{Quot}_{\mathcal{F}/X/B}) = \left\{ \begin{array}{l} (h, \mathcal{F}_T \rightarrow \mathcal{Q}) \text{ where } h : T \rightarrow B \text{ and} \\ \mathcal{Q} \text{ is of finite presentation and} \\ \text{flat over } T \text{ with support proper over } T \end{array} \right\}$$

where $\mathcal{F}_T$ denotes the pullback of $\mathcal{F}$ to the algebraic space $X \times_{B,h} T$.

Proof. Observe that the left and right hand side of the equality are subsets of the left and right hand side of the second equality in Lemma 7.5. To see that these subsets correspond under the identification given in the proof of that lemma it suffices to show: given $h : T \rightarrow B$, a surjective étale morphism $U \rightarrow T$, a finite type quasi-coherent $\mathcal{O}_{X_T}$ -module $\mathcal{Q}$ the following are equivalent

- (1) the scheme theoretic support of $\mathcal{Q}$ is proper over T , and
- (2) the scheme theoretic support of $(X_U \rightarrow X_T)^* \mathcal{Q}$ is proper over U .

This follows from Descent on Spaces, Lemma 11.19 combined with Morphisms of Spaces, Lemma 30.10 which shows that taking scheme theoretic support commutes with flat base change. $\square$

09TU **Proposition 8.4.** *Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . Let $\mathcal{F}$ be a quasi-coherent sheaf on X . If f is of finite presentation and separated, then $\text{Quot}_{\mathcal{F}/X/B}$ is an algebraic space. If $\mathcal{F}$ is of finite presentation, then $\text{Quot}_{\mathcal{F}/X/B} \rightarrow B$ is locally of finite presentation.*

Proof. By Lemma 8.2 we have that $\text{Quot}_{\mathcal{F}/X/B}$ is a sheaf in the fppf topology. Let $\text{Quot}_{\mathcal{F}/X/B}$ be the stack in groupoids corresponding to $\text{Quot}_{\mathcal{F}/X/S}$, see Algebraic

Stacks, Section 7. By Algebraic Stacks, Proposition 13.3 it suffices to show that $Quot_{\mathcal{F}/X/B}$ is an algebraic stack. Consider the 1-morphism of stacks in groupoids

$$Quot_{\mathcal{F}/X/S} \longrightarrow Coh_{X/B}$$

on $(Sch/S)_{fppf}$ which associates to the quotient $\mathcal{F}_T \rightarrow \mathcal{Q}$ the module $\mathcal{Q}$. By Theorem 6.1 we know that $Coh_{X/B}$ is an algebraic stack. By Algebraic Stacks, Lemma 15.4 it suffices to show that this 1-morphism is representable by algebraic spaces.

Let T be a scheme over S and let the object $(h, \mathcal{G})$ of $Coh_{X/B}$ over T correspond to a 1-morphism $\xi : (Sch/T)_{fppf} \rightarrow Coh_{X/B}$. The 2-fibre product

$$\mathcal{Z} = (Sch/T)_{fppf} \times_{\xi, Coh_{X/B}} Quot_{\mathcal{F}/X/S}$$

is a stack in setoids, see Stacks, Lemma 6.7. The corresponding sheaf of sets (i.e., functor, see Stacks, Lemmas 6.7 and 6.2) assigns to a scheme T'/T the set of surjections $u : \mathcal{F}_{T'} \rightarrow \mathcal{G}_{T'}$ of quasi-coherent modules on $X_{T'}$. Thus we see that $\mathcal{Z}$ is representable by an open subspace (by Flatness on Spaces, Lemma 9.3) of the algebraic space $Hom(\mathcal{F}_T, \mathcal{G})$ from Proposition 3.10. $\square$

0CZW Remark 8.5 (Quot via Artin's axioms). Let S be a Noetherian scheme all of whose local rings are G-rings. Let X be an algebraic space over S whose structure morphism $f : X \rightarrow S$ is of finite presentation and separated. Let $\mathcal{F}$ be a finitely presented quasi-coherent sheaf on X flat over S . In this remark we sketch how one can use Artin's axioms to prove that $Quot_{\mathcal{F}/X/S}$ is an algebraic space locally of finite presentation over S and avoid using the algebraicity of the stack of coherent sheaves as was done in the proof of Proposition 8.4.

We check the conditions listed in Artin's Axioms, Proposition 16.1. Representability of the diagonal of $Quot_{\mathcal{F}/X/S}$ can be seen as follows: suppose we have two quotients $\mathcal{F}_T \rightarrow \mathcal{Q}_i$, $i = 1, 2$. Denote $\mathcal{K}_1$ the kernel of the first one. Then we have to show that the locus of T over which $u : \mathcal{K}_1 \rightarrow \mathcal{Q}_2$ becomes zero is representable. This follows for example from Flatness on Spaces, Lemma 8.6 or from a discussion of the Hom sheaf earlier in this chapter. Axioms [0] (sheaf), [1] (limits), [2] (Rim-Schlessinger) follow from Lemmas 8.2, 7.7, and 7.8 (plus some extra work to deal with the properness condition). Axiom [3] (finite dimensionality of tangent spaces) follows from the description of the infinitesimal deformations in Remark 7.10 and finiteness of cohomology of coherent sheaves on proper algebraic spaces over fields (Cohomology of Spaces, Lemma 20.2). Axiom [4] (effectiveness of formal objects) follows from Grothendieck's existence theorem (More on Morphisms of Spaces, Theorem 42.11). As usual, the trickiest to verify is axiom [5] (openness of versality). One can for example use the obstruction theory described in Remark 7.9 and the description of deformations in Remark 7.10 to do this using the criterion in Artin's Axioms, Lemma 22.2. Please compare with the second proof of Lemma 5.11.

9. The Hilbert functor

0CZX In this section we prove the Hilb functor is an algebraic space.

0CZY Situation 9.1. Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . Assume that f is of finite presentation. For any scheme T over B

we will denote X_T the base change of X to T . Given such a T we set

$$\mathrm{Hilb}_{X/B}(T) = \left\{ \begin{array}{l} \text{closed subspaces } Z \subset X_T \text{ such that } Z \rightarrow T \\ \text{is of finite presentation, flat, and proper} \end{array} \right\}$$

Since base change preserves the required properties (Spaces, Lemma 12.3 and Morphisms of Spaces, Lemmas 28.3, 30.4, and 40.3) we obtain a functor

$$0CZZ \quad (9.1.1) \quad \mathrm{Hilb}_{X/B} : (\mathrm{Sch}/B)^{opp} \longrightarrow \mathrm{Sets}$$

This is the *Hilbert functor* associated to X/B .

In Situation 9.1 we sometimes think of $\mathrm{Hilb}_{X/B}$ as a functor $(\mathrm{Sch}/S)^{opp} \rightarrow \mathrm{Sets}$ endowed with a morphism $\mathrm{Hilb}_{X/S} \rightarrow B$. Namely, if T is a scheme over S , then an element of $\mathrm{Hilb}_{X/B}(T)$ is a pair (h, Z) where h is a morphism $h : T \rightarrow B$ and $Z \subset X_T = X \times_{B,h} T$ is a closed subscheme, flat, proper, and of finite presentation over T . In particular, when we say that $\mathrm{Hilb}_{X/B}$ is an algebraic space, we mean that the corresponding functor $(\mathrm{Sch}/S)^{opp} \rightarrow \mathrm{Sets}$ is an algebraic space.

Of course the Hilbert functor is just a special case of the Quot functor.

0D00 **Lemma 9.2.** *In Situation 9.1 we have $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$.*

Proof. Let T be a scheme over B . Given an element $Z \in \mathrm{Hilb}_{X/B}(T)$ we can consider the quotient $\mathcal{O}_{X_T} \rightarrow i_*\mathcal{O}_Z$ where $i : Z \rightarrow X_T$ is the inclusion morphism. Note that $i_*\mathcal{O}_Z$ is quasi-coherent. Since $Z \rightarrow T$ and $X_T \rightarrow T$ are of finite presentation, we see that i is of finite presentation (Morphisms of Spaces, Lemma 28.9), hence $i_*\mathcal{O}_Z$ is an $\mathcal{O}_{X_T}$ -module of finite presentation (Descent on Spaces, Lemma 6.7). Since $Z \rightarrow T$ is proper we see that $i_*\mathcal{O}_Z$ has support proper over T (as defined in Derived Categories of Spaces, Section 7). Since $\mathcal{O}_Z$ is flat over T and i is affine, we see that $i_*\mathcal{O}_Z$ is flat over T (small argument omitted). Hence $\mathcal{O}_{X_T} \rightarrow i_*\mathcal{O}_Z$ is an element of $\mathrm{Quot}_{\mathcal{O}_X/X/B}(T)$.

Conversely, given an element $\mathcal{O}_{X_T} \rightarrow \mathcal{Q}$ of $\mathrm{Quot}_{\mathcal{O}_X/X/B}(T)$, we can consider the closed immersion $i : Z \rightarrow X_T$ corresponding to the quasi-coherent ideal sheaf $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{X_T} \rightarrow \mathcal{Q})$ (Morphisms of Spaces, Lemma 13.1). By construction of Z we see that $\mathcal{Q} = i_*\mathcal{O}_Z$. Then we can read the arguments given above backwards to see that Z defines an element of $\mathrm{Hilb}_{X/B}(T)$. For example, $\mathcal{I}$ is quasi-coherent of finite type (Modules on Sites, Lemma 24.1) hence $i : Z \rightarrow X_T$ is of finite presentation (Morphisms of Spaces, Lemma 28.12) hence $Z \rightarrow T$ is of finite presentation (Morphisms of Spaces, Lemma 28.2). Properness of $Z \rightarrow T$ follows from the discussion in Derived Categories of Spaces, Section 7. Flatness of $Z \rightarrow T$ follows from flatness of $\mathcal{Q}$ over T .

We omit the (immediate) verification that the two constructions given above are mutually inverse. $\square$

Sanity check: $\mathrm{Hilb}_{X/B}$ sheaf plays the same role among algebraic spaces over S .

0D3W **Lemma 9.3.** *In Situation 9.1. Let T be an algebraic space over S . We have*

$$\mathrm{Mor}_{\mathrm{Sh}((\mathrm{Sch}/S)_{fppf})}(T, \mathrm{Hilb}_{X/B}) = \left\{ \begin{array}{l} (h, Z) \text{ where } h : T \rightarrow B, Z \subset X_T \\ \text{finite presentation, flat, proper over } T \end{array} \right\}$$

where $X_T = X \times_{B,h} T$.

Proof. By Lemma 9.2 we have $\mathrm{Hilb}_{X/B} = \mathrm{Quot}_{\mathcal{O}_X/X/B}$. Thus we can apply Lemma 8.3 to see that the left hand side is bijective with the set of surjections $\mathcal{O}_{X_T} \rightarrow \mathcal{Q}$ which are finitely presented, flat over T , and have support proper over T . Arguing exactly as in the proof of Lemma 9.2 we see that such quotients correspond exactly to the closed immersions $Z \rightarrow X_T$ such that $Z \rightarrow T$ is proper, flat, and of finite presentation. $\square$

0D01 **Proposition 9.4.** *Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . If f is of finite presentation and separated, then $\mathrm{Hilb}_{X/B}$ is an algebraic space locally of finite presentation over B .*

Proof. Immediate consequence of Lemma 9.2 and Proposition 8.4. $\square$

10. The Picard stack

0D02 The Picard stack for a morphism of algebraic spaces was introduced in Examples of Stacks, Section 16. We will deduce it is an open substack of the stack of coherent sheaves (in good cases) from the following lemma.

0D03 **Lemma 10.1.** *Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S which is flat, of finite presentation, and proper. The natural map*

$$\mathrm{Pic}_{X/B} \longrightarrow \mathrm{Coh}_{X/B}$$

is representable by open immersions.

Proof. Observe that the map simply sends a triple $(T, g, \mathcal{L})$ as in Examples of Stacks, Section 16 to the same triple $(T, g, \mathcal{L})$ but where now we view this as a triple of the kind described in Situation 5.1. This works because the invertible $\mathcal{O}_{X_T}$ -module $\mathcal{L}$ is certainly a finitely presented $\mathcal{O}_{X_T}$ -module, it is flat over T because $X_T \rightarrow T$ is flat, and the support is proper over T as $X_T \rightarrow T$ is proper (Morphisms of Spaces, Lemmas 30.4 and 40.3). Thus the statement makes sense.

Having said this, it is clear that the content of the lemma is the following: given an object $(T, g, \mathcal{F})$ of $\mathrm{Coh}_{X/B}$ there is an open subscheme $U \subset T$ such that for a morphism of schemes $T' \rightarrow T$ the following are equivalent

- (a) $T' \rightarrow T$ factors through U ,
- (b) the pullback $\mathcal{F}_{T'}$ of $\mathcal{F}$ by $X_{T'} \rightarrow X_T$ is invertible.

Let $W \subset |X_T|$ be the set of points $x \in |X_T|$ such that $\mathcal{F}$ is locally free in a neighbourhood of x . By More on Morphisms of Spaces, Lemma 23.8. W is open and formation of W commutes with arbitrary base change. Clearly, if $T' \rightarrow T$ satisfies (b), then $|X_{T'}| \rightarrow |X_T|$ maps into W . Hence we may replace T by the open $T \setminus f_T(|X_T| \setminus W)$ in order to construct U . After doing so we reach the situation where $\mathcal{F}$ is finite locally free. In this case we get a disjoint union decomposition $X_T = X_0 \amalg X_1 \amalg X_2 \amalg \dots$ into open and closed subspaces such that the restriction of $\mathcal{F}$ is locally free of rank i on X_i . Then clearly

$$U = T \setminus f_T(|X_0| \cup |X_2| \cup |X_3| \cup \dots)$$

works. (Note that if we assume that T is quasi-compact, then X_T is quasi-compact hence only a finite number of X_i are nonempty and so U is indeed open.) $\square$

0D04 **Proposition 10.2.** *Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . If f is flat, of finite presentation, and proper, then $\mathrm{Pic}_{X/B}$ is an algebraic stack.*

Proof. Immediate consequence of Lemma 10.1, Algebraic Stacks, Lemma 15.4 and either Theorem 5.12 or Theorem 6.1 $\square$

11. The Picard functor

0D24 In this section we revisit the Picard functor discussed in Picard Schemes of Curves, Section 4. The discussion will be more general as we want to study the Picard functor of a morphism of algebraic spaces as in the section on the Picard stack, see Section 10.

Let S be a scheme and let X be an algebraic space over S . An invertible sheaf on X is an invertible $\mathcal{O}_X$ -module on $X_{\acute{e}tale}$, see Modules on Sites, Definition 32.1. The group of isomorphism classes of invertible modules is denoted $\text{Pic}(X)$, see Modules on Sites, Definition 32.6. Given a morphism $f : X \rightarrow Y$ of algebraic spaces over S pullback defines a group homomorphism $\text{Pic}(Y) \rightarrow \text{Pic}(X)$. The assignment $X \rightsquigarrow \text{Pic}(X)$ is a contravariant functor from the category of schemes to the category of abelian groups. This functor is not representable, but it turns out that a relative variant of this construction sometimes is representable.

0D25 **Situation 11.1.** Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . We define

$$\text{Pic}_{X/B} : (\text{Sch}/B)^{opp} \longrightarrow \text{Sets}$$

as the fppf sheafification of the functor which to a scheme T over B associates the group $\text{Pic}(X_T)$.

In Situation 11.1 we sometimes think of $\text{Pic}_{X/B}$ as a functor $(\text{Sch}/S)^{opp} \rightarrow \text{Sets}$ endowed with a morphism $\text{Pic}_{X/B} \rightarrow B$. In this point of view, we define $\text{Pic}_{X/B}$ to be the fppf sheafification of the functor

$$T/S \longmapsto \{(h, \mathcal{L}) \mid h : T \rightarrow B, \mathcal{L} \in \text{Pic}(X \times_{B,h} T)\}$$

In particular, when we say that $\text{Pic}_{X/B}$ is an algebraic space, we mean that the corresponding functor $(\text{Sch}/S)^{opp} \rightarrow \text{Sets}$ is an algebraic space.

An often used remark is that if T is a scheme over B , then $\text{Pic}_{X_T/T}$ is the restriction of $\text{Pic}_{X/B}$ to $(\text{Sch}/T)_{fppf}$.

0D26 **Lemma 11.2.** *In Situation 11.1 the functor $\text{Pic}_{X/B}$ is the sheafification of the functor $T \mapsto \text{Ob}(\text{Pic}_{X/B,T})/\cong$.*

Proof. Since the fibre category $\text{Pic}_{X/B,T}$ of the Picard stack $\mathcal{P}ic_{X/B}$ over T is the category of invertible sheaves on X_T (see Section 10 and Examples of Stacks, Section 16) this is immediate from the definitions. $\square$

It turns out to be nontrivial to see what the value of $\text{Pic}_{X/B}$ is on schemes T over B . Here is a lemma that helps with this task.

0D27 **Lemma 11.3.** *In Situation 11.1. If $\mathcal{O}_T \rightarrow f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all schemes T over B , then*

$$0 \rightarrow \text{Pic}(T) \rightarrow \text{Pic}(X_T) \rightarrow \text{Pic}_{X/B}(T)$$

is an exact sequence for all T .

Proof. We may replace B by T and X by X_T and assume that $B = T$ to simplify the notation. Let $\mathcal{N}$ be an invertible $\mathcal{O}_B$ -module. If $f^*\mathcal{N} \cong \mathcal{O}_X$, then we see that $f_*f^*\mathcal{N} \cong f_*\mathcal{O}_X \cong \mathcal{O}_B$ by assumption. Since $\mathcal{N}$ is locally trivial, we see that the canonical map $\mathcal{N} \rightarrow f_*f^*\mathcal{N}$ is locally an isomorphism (because $\mathcal{O}_B \rightarrow f_*f^*\mathcal{O}_B$ is an isomorphism by assumption). Hence we conclude that $\mathcal{N} \rightarrow f_*f^*\mathcal{N} \rightarrow \mathcal{O}_B$ is an isomorphism and we see that $\mathcal{N}$ is trivial. This proves the first arrow is injective.

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$ -module which is in the kernel of $\text{Pic}(X) \rightarrow \text{Pic}_{X/B}(B)$. Then there exists an fppf covering $\{B_i \rightarrow B\}$ such that $\mathcal{L}$ pulls back to the trivial invertible sheaf on X_{B_i} . Choose a trivializing section s_i . Then $\text{pr}_0^*s_i$ and $\text{pr}_1^*s_j$ are both trivialising sections of $\mathcal{L}$ over $X_{B_i \times_B B_j}$ and hence differ by a multiplicative unit

$$f_{ij} \in \Gamma(X_{S_i \times_B B_j}, \mathcal{O}_{X_{B_i \times_B B_j}}^*) = \Gamma(B_i \times_B B_j, \mathcal{O}_{B_i \times_N B_j}^*)$$

(equality by our assumption on pushforward of structure sheaves). Of course these elements satisfy the cocycle condition on $B_i \times_B B_j \times_B B_k$, hence they define a descent datum on invertible sheaves for the fppf covering $\{B_i \rightarrow B\}$. By Descent, Proposition 5.2 there is an invertible $\mathcal{O}_B$ -module $\mathcal{N}$ with trivializations over B_i whose associated descent datum is $\{f_{ij}\}$. (The proposition applies because B is a scheme by the replacement performed at the start of the proof.) Then $f^*\mathcal{N} \cong \mathcal{L}$ as the functor from descent data to modules is fully faithful. $\square$

0D28 **Lemma 11.4.** *In Situation 11.1 let $\sigma : B \rightarrow X$ be a section. Assume that $\mathcal{O}_T \rightarrow f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all T over B . Then*

$$0 \rightarrow \text{Pic}(T) \rightarrow \text{Pic}(X_T) \rightarrow \text{Pic}_{X/B}(T) \rightarrow 0$$

is a split exact sequence with splitting given by $\sigma_T^ : \text{Pic}(X_T) \rightarrow \text{Pic}(T)$.*

Proof. Denote $K(T) = \text{Ker}(\sigma_T^* : \text{Pic}(X_T) \rightarrow \text{Pic}(T))$. Since σ is a section of f we see that $\text{Pic}(X_T)$ is the direct sum of $\text{Pic}(T)$ and $K(T)$. Thus by Lemma 11.3 we see that $K(T) \subset \text{Pic}_{X/B}(T)$ for all T . Moreover, it is clear from the construction that $\text{Pic}_{X/B}$ is the sheafification of the presheaf K . To finish the proof it suffices to show that K satisfies the sheaf condition for fppf coverings which we do in the next paragraph.

Let $\{T_i \rightarrow T\}$ be an fppf covering. Let $\mathcal{L}_i$ be elements of $K(T_i)$ which map to the same elements of $K(T_i \times_T T_j)$ for all i and j . Choose an isomorphism $\alpha_i : \mathcal{O}_{T_i} \rightarrow \sigma_{T_i}^*\mathcal{L}_i$ for all i . Choose an isomorphism

$$\varphi_{ij} : \mathcal{L}_i|_{X_{T_i \times_T T_j}} \longrightarrow \mathcal{L}_j|_{X_{T_i \times_T T_j}}$$

If the map

$$\alpha_j|_{T_i \times_T T_j} \circ \sigma_{T_i \times_T T_j}^* \varphi_{ij} \circ \alpha_i|_{T_i \times_T T_j} : \mathcal{O}_{T_i \times_T T_j} \rightarrow \mathcal{O}_{T_i \times_T T_j}$$

is not equal to multiplication by 1 but some u_{ij} , then we can scale φ_{ij} by u_{ij}^{-1} to correct this. Having done this, consider the self map

$$\varphi_{ki}|_{X_{T_i \times_T T_j \times_T T_k}} \circ \varphi_{jk}|_{X_{T_i \times_T T_j \times_T T_k}} \circ \varphi_{ij}|_{X_{T_i \times_T T_j \times_T T_k}} \quad \text{on} \quad \mathcal{L}_i|_{X_{T_i \times_T T_j \times_T T_k}}$$

which is given by multiplication by some section f_{ijk} of the structure sheaf of $X_{T_i \times_T T_j \times_T T_k}$. By our choice of φ_{ij} we see that the pullback of this map by σ is equal to multiplication by 1. By our assumption on functions on X , we see that $f_{ijk} = 1$. Thus we obtain a descent datum for the fppf covering $\{X_{T_i} \rightarrow X\}$. By Descent on Spaces, Proposition 4.1 there is an invertible $\mathcal{O}_{X_T}$ -module $\mathcal{L}$ and an

isomorphism $\alpha : \mathcal{O}_T \rightarrow \sigma_T^* \mathcal{L}$ whose pullback to X_{T_i} recovers $(\mathcal{L}_i, \alpha_i)$ (small detail omitted). Thus $\mathcal{L}$ defines an object of $K(T)$ as desired. $\square$

In Situation 11.1 let $\sigma : B \rightarrow X$ be a section. We denote $\mathcal{P}ic_{X/B, \sigma}$ the category defined as follows:

- (1) An object is a quadruple $(T, h, \mathcal{L}, \alpha)$, where $(T, h, \mathcal{L})$ is an object of $\mathcal{P}ic_{X/B}$ over T and $\alpha : \mathcal{O}_T \rightarrow \sigma_T^* \mathcal{L}$ is an isomorphism.
- (2) A morphism $(g, \varphi) : (T, h, \mathcal{L}, \alpha) \rightarrow (T', h', \mathcal{L}', \alpha')$ is given by a morphism of schemes $g : T \rightarrow T'$ with $h = h' \circ g$ and an isomorphism $\varphi : (g')^* \mathcal{L}' \rightarrow \mathcal{L}$ such that $\sigma_T^* \varphi \circ g^* \alpha' = \alpha$. Here $g' : X_{T'} \rightarrow X_T$ is the base change of g .

There is a natural faithful forgetful functor

$$\mathcal{P}ic_{X/B, \sigma} \longrightarrow \mathcal{P}ic_{X/B}$$

In this way we view $\mathcal{P}ic_{X/B, \sigma}$ as a category over $(Sch/S)_{fppf}$.

0D29 **Lemma 11.5.** *In Situation 11.1 let $\sigma : B \rightarrow X$ be a section. Then $\mathcal{P}ic_{X/B, \sigma}$ as defined above is a stack in groupoids over $(Sch/S)_{fppf}$.*

Proof. We already know that $\mathcal{P}ic_{X/B}$ is a stack in groupoids over $(Sch/S)_{fppf}$ by Examples of Stacks, Lemma 16.1. Let us show descent for objects for $\mathcal{P}ic_{X/B, \sigma}$. Let $\{T_i \rightarrow T\}$ be an fppf covering and let $\xi_i = (T_i, h_i, \mathcal{L}_i, \alpha_i)$ be an object of $\mathcal{P}ic_{X/B, \sigma}$ lying over T_i , and let $\varphi_{ij} : \text{pr}_0^* \xi_i \rightarrow \text{pr}_1^* \xi_j$ be a descent datum. Applying the result for $\mathcal{P}ic_{X/B}$ we see that we may assume we have an object $(T, h, \mathcal{L})$ of $\mathcal{P}ic_{X/B}$ over T which pulls back to ξ_i for all i . Then we get

$$\alpha_i : \mathcal{O}_{T_i} \rightarrow \sigma_{T_i}^* \mathcal{L}_i = (T_i \rightarrow T)^* \sigma_T^* \mathcal{L}$$

Since the maps φ_{ij} are compatible with the α_i we see that α_i and α_j pullback to the same map on $T_i \times_T T_j$. By descent of quasi-coherent sheaves (Descent, Proposition 5.2, we see that the α_i are the restriction of a single map $\alpha : \mathcal{O}_T \rightarrow \sigma_T^* \mathcal{L}$ as desired. We omit the proof of descent for morphisms. $\square$

0D2A **Lemma 11.6.** *In Situation 11.1 let $\sigma : B \rightarrow X$ be a section. The morphism $\mathcal{P}ic_{X/B, \sigma} \rightarrow \mathcal{P}ic_{X/B}$ is representable, surjective, and smooth.*

Proof. Let T be a scheme and let $(Sch/T)_{fppf} \rightarrow \mathcal{P}ic_{X/B}$ be given by the object $\xi = (T, h, \mathcal{L})$ of $\mathcal{P}ic_{X/B}$ over T . We have to show that

$$(Sch/T)_{fppf} \times_{\xi, \mathcal{P}ic_{X/B}} \mathcal{P}ic_{X/B, \sigma}$$

is representable by a scheme V and that the corresponding morphism $V \rightarrow T$ is surjective and smooth. See Algebraic Stacks, Sections 6, 9, and 10. The forgetful functor $\mathcal{P}ic_{X/B, \sigma} \rightarrow \mathcal{P}ic_{X/B}$ is faithful on fibre categories and for T'/T the set of isomorphism classes is the set of isomorphisms

$$\alpha' : \mathcal{O}_{T'} \longrightarrow (T' \rightarrow T)^* \sigma_T^* \mathcal{L}$$

See Algebraic Stacks, Lemma 9.2. We know this functor is representable by an affine scheme U of finite presentation over T by Proposition 4.3 (applied to $\text{id} : T \rightarrow T$ and $\mathcal{O}_T$ and $\sigma_T^* \mathcal{L}$). Working Zariski locally on T we may assume that $\sigma_T^* \mathcal{L}$ is isomorphic to $\mathcal{O}_T$ and then we see that our functor is representable by $\mathbf{G}_m \times T$ over T . Hence $U \rightarrow T$ Zariski locally on T looks like the projection $\mathbf{G}_m \times T \rightarrow T$ which is indeed smooth and surjective. $\square$

0D2B **Lemma 11.7.** *In Situation 11.1 let $\sigma : B \rightarrow X$ be a section. If $\mathcal{O}_T \rightarrow f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all T over B , then $\mathcal{P}ic_{X/B,\sigma} \rightarrow (Sch/S)_{fppf}$ is fibred in setoids with set of isomorphism classes over T given by*

$$\coprod_{h:T \rightarrow B} \text{Ker}(\sigma_T^* : \text{Pic}(X \times_{B,h} T) \rightarrow \text{Pic}(T))$$

Proof. If $\xi = (T, h, \mathcal{L}, \alpha)$ is an object of $\mathcal{P}ic_{X/B,\sigma}$ over T , then an automorphism φ of ξ is given by multiplication with an invertible global section u of the structure sheaf of X_T such that moreover $\sigma_T^* u = 1$. Then $u = 1$ by our assumption that $\mathcal{O}_T \rightarrow f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. Hence $\mathcal{P}ic_{X/B,\sigma}$ is fibred in setoids over $(Sch/S)_{fppf}$. Given T and $h : T \rightarrow B$ the set of isomorphism classes of pairs $(\mathcal{L}, \alpha)$ is the same as the set of isomorphism classes of $\mathcal{L}$ with $\sigma_T^* \mathcal{L} \cong \mathcal{O}_T$ (isomorphism not specified). This is clear because any two choices of α differ by a global unit on T and this is the same thing as a global unit on X_T . $\square$

0D2C **Proposition 11.8.** *Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . Assume that*

- (1) *f is flat, of finite presentation, and proper, and*
- (2) *$\mathcal{O}_T \rightarrow f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for all schemes T over B .*

Then $\mathcal{P}ic_{X/B}$ is an algebraic space.

In the situation of the proposition the algebraic stack $\mathcal{P}ic_{X/B}$ is a gerbe over the algebraic space $\text{Pic}_{X/B}$. After developing the general theory of gerbes, this provides a shorter proof of the proposition (but using more general theory).

Proof. There exists a surjective, flat, finitely presented morphism $B' \rightarrow B$ of algebraic spaces such that the base change $X' = X \times_B B'$ over B' has a section: namely, we can take $B' = X$. Observe that $\text{Pic}_{X'/B'} = B' \times_B \text{Pic}_{X/B}$. Hence $\text{Pic}_{X'/B'} \rightarrow \text{Pic}_{X/B}$ is representable by algebraic spaces, surjective, flat, and finitely presented. Hence, if we can show that $\text{Pic}_{X'/B'}$ is an algebraic space, then it follows that $\text{Pic}_{X/B}$ is an algebraic space by Bootstrap, Theorem 10.1. In this way we reduce to the case described in the next paragraph.

In addition to the assumptions of the proposition, assume that we have a section $\sigma : B \rightarrow X$. By Proposition 10.2 we see that $\mathcal{P}ic_{X/B}$ is an algebraic stack. By Lemma 11.6 and Algebraic Stacks, Lemma 15.4 we see that $\mathcal{P}ic_{X/B,\sigma}$ is an algebraic stack. By Lemma 11.7 and Algebraic Stacks, Lemma 8.2 we see that $T \mapsto \text{Ker}(\sigma_T^* : \text{Pic}(X_T) \rightarrow \text{Pic}(T))$ is an algebraic space. By Lemma 11.4 this functor is the same as $\text{Pic}_{X/B}$. $\square$

0D2D **Lemma 11.9.** *With assumptions and notation as in Proposition 11.8. Then the diagonal $\text{Pic}_{X/B} \rightarrow \text{Pic}_{X/B} \times_B \text{Pic}_{X/B}$ is representable by immersions. In other words, $\text{Pic}_{X/B} \rightarrow B$ is locally separated.*

Proof. Let T be a scheme over B and let $s, t \in \text{Pic}_{X/B}(T)$. We want to show that there exists a locally closed subscheme $Z \subset T$ such that $s|_Z = t|_Z$ and such that a morphism $T' \rightarrow T$ factors through Z if and only if $s|_{T'} = t|_{T'}$.

We first reduce the general problem to the case where s and t come from invertible modules on X_T . We suggest the reader skip this step. Choose an fppf covering $\{T_i \rightarrow T\}_{i \in I}$ such that $s|_{T_i}$ and $t|_{T_i}$ come from $\text{Pic}(X_{T_i})$ for all i . Suppose that we can show the result for all the pairs $s|_{T_i}, t|_{T_i}$. Then we obtain locally closed subschemes $Z_i \subset T_i$ with the desired universal property. It follows that Z_i and

Z_j have the same scheme theoretic inverse image in $T_i \times_T T_j$. This determines a descent datum on Z_i/T_i . Since $Z_i \rightarrow T_i$ is locally quasi-finite, it follows from More on Morphisms, Lemma 57.1 that we obtain a locally quasi-finite morphism $Z \rightarrow T$ recovering $Z_i \rightarrow T_i$ by base change. Then $Z \rightarrow T$ is an immersion by Descent, Lemma 24.1. Finally, because $\text{Pic}_{X/B}$ is an fppf sheaf, we conclude that $s|_Z = t|_Z$ and that Z satisfies the universal property mentioned above.

Assume s and t come from invertible modules $\mathcal{V}, \mathcal{W}$ on X_T . Set $\mathcal{L} = \mathcal{V} \otimes \mathcal{W}^{\otimes -1}$. We are looking for a locally closed subscheme Z of T such that $T' \rightarrow T$ factors through Z if and only if $\mathcal{L}_{X_{T'}}$ is the pullback of an invertible sheaf on T' , see Lemma 11.3. Hence the existence of Z follows from More on Morphisms of Spaces, Lemma 53.1. $\square$

12. Relative morphisms

0D19 We continue the discussion from Criteria for Representability, Section 10. In that section, starting with a scheme S and morphisms of algebraic spaces $Z \rightarrow B$ and $X \rightarrow B$ over S we constructed a functor

$$\text{Mor}_B(Z, X) : (\text{Sch}/B)^{\text{opp}} \longrightarrow \text{Sets}, \quad T \longmapsto \{f : Z_T \rightarrow X_T\}$$

We sometimes think of $\text{Mor}_B(Z, X)$ as a functor $(\text{Sch}/S)^{\text{opp}} \rightarrow \text{Sets}$ endowed with a morphism $\text{Mor}_B(Z, X) \rightarrow B$. Namely, if T is a scheme over S , then an element of $\text{Mor}_B(Z, X)(T)$ is a pair (f, h) where h is a morphism $h : T \rightarrow B$ and $f : Z \times_{B, h} T \rightarrow X \times_{B, h} T$ is a morphism of algebraic spaces over T . In particular, when we say that $\text{Mor}_B(Z, X)$ is an algebraic space, we mean that the corresponding functor $(\text{Sch}/S)^{\text{opp}} \rightarrow \text{Sets}$ is an algebraic space.

0D1A **Lemma 12.1.** *Let S be a scheme. Consider morphisms of algebraic spaces $Z \rightarrow B$ and $X \rightarrow B$ over S . If $X \rightarrow B$ is separated and $Z \rightarrow B$ is of finite presentation, flat, and proper, then there is a natural injective transformation of functors*

$$\text{Mor}_B(Z, X) \longrightarrow \text{Hilb}_{Z \times_B X/B}$$

which maps a morphism $f : Z_T \rightarrow X_T$ to its graph.

Proof. Given a scheme T over B and a morphism $f_T : Z_T \rightarrow X_T$ over T , the graph of f is the morphism $\Gamma_f = (\text{id}, f) : Z_T \rightarrow Z_T \times_T X_T = (Z \times_B X)_T$. Recall that being separated, flat, proper, or finite presentation are properties of morphisms of algebraic spaces which are stable under base change (Morphisms of Spaces, Lemmas 4.4, 30.4, 40.3, and 28.3). Hence Γ_f is a closed immersion by Morphisms of Spaces, Lemma 4.6. Moreover, $\Gamma_f(Z_T)$ is flat, proper, and of finite presentation over T . Thus $\Gamma_f(Z_T)$ defines an element of $\text{Hilb}_{Z \times_B X/B}(T)$. To show the transformation is injective it suffices to show that two morphisms with the same graph are the same. This is true because if $Y \subset (Z \times_B X)_T$ is the graph of a morphism f , then we can recover f by using the inverse of $\text{pr}_1|_Y : Y \rightarrow Z_T$ composed with $\text{pr}_2|_Y$. $\square$

0D1B **Lemma 12.2.** *Assumption and notation as in Lemma 12.1. The transformation $\text{Mor}_B(Z, X) \longrightarrow \text{Hilb}_{Z \times_B X/B}$ is representable by open immersions.*

Proof. Let T be a scheme over B and let $Y \subset (Z \times_B X)_T$ be an element of $\text{Hilb}_{Z \times_B X/B}(T)$. Then we see that Y is the graph of a morphism $Z_T \rightarrow X_T$ over T if and only if $k = \text{pr}_1|_Y : Y \rightarrow Z_T$ is an isomorphism. By More on Morphisms of Spaces, Lemma 49.6 there exists an open subscheme $V \subset T$ such that for any

morphism of schemes $T' \rightarrow T$ we have $k_{T'} : Y_{T'} \rightarrow Z_{T'}$ is an isomorphism if and only if $T' \rightarrow T$ factors through V . This proves the lemma. $\square$

0D1C **Proposition 12.3.** *Let S be a scheme. Let $Z \rightarrow B$ and $X \rightarrow B$ be morphisms of algebraic spaces over S . Assume $X \rightarrow B$ is of finite presentation and separated and $Z \rightarrow B$ is of finite presentation, flat, and proper. Then $\text{Mor}_B(Z, X)$ is an algebraic space locally of finite presentation over B .*

Proof. Immediate consequence of Lemma 12.2 and Proposition 9.4. $\square$

13. The stack of algebraic spaces

0D1D This section continues the discussion started in Examples of Stacks, Sections 7, 8, and 12. Working over $\mathbf{Z}$, the discussion therein shows that we have a stack in groupoids

$$p'_{ft} : \mathcal{Spaces}'_{ft} \longrightarrow \mathcal{Sch}_{fppf}$$

parametrizing (nonflat) families of finite type algebraic spaces. More precisely, an object³ of $\mathcal{Spaces}'_{ft}$ is a finite type morphism $X \rightarrow S$ from an algebraic space X to a scheme S and a morphism $(X' \rightarrow S') \rightarrow (X \rightarrow S)$ is given by a pair (f, g) where $f : X' \rightarrow X$ is a morphism of algebraic spaces and $g : S' \rightarrow S$ is a morphism of schemes which fit into a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad f \quad} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\quad g \quad} & S \end{array}$$

inducing an isomorphism $X' \rightarrow S' \times_S X$, in other words, the diagram is cartesian in the category of algebraic spaces. The functor p'_{ft} sends $(X \rightarrow S)$ to S and sends (f, g) to g . We define a full subcategory

$$\mathcal{Spaces}'_{fp, flat, proper} \subset \mathcal{Spaces}'_{ft}$$

consisting of objects $X \rightarrow S$ of $\mathcal{Spaces}'_{ft}$ such that $X \rightarrow S$ is of finite presentation, flat, and proper. We denote

$$p'_{fp, flat, proper} : \mathcal{Spaces}'_{fp, flat, proper} \longrightarrow \mathcal{Sch}_{fppf}$$

the restriction of the functor p'_{ft} to the indicated subcategory. We first review the results already obtained in the references listed above, and then we start adding further results.

0D1E **Lemma 13.1.** *The category $\mathcal{Spaces}'_{ft}$ is fibred in groupoids over $\mathcal{Sch}_{fppf}$. The same is true for $\mathcal{Spaces}'_{fp, flat, proper}$.*

Proof. We have seen this in Examples of Stacks, Section 12 for the case of $\mathcal{Spaces}'_{ft}$ and this easily implies the result for the other case. However, let us also prove this directly by checking conditions (1) and (2) of Categories, Definition 35.1.

Condition (1). Let $X \rightarrow S$ be an object of $\mathcal{Spaces}'_{ft}$ and let $S' \rightarrow S$ be a morphism of schemes. Then we set $X' = S' \times_S X$. Note that $X' \rightarrow S'$ is of finite type by Morphisms of Spaces, Lemma 23.3. to obtain a morphism $(X' \rightarrow S') \rightarrow (X \rightarrow S)$ lying over $S' \rightarrow S$. Argue similarly for the other case using Morphisms of Spaces, Lemmas 28.3, 30.4, and 40.3.

³We always perform a replacement as in Examples of Stacks, Lemma 8.2.

Condition (2). Consider morphisms $(f, g) : (X' \rightarrow S') \rightarrow (X \rightarrow S)$ and $(a, b) : (Y \rightarrow T) \rightarrow (X \rightarrow S)$ of $\mathcal{Spaces}'_{ft}$. Given a morphism $h : T \rightarrow S'$ with $g \circ h = b$ we have to show there is a unique morphism $(k, h) : (Y \rightarrow T) \rightarrow (X' \rightarrow S')$ of $\mathcal{Spaces}'_{ft}$ such that $(f, g) \circ (k, h) = (a, b)$. This is clear from the fact that $X' = S' \times_S X$. The same therefore works for any full subcategory of $\mathcal{Spaces}'_{ft}$ satisfying (1). $\square$

0D1F **Lemma 13.2.** *The diagonal*

$$\Delta : \mathcal{Spaces}'_{fp, flat, proper} \longrightarrow \mathcal{Spaces}'_{fp, flat, proper} \times \mathcal{Spaces}'_{fp, flat, proper}$$

is representable by algebraic spaces.

Proof. We will use criterion (2) of Algebraic Stacks, Lemma 10.11. Let S be a scheme and let X and Y be algebraic spaces of finite presentation over S , flat over S , and proper over S . We have to show that the functor

$$Isom_S(X, Y) : (Sch/S)_{fppf} \longrightarrow Sets, \quad T \longmapsto \{f : X_T \rightarrow Y_T \text{ isomorphism}\}$$

is an algebraic space. An elementary argument shows that $Isom_S(X, Y)$ sits in a fibre product

$$\begin{array}{ccc} Isom_S(X, Y) & \xrightarrow{\hspace{2cm}} & S \\ \downarrow & & \downarrow (id, id) \\ Mor_S(X, Y) \times Mor_S(Y, X) & \longrightarrow & Mor_S(X, X) \times Mor_S(Y, Y) \end{array}$$

The bottom arrow sends (φ, ψ) to $(\psi \circ \varphi, \varphi \circ \psi)$. By Proposition 12.3 the functors on the bottom row are algebraic spaces over S . Hence the result follows from the fact that the category of algebraic spaces over S has fibre products. $\square$

0D1G **Lemma 13.3.** *The category $\mathcal{Spaces}'_{ft}$ is a stack in groupoids over Sch_{fppf} . The same is true for $\mathcal{Spaces}'_{fp, flat, proper}$.*

Proof. The reason this lemma holds is the slogan: any fppf descent datum for algebraic spaces is effective, see Bootstrap, Section 11. More precisely, the lemma for $\mathcal{Spaces}'_{ft}$ follows from Examples of Stacks, Lemma 8.1 as we saw in Examples of Stacks, Section 12. However, let us review the proof. We need to check conditions (1), (2), and (3) of Stacks, Definition 5.1.

Property (1) we have seen in Lemma 13.1.

Property (2) follows from Lemma 13.2 in the case of $\mathcal{Spaces}'_{fp, flat, proper}$. In the case of $\mathcal{Spaces}'_{ft}$ it follows from Examples of Stacks, Lemma 7.2 (and this is really the “correct” reference).

Condition (3) for $\mathcal{Spaces}'_{ft}$ is checked as follows. Suppose given

- (1) an fppf covering $\{U_i \rightarrow U\}_{i \in I}$ in Sch_{fppf} ,
- (2) for each $i \in I$ an algebraic space X_i of finite type over U_i , and
- (3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \rightarrow U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$.

We have to show there exists an algebraic space X of finite type over U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij} . This follows from Bootstrap, Lemma 11.3 part (2). By Descent on Spaces, Lemma 11.11 we see that $X \rightarrow U$ is of finite type. In the case of $\mathcal{Spaces}'_{fp, flat, proper}$ one additionally uses Descent on Spaces, Lemma 11.12, 11.13, and 11.19 in the last step. $\square$

Sanity check: the stacks $\mathcal{Spaces}'_{ft}$ and $\mathcal{Spaces}'_{fp,flat,proper}$ play the same role among algebraic spaces.

0E93 **Lemma 13.4.** *Let T be an algebraic space over $\mathbf{Z}$. Let $\mathcal{S}_T$ denote the corresponding algebraic stack (Algebraic Stacks, Sections 7, 8, and 13). We have an equivalence of categories*

$$\left\{ \begin{array}{l} \text{morphisms of algebraic spaces} \\ X \rightarrow T \text{ of finite type} \end{array} \right\} \longrightarrow \text{Mor}_{\text{Cat}/\text{Sch}_{fppf}}(\mathcal{S}_T, \mathcal{Spaces}'_{ft})$$

and an equivalence of categories

$$\left\{ \begin{array}{l} \text{morphisms of algebraic spaces } X \rightarrow T \\ \text{of finite presentation, flat, and proper} \end{array} \right\} \longrightarrow \text{Mor}_{\text{Cat}/\text{Sch}_{fppf}}(\mathcal{S}_T, \mathcal{Spaces}'_{fp,flat,proper})$$

Proof. We are going to deduce this lemma from the fact that it holds for schemes (essentially by construction of the stacks) and the fact that fppf descent data for algebraic spaces over algebraic spaces are effective. We strongly encourage the reader to skip the proof.

The construction from left to right in either arrow is straightforward: given $X \rightarrow T$ of finite type the functor $\mathcal{S}_T \rightarrow \mathcal{Spaces}'_{ft}$ assigns to U/T the base change $X_U \rightarrow U$. We will explain how to construct a quasi-inverse.

If T is a scheme, then there is a quasi-inverse by the 2-Yoneda lemma, see Categories, Lemma 41.2. Let $p : U \rightarrow T$ be a surjective étale morphism where U is a scheme. Let $R = U \times_T U$ with projections $s, t : R \rightarrow U$. Observe that we obtain morphisms

$$\begin{array}{ccccc} \mathcal{S}_{U \times_T U \times_T U} & \xrightarrow{\quad} & \mathcal{S}_R & \xrightarrow{\quad} & \mathcal{S}_U \longrightarrow \mathcal{S}_T \\ & \searrow & \downarrow & \searrow & \\ & & \mathcal{S}_R & \xrightarrow{\quad} & \mathcal{S}_U \end{array}$$

satisfying various compatibilities (on the nose).

Let $G : \mathcal{S}_T \rightarrow \mathcal{Spaces}'_{ft}$ be a functor over Sch_{fppf} . The restriction of G to $\mathcal{S}_U$ via the map displayed above corresponds to a finite type morphism $X_U \rightarrow U$ of algebraic spaces via the 2-Yoneda lemma. Since $p \circ s = p \circ t$ we see that $R \times_{s,U} X_U$ and $R \times_{t,U} X_U$ both correspond to the restriction of G to $\mathcal{S}_R$. Thus we obtain a canonical isomorphism $\varphi : X_U \times_{U,t} R \rightarrow R \times_{s,U} X_U$ over R . This isomorphism satisfies the cocycle condition by the various compatibilities of the diagram given above. Thus a descent datum which is effective by Bootstrap, Lemma 11.3 part (2). In other words, we obtain an object $X \rightarrow T$ of the right hand side category. We omit checking the construction $G \rightsquigarrow X$ is functorial and that it is quasi-inverse to the other construction. In the case of $\mathcal{Spaces}'_{fp,flat,proper}$ one additionally uses Descent on Spaces, Lemma 11.12, 11.13, and 11.19 in the last step to see that $X \rightarrow T$ is of finite presentation, flat, and proper. $\square$

0D1H **Remark 13.5.** Let B be an algebraic space over $\text{Spec}(\mathbf{Z})$. Let $B\text{-}\mathcal{Spaces}'_{ft}$ be the category consisting of pairs $(X \rightarrow S, h : S \rightarrow B)$ where $X \rightarrow S$ is an object of $\mathcal{Spaces}'_{ft}$ and $h : S \rightarrow B$ is a morphism. A morphism $(X' \rightarrow S', h') \rightarrow (X \rightarrow S, h)$ in $B\text{-}\mathcal{Spaces}'_{ft}$ is a morphism (f, g) in $\mathcal{Spaces}'_{ft}$ such that $h \circ g = h'$. In this situation the diagram

$$\begin{array}{ccc} B\text{-}\mathcal{Spaces}'_{ft} & \longrightarrow & \mathcal{Spaces}'_{ft} \\ \downarrow & & \downarrow \\ (\text{Sch}/B)_{fppf} & \longrightarrow & \text{Sch}_{fppf} \end{array}$$

is 2-fibre product square. This trivial remark will occasionally be useful to deduce results from the absolute case $\mathcal{Spaces}'_{ft}$ to the case of families over a given base algebraic space. Of course, a similar construction works for $B\text{-}\mathcal{Spaces}'_{fp,flat,proper}$

0D1I **Lemma 13.6.** *The stack $p'_{fp,flat,proper} : \mathcal{Spaces}'_{fp,flat,proper} \rightarrow \mathcal{Sch}_{fppf}$ is limit preserving (Artin's Axioms, Definition 11.1).*

Proof. Let $T = \lim T_i$ be the limits of a directed inverse system of affine schemes. By Limits of Spaces, Lemma 7.1 the category of algebraic spaces of finite presentation over T is the colimit of the categories of algebraic spaces of finite presentation over T_i . To finish the proof use that flatness and properness descends through the limit, see Limits of Spaces, Lemmas 6.12 and 6.13. $\square$

0D1J **Lemma 13.7.** *Let*

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

be a pushout in the category of schemes where $T \rightarrow T'$ is a thickening and $T \rightarrow S$ is affine, see More on Morphisms, Lemma 14.3. Then the functor on fibre categories

$$\begin{array}{c} \mathcal{Spaces}'_{fp,flat,proper,S'} \\ \downarrow \\ \mathcal{Spaces}'_{fp,flat,proper,S} \times_{\mathcal{Spaces}'_{fp,flat,proper,T}} \mathcal{Spaces}'_{fp,flat,proper,T'} \end{array}$$

is an equivalence.

Proof. The functor is an equivalence if we drop “proper” from the list of conditions and replace “of finite presentation” by “locally of finite presentation”, see Pushouts of Spaces, Lemma 6.7. Thus it suffices to show that given a morphism $X' \rightarrow S'$ of an algebraic space to S' which is flat and locally of finite presentation, then $X' \rightarrow S'$ is proper if and only if $S \times_{S'} X' \rightarrow S$ and $T' \times_{S'} X' \rightarrow T'$ are proper. One implication follows from the fact that properness is preserved under base change (Morphisms of Spaces, Lemma 40.3) and the other from the fact that properness of $S \times_{S'} X' \rightarrow S$ implies properness of $X' \rightarrow S'$ by More on Morphisms of Spaces, Lemma 10.2. $\square$

0D1K **Lemma 13.8.** *Let k be a field and let $x = (X \rightarrow \text{Spec}(k))$ be an object of $\mathcal{X} = \mathcal{Spaces}'_{fp,flat,proper}$ over $\text{Spec}(k)$.*

- (1) *If k is of finite type over $\mathbf{Z}$, then the vector spaces $T\mathcal{F}_{\mathcal{X},k,x}$ and $\text{Inf}(\mathcal{F}_{\mathcal{X},k,x})$ (see Artin's Axioms, Section 8) are finite dimensional, and*
- (2) *in general the vector spaces $T_x(k)$ and $\text{Inf}_x(k)$ (see Artin's Axioms, Section 21) are finite dimensional.*

Proof. The discussion in Artin's Axioms, Section 8 only applies to fields of finite type over the base scheme $\text{Spec}(\mathbf{Z})$. Our stack satisfies (RS*) by Lemma 13.7 and we may apply Artin's Axioms, Lemma 21.2 to get the vector spaces $T_x(k)$ and $\text{Inf}_x(k)$ mentioned in (2). Moreover, in the finite type case these spaces agree with the ones mentioned in (1) by Artin's Axioms, Remark 21.7. With this out of the way we can start the proof. Observe that the first order thickening $\text{Spec}(k) \rightarrow \text{Spec}(k[\epsilon]) = \text{Spec}(k[k])$ has conormal module k . Hence the formula in Deformation

Theory, Lemma 14.2 describing infinitesimal deformations of X and infinitesimal automorphisms of X become

$$T_x(k) = \text{Ext}_{\mathcal{O}_X}^1(NL_{X/k}, \mathcal{O}_X) \quad \text{and} \quad \text{Inf}_x(k) = \text{Ext}_{\mathcal{O}_X}^0(NL_{X/k}, \mathcal{O}_X)$$

By More on Morphisms of Spaces, Lemma 21.5 and the fact that X is Noetherian, we see that $NL_{X/k}$ has coherent cohomology sheaves zero except in degrees 0 and -1 . By Derived Categories of Spaces, Lemma 8.4 the displayed Ext-groups are finite k -vector spaces and the proof is complete. $\square$

Beware that openness of versality (as proved in the next lemma) is a bit strange because our stack does not satisfy formal effectiveness, see Examples, Section 71. Later we will apply the openness of versality to suitable substacks of $\mathcal{S}paces'_{fp, flat, proper}$ which do satisfy formal effectiveness to conclude that these stacks are algebraic.

0D3X **Lemma 13.9.** *The stack in groupoids $\mathcal{X} = \mathcal{S}paces'_{fp, flat, proper}$ satisfies openness of versality over $\text{Spec}(\mathbf{Z})$. Similarly, after base change (Remark 13.5) openness of versality holds over any Noetherian base scheme S .*

Proof. For the “usual” proof of this fact, please see the discussion in the remark following this proof. We will prove this using Artin’s Axioms, Lemma 20.3. We have already seen that $\mathcal{X}$ has diagonal representable by algebraic spaces, has (RS^*) , and is limit preserving, see Lemmas 13.2, 13.7, and 13.6. Hence we only need to see that $\mathcal{X}$ satisfies the strong formal effectiveness formulated in Artin’s Axioms, Lemma 20.3.

Let (R_n) be an inverse system of rings such that $R_n \rightarrow R_m$ is surjective with square zero kernel for all $n \geq m$. Let $X_n \rightarrow \text{Spec}(R_n)$ be a finitely presented, flat, proper morphism where X_n is an algebraic space and let $X_{n+1} \rightarrow X_n$ be a morphism over $\text{Spec}(R_{n+1})$ inducing an isomorphism $X_n = X_{n+1} \times_{\text{Spec}(R_{n+1})} \text{Spec}(R_n)$. We have to find a flat, proper, finitely presented morphism $X \rightarrow \text{Spec}(\lim R_n)$ whose source is an algebraic space such that X_n is the base change of X for all n .

Let $I_n = \text{Ker}(R_n \rightarrow R_1)$. We may think of $(X_1 \subset X_n) \rightarrow (\text{Spec}(R_1) \subset \text{Spec}(R_n))$ as a morphism of first order thickenings. (Please read some of the material on thickenings of algebraic spaces in More on Morphisms of Spaces, Section 9 before continuing.) The structure sheaf of X_n is an extension

$$0 \rightarrow \mathcal{O}_{X_1} \otimes_{R_1} I_n \rightarrow \mathcal{O}_{X_n} \rightarrow \mathcal{O}_{X_1} \rightarrow 0$$

over $0 \rightarrow I_n \rightarrow R_n \rightarrow R_1$, see More on Morphisms of Spaces, Lemma 18.1. Let’s consider the extension

$$0 \rightarrow \lim \mathcal{O}_{X_1} \otimes_{R_1} I_n \rightarrow \lim \mathcal{O}_{X_n} \rightarrow \mathcal{O}_{X_1} \rightarrow 0$$

over $0 \rightarrow \lim I_n \rightarrow \lim R_n \rightarrow R_1 \rightarrow 0$. The displayed sequence is exact as the $R^1 \lim$ of the system of kernels is zero by Derived Categories of Spaces, Lemma 5.4. Observe that the map

$$\mathcal{O}_{X_1} \otimes_{R_1} \lim I_n \longrightarrow \lim \mathcal{O}_{X_1} \otimes_{R_1} I_n$$

induces an isomorphism upon applying the functor DQ_X , see Derived Categories of Spaces, Lemma 25.6. Hence we obtain a unique extension

$$0 \rightarrow \mathcal{O}_{X_1} \otimes_{R_1} \lim I_n \rightarrow \mathcal{O}' \rightarrow \mathcal{O}_{X_1} \rightarrow 0$$

over $0 \rightarrow \lim I_n \rightarrow \lim R_n \rightarrow R_1 \rightarrow 0$ by the equivalence of categories of Deformation Theory, Lemma 14.4. The sheaf $\mathcal{O}'$ determines a first order thickening of

algebraic spaces $X_1 \subset X$ over $\mathrm{Spec}(R_1) \subset \mathrm{Spec}(\lim R_n)$ by More on Morphisms of Spaces, Lemma 9.7. Observe that $X \rightarrow \mathrm{Spec}(\lim R_n)$ is flat by the already used More on Morphisms of Spaces, Lemma 18.1. By More on Morphisms of Spaces, Lemma 18.3 we see that $X \rightarrow \mathrm{Spec}(\lim R_n)$ is proper and of finite presentation. This finishes the proof. $\square$

0D1P **Remark 13.10.** Lemma 13.9 can also be shown using either Artin's Axioms, Lemma 24.4 (as in the first proof of Lemma 5.11), or using an obstruction theory as in Artin's Axioms, Lemma 22.2 (as in the second proof of Lemma 5.11). In both cases one uses the deformation and obstruction theory developed in Cotangent, Section 23 to translate the needed properties of deformations and obstructions into Ext-groups to which Derived Categories of Spaces, Lemma 23.3 can be applied. The second method (using an obstruction theory and therefore using the full cotangent complex) is perhaps the "standard" method used in most references.

14. The stack of polarized proper schemes

0D1L To study the stack of polarized proper schemes it suffices to work over $\mathbf{Z}$ as we can later pullback to any scheme or algebraic space we want (see Remark 14.5).

0D1M **Situation 14.1.** We define a category *Polarized* as follows. Objects are pairs $(X \rightarrow S, \mathcal{L})$ where

- (1) $X \rightarrow S$ is a morphism of schemes which is proper, flat, and of finite presentation, and
- (2) $\mathcal{L}$ is an invertible $\mathcal{O}_X$ -module which is relatively ample on X/S (Morphisms, Definition 37.1).

A morphism $(X' \rightarrow S', \mathcal{L}') \rightarrow (X \rightarrow S, \mathcal{L})$ between objects is given by a triple (f, g, φ) where $f : X' \rightarrow X$ and $g : S' \rightarrow S$ are morphisms of schemes which fit into a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ \downarrow & f & \downarrow \\ S' & \xrightarrow{\quad g} & S \end{array}$$

inducing an isomorphism $X' \rightarrow S' \times_S X$, in other words, the diagram is cartesian, and $\varphi : f^* \mathcal{L} \rightarrow \mathcal{L}'$ is an isomorphism. Composition is defined in the obvious manner (see Examples of Stacks, Sections 7 and 4). The forgetful functor

$$p : \mathcal{Polarized} \longrightarrow \mathcal{Sch}_{fppf}, \quad (X \rightarrow S, \mathcal{L}) \longmapsto S$$

is how we view *Polarized* as a category over $\mathcal{Sch}_{fppf}$ (see Section 2 for notation).

In the previous section we have done a substantial amount of work on the stack $\mathcal{Spaces}'_{fp, flat, proper}$ of finitely presented, flat, proper algebraic spaces. To use this material we consider the forgetful functor

$$0D3Y \quad (14.1.1) \quad \mathcal{Polarized} \longrightarrow \mathcal{Spaces}'_{fp, flat, proper}, \quad (X \rightarrow S, \mathcal{L}) \longmapsto (X \rightarrow S)$$

This functor will be a useful tool in what follows. Observe that if $(X \rightarrow S)$ is in the essential image of (14.1.1), then X and S are schemes.

0D3Z **Lemma 14.2.** *The category *Polarized* is fibred in groupoids over $\mathcal{Spaces}'_{fp, flat, proper}$. The category *Polarized* is fibred in groupoids over $\mathcal{Sch}_{fppf}$.*

Proof. We check conditions (1) and (2) of Categories, Definition 35.1.

Condition (1). Let $(X \rightarrow S, \mathcal{L})$ be an object of $\mathcal{Polarized}$ and let $(X' \rightarrow S') \rightarrow (X \rightarrow S)$ be a morphism of $\mathcal{Spaces}'_{fp, flat, proper}$. Then we let $\mathcal{L}'$ be the pullback of $\mathcal{L}$ to X' . Observe that X, S, S' are schemes, hence X' is a scheme as well (as the fibre product of schemes). Then $\mathcal{L}'$ is ample on X'/S' by Morphisms, Lemma 37.9. In this way we obtain a morphism $(X' \rightarrow S', \mathcal{L}') \rightarrow (X \rightarrow S, \mathcal{L})$ lying over $(X' \rightarrow S') \rightarrow (X \rightarrow S)$.

Condition (2). Consider morphisms $(f, g, \varphi) : (X' \rightarrow S', \mathcal{L}') \rightarrow (X \rightarrow S, \mathcal{L})$ and $(a, b, \psi) : (Y \rightarrow T, \mathcal{N}) \rightarrow (X \rightarrow S, \mathcal{L})$ of $\mathcal{Polarized}$. Given a morphism $(k, h) : (Y \rightarrow T) \rightarrow (X' \rightarrow S')$ of $\mathcal{Spaces}'_{fp, flat, proper}$ with $(f, g) \circ (k, h) = (a, b)$ we have to show there is a unique morphism $(k, h, \chi) : (Y \rightarrow T, \mathcal{N}) \rightarrow (X' \rightarrow S', \mathcal{L}')$ of $\mathcal{Polarized}$ such that $(f, g, \varphi) \circ (k, h, \chi) = (a, b, \psi)$. We can just take

$$\chi = \psi \circ (k^* \varphi)^{-1}$$

This proves condition (2). A composition of functors defining fibred categories defines a fibred category, see Categories, Lemma 33.12. This we see that $\mathcal{Polarized}$ is fibred in groupoids over Sch_{fppf} (strictly speaking we should check the fibre categories are groupoids and apply Categories, Lemma 35.2). $\square$

0D40 **Lemma 14.3.** *The category $\mathcal{Polarized}$ is a stack in groupoids over $\mathcal{Spaces}'_{fp, flat, proper}$ (endowed with the inherited topology, see Stacks, Definition 10.2). The category $\mathcal{Polarized}$ is a stack in groupoids over Sch_{fppf} .*

Proof. We prove $\mathcal{Polarized}$ is a stack in groupoids over $\mathcal{Spaces}'_{fp, flat, proper}$ by checking conditions (1), (2), and (3) of Stacks, Definition 5.1. We have already seen (1) in Lemma 14.2.

A covering of $\mathcal{Spaces}'_{fp, flat, proper}$ comes about in the following manner: Let $X \rightarrow S$ be an object of $\mathcal{Spaces}'_{fp, flat, proper}$. Suppose that $\{S_i \rightarrow S\}_{i \in I}$ is a covering of Sch_{fppf} . Set $X_i = S_i \times_S X$. Then $\{(X_i \rightarrow S_i) \rightarrow (X \rightarrow S)\}_{i \in I}$ is a covering of $\mathcal{Spaces}'_{fp, flat, proper}$ and every covering of $\mathcal{Spaces}'_{fp, flat, proper}$ is isomorphic to one of these. Set $S_{ij} = S_i \times_S S_j$ and $X_{ij} = S_{ij} \times_S X$ so that $(X_{ij} \rightarrow S_{ij}) = (X_i \rightarrow S_i) \times_{(X \rightarrow S)} (X_j \rightarrow S_j)$. Next, suppose that $\mathcal{L}, \mathcal{N}$ are ample invertible sheaves on X/S so that $(X \rightarrow S, \mathcal{L})$ and $(X \rightarrow S, \mathcal{N})$ are two objects of $\mathcal{Polarized}$ over the object $(X \rightarrow S)$. To check descent for morphisms, we assume we have morphisms $(\text{id}, \text{id}, \varphi_i)$ from $(X_i \rightarrow S_i, \mathcal{L}|_{X_i})$ to $(X_i \rightarrow S_i, \mathcal{N}|_{X_i})$ whose base changes to morphisms from $(X_{ij} \rightarrow S_{ij}, \mathcal{L}|_{X_{ij}})$ to $(X_{ij} \rightarrow S_{ij}, \mathcal{N}|_{X_{ij}})$ agree. Then $\varphi_i : \mathcal{L}|_{X_i} \rightarrow \mathcal{N}|_{X_i}$ are isomorphisms of invertible modules over X_i such that φ_i and φ_j restrict to the same isomorphisms over X_{ij} . By descent for quasi-coherent sheaves (Descent on Spaces, Proposition 4.1) we obtain a unique isomorphism $\varphi : \mathcal{L} \rightarrow \mathcal{N}$ whose restriction to X_i recovers φ_i .

Decent for objects is proved in exactly the same manner. Namely, suppose that $\{(X_i \rightarrow S_i) \rightarrow (X \rightarrow S)\}_{i \in I}$ is a covering of $\mathcal{Spaces}'_{fp, flat, proper}$ as above. Suppose we have objects $(X_i \rightarrow S_i, \mathcal{L}_i)$ of $\mathcal{Polarized}$ lying over $(X_i \rightarrow S_i)$ and a descent datum

$$(\text{id}, \text{id}, \varphi_{ij}) : (X_{ij} \rightarrow S_{ij}, \mathcal{L}_i|_{X_{ij}}) \rightarrow (X_{ij} \rightarrow S_{ij}, \mathcal{L}_j|_{X_{ij}})$$

satisfying the obvious cocycle condition over $(X_{ijk} \rightarrow S_{ijk})$ for every triple of indices. Then by descent for quasi-coherent sheaves (Descent on Spaces, Proposition 4.1) we obtain a unique invertible $\mathcal{O}_X$ -module $\mathcal{L}$ and isomorphisms $\mathcal{L}|_{X_i} \rightarrow \mathcal{L}_i$

recovering the descent datum φ_{ij} . To show that $(X \rightarrow S, \mathcal{L})$ is an object of $\mathcal{Polarized}$ we have to prove that $\mathcal{L}$ is ample. This follows from Descent on Spaces, Lemma 13.1.

Since we already have seen that $\mathcal{Spaces}'_{fp,flat,proper}$ is a stack in groupoids over Sch_{fppf} (Lemma 13.3) it now follows formally that $\mathcal{Polarized}$ is a stack in groupoids over Sch_{fppf} . See Stacks, Lemma 10.6. $\square$

Sanity check: the stack $\mathcal{Polarized}$ plays the same role among algebraic spaces.

0E94 **Lemma 14.4.** *Let T be an algebraic space over $\mathbf{Z}$. Let $\mathcal{S}_T$ denote the corresponding algebraic stack (Algebraic Stacks, Sections 7, 8, and 13). We have an equivalence of categories*

$$\left\{ \begin{array}{l} (X \rightarrow T, \mathcal{L}) \text{ where } X \rightarrow T \text{ is a morphism} \\ \text{of algebraic spaces, is proper, flat, and of} \\ \text{finite presentation and } \mathcal{L} \text{ ample on } X/T \end{array} \right\} \longrightarrow \text{Mor}_{Cat/Sch_{fppf}}(\mathcal{S}_T, \mathcal{Polarized})$$

Proof. Omitted. Hints: Argue exactly as in the proof of Lemma 13.4 and use Descent on Spaces, Proposition 4.1 to descent the invertible sheaf in the construction of the quasi-inverse functor. The relative ampleness property descends by Descent on Spaces, Lemma 13.1. $\square$

0D1N **Remark 14.5.** Let B be an algebraic space over $\text{Spec}(\mathbf{Z})$. Let $B\text{-}\mathcal{Polarized}$ be the category consisting of triples $(X \rightarrow S, \mathcal{L}, h : S \rightarrow B)$ where $(X \rightarrow S, \mathcal{L})$ is an object of $\mathcal{Polarized}$ and $h : S \rightarrow B$ is a morphism. A morphism $(X' \rightarrow S', \mathcal{L}', h') \rightarrow (X \rightarrow S, \mathcal{L}, h)$ in $B\text{-}\mathcal{Polarized}$ is a morphism (f, g, φ) in $\mathcal{Polarized}$ such that $h \circ g = h'$. In this situation the diagram

$$\begin{array}{ccc} B\text{-}\mathcal{Polarized} & \longrightarrow & \mathcal{Polarized} \\ \downarrow & & \downarrow \\ (Sch/B)_{fppf} & \longrightarrow & Sch_{fppf} \end{array}$$

is 2-fibre product square. This trivial remark will occasionally be useful to deduce results from the absolute case $\mathcal{Polarized}$ to the case of families over a given base algebraic space.

0D41 **Lemma 14.6.** *The functor (14.1.1) defines a 1-morphism*

$$\mathcal{Polarized} \rightarrow \mathcal{Spaces}'_{fp,flat,proper}$$

of stacks in groupoids over Sch_{fppf} which is algebraic in the sense of Criteria for Representability, Definition 8.1.

Proof. By Lemmas 13.3 and 14.3 the statement makes sense. To prove it, we choose a scheme S and an object $\xi = (X \rightarrow S)$ of $\mathcal{Spaces}'_{fp,flat,proper}$ over S . We have to show that

$$\mathcal{X} = (Sch/S)_{fppf} \times_{\xi, \mathcal{Spaces}'_{fp,flat,proper}} \mathcal{Polarized}$$

is an algebraic stack over S . Observe that an object of $\mathcal{X}$ is given by a pair $(T/S, \mathcal{L})$ where T is a scheme over S and $\mathcal{L}$ is an invertible $\mathcal{O}_{X_T}$ -module which is ample on X_T/T . Morphisms are defined in the obvious manner. In particular, we see immediately that we have an inclusion

$$\mathcal{X} \subset \text{Pic}_{X/S}$$

of categories over $(Sch/S)_{fppf}$, inducing equality on morphism sets. Since $\mathcal{P}ic_{X/S}$ is an algebraic stack by Proposition 10.2 it suffices to show that the inclusion above is representable by open immersions. This is exactly the content of Descent on Spaces, Lemma 13.2. $\square$

0D42 **Lemma 14.7.** *The diagonal*

$$\Delta : \mathcal{P}olarized \longrightarrow \mathcal{P}olarized \times \mathcal{P}olarized$$

is representable by algebraic spaces.

Proof. This is a formal consequence of Lemmas 14.6 and 13.2. See Criteria for Representability, Lemma 8.4. $\square$

0D43 **Lemma 14.8.** *The stack in groupoids $\mathcal{P}olarized$ is limit preserving (Artin's Axioms, Definition 11.1).*

Proof. Let I be a directed set and let $(A_i, \varphi_{ii'})$ be a system of rings over I . Set $S = \text{Spec}(A)$ and $S_i = \text{Spec}(A_i)$. We have to show that on fibre categories we have

$$\mathcal{P}olarized_S = \text{colim } \mathcal{P}olarized_{S_i}$$

We know that the category of schemes of finite presentation over S is the colimit of the category of schemes of finite presentation over S_i , see Limits, Lemma 10.1. Moreover, given $X_i \rightarrow S_i$ of finite presentation, with limit $X \rightarrow S$, then the category of invertible $\mathcal{O}_X$ -modules $\mathcal{L}$ is the colimit of the categories of invertible $\mathcal{O}_{X_i}$ -modules $\mathcal{L}_i$, see Limits, Lemma 10.2 and 10.3. If $X \rightarrow S$ is proper and flat, then for sufficiently large i the morphism $X_i \rightarrow S_i$ is proper and flat too, see Limits, Lemmas 13.1 and 8.7. Finally, if $\mathcal{L}$ is ample on X then $\mathcal{L}_i$ is ample on X_i for i sufficiently large, see Limits, Lemma 4.15. Putting everything together finishes the proof. $\square$

0D44 **Lemma 14.9.** *In Situation 5.1. Let*

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

be a pushout in the category of schemes where $T \rightarrow T'$ is a thickening and $T \rightarrow S$ is affine, see More on Morphisms, Lemma 14.3. Then the functor on fibre categories

$$\mathcal{P}olarized_{S'} \longrightarrow \mathcal{P}olarized_S \times_{\mathcal{P}olarized_T} \mathcal{P}olarized_{T'}$$

is an equivalence.

Proof. By More on Morphisms, Lemma 14.6 there is an equivalence

$$flat\text{-}lfp_{S'} \longrightarrow flat\text{-}lfp_S \times_{flat\text{-}lfp_T} flat\text{-}lfp_{T'}$$

where $flat\text{-}lfp_S$ signifies the category of schemes flat and locally of finite presentation over S . Let X'/S' on the left hand side correspond to the triple $(X/S, Y'/T', \varphi)$ on the right hand side. Set $Y = T \times_{T'} Y'$ which is isomorphic with $T \times_S X$ via φ . Then More on Morphisms, Lemma 14.5 shows that we have an equivalence

$$QCoh\text{-}flat_{X'/S'} \longrightarrow QCoh\text{-}flat_{X/S} \times_{QCoh\text{-}flat_{Y/T}} QCoh\text{-}flat_{Y'/T'}$$

where $QCoh\text{-}flat_{X/S}$ signifies the category of quasi-coherent $\mathcal{O}_X$ -modules flat over S . Since $X \rightarrow S$, $Y \rightarrow T$, $X' \rightarrow S'$, $Y' \rightarrow T'$ are flat, this will in particular apply to invertible modules to give an equivalence of categories

$$Pic(X') \longrightarrow Pic(X) \times_{Pic(Y)} Pic(Y')$$

where $Pic(X)$ signifies the category of invertible $\mathcal{O}_X$ -modules. There is a small point here: one has to show that if an object $\mathcal{F}'$ of $QCoh\text{-}flat_{X'/S'}$ pulls back to invertible modules on X and Y' , then $\mathcal{F}'$ is an invertible $\mathcal{O}_{X'}$ -module. It follows from the cited lemma that $\mathcal{F}'$ is an $\mathcal{O}_{X'}$ -module of finite presentation. By More on Morphisms, Lemma 16.7 it suffices to check the restriction of $\mathcal{F}'$ to fibres of $X' \rightarrow S'$ is invertible. But the fibres of $X' \rightarrow S'$ are the same as the fibres of $X \rightarrow S$ and hence these restrictions are invertible.

Having said the above we obtain an equivalence of categories if we drop the assumption (for the category of objects over S) that $X \rightarrow S$ be proper and the assumption that $\mathcal{L}$ be ample. Now it is clear that if $X' \rightarrow S'$ is proper, then $X \rightarrow S$ and $Y' \rightarrow T'$ are proper (Morphisms, Lemma 41.5). Conversely, if $X \rightarrow S$ and $Y' \rightarrow T'$ are proper, then $X' \rightarrow S'$ is proper by More on Morphisms, Lemma 3.3. Similarly, if $\mathcal{L}'$ is ample on X'/S' , then $\mathcal{L}'|_X$ is ample on X/S and $\mathcal{L}'|_{Y'}$ is ample on Y'/T' (Morphisms, Lemma 37.9). Finally, if $\mathcal{L}'|_X$ is ample on X/S and $\mathcal{L}'|_{Y'}$ is ample on Y'/T' , then $\mathcal{L}'$ is ample on X'/S' by More on Morphisms, Lemma 3.2. $\square$

0D4S **Lemma 14.10.** *Let k be a field and let $x = (X \rightarrow \text{Spec}(k), \mathcal{L})$ be an object of $\mathcal{X} = \text{Polarized over Spec}(k)$.*

- (1) *If k is of finite type over $\mathbf{Z}$, then the vector spaces $T\mathcal{F}_{\mathcal{X},k,x}$ and $\text{Inf}(\mathcal{F}_{\mathcal{X},k,x})$ (see Artin's Axioms, Section 8) are finite dimensional, and*
- (2) *in general the vector spaces $T_x(k)$ and $\text{Inf}_x(k)$ (see Artin's Axioms, Section 21) are finite dimensional.*

Proof. The discussion in Artin's Axioms, Section 8 only applies to fields of finite type over the base scheme $\text{Spec}(\mathbf{Z})$. Our stack satisfies (RS*) by Lemma 14.9 and we may apply Artin's Axioms, Lemma 21.2 to get the vector spaces $T_x(k)$ and $\text{Inf}_x(k)$ mentioned in (2). Moreover, in the finite type case these spaces agree with the ones mentioned in part (1) by Artin's Axioms, Remark 21.7. With this out of the way we can start the proof.

One proof is to use an argument as in the proof of Lemma 13.8; this would require us to develop a deformation theory for pairs consisting of a scheme and a quasi-coherent module. Another proof would be the use the result from Lemma 13.8, the algebraicity of $\text{Polarized} \rightarrow \text{Spaces}'_{fp,flat,proper}$, and a computation of the deformation space of an invertible module. However, what we will do instead is to translate the question into a deformation question on graded k -algebras and deduce the result that way.

Let $\mathcal{C}_k$ be the category of Artinian local k -algebras A with residue field k . We get a predeformation category $p : \mathcal{F} \rightarrow \mathcal{C}_k$ from our object x of $\mathcal{X}$ over k , see Artin's Axioms, Section 3. Thus $\mathcal{F}(A)$ is the category of triples $(X_A, \mathcal{L}_A, \alpha)$, where $(X_A, \mathcal{L}_A)$ is an object of Polarized over A and α is an isomorphism $(X_A, \mathcal{L}_A) \times_{\text{Spec}(A)} \text{Spec}(k) \cong (X, \mathcal{L})$. On the other hand, let $q : \mathcal{G} \rightarrow \mathcal{C}_k$ be the category cofibred in groupoids defined in Deformation Problems, Example 7.1. Choose $d_0 \gg 0$ (we'll

see below how large). Let P be the graded k -algebra

$$P = k \oplus \bigoplus_{d \geq d_0} H^0(X, \mathcal{L}^{\otimes d})$$

Then $y = (k, P)$ is an object of $\mathcal{G}(k)$. Let $\mathcal{G}_y$ be the predeformation category of Formal Deformation Theory, Remark 6.4. Given $(X_A, \mathcal{F}_A, \alpha)$ as above we set

$$Q = A \oplus \bigoplus_{d \geq d_0} H^0(X_A, \mathcal{L}_A^{\otimes d})$$

The isomorphism α induces a map $\beta : Q \rightarrow P$. By deformation theory of projective schemes (More on Morphisms, Lemma 10.6) we obtain a 1-morphism

$$\mathcal{F} \longrightarrow \mathcal{G}_y, \quad (X_A, \mathcal{F}_A, \alpha) \longmapsto (Q, \beta : Q \rightarrow P)$$

of categories cofibred in groupoids over $\mathcal{C}_k$. In fact, this functor is an equivalence with quasi-inverse given by $Q \mapsto \underline{\text{Proj}}_A(Q)$. Namely, the scheme $X_A = \underline{\text{Proj}}_A(Q)$ is flat over A by Divisors, Lemma 30.6. Set $\mathcal{L}_A = \mathcal{O}_{X_A}(1)$; this is flat over A by the same lemma. We get an isomorphism $(X_A, \mathcal{L}_A) \times_{\text{Spec}(A)} \text{Spec}(k) = (X, \mathcal{L})$ from β . Then we can deduce all the desired properties of the pair $(X_A, \mathcal{L}_A)$ from the corresponding properties of $(X, \mathcal{L})$ using the techniques in More on Morphisms, Sections 3 and 10. Some details omitted.

In conclusion, we see that $T\mathcal{F} = T\mathcal{G}_y = T_y\mathcal{G}$ and $\text{Inf}(\mathcal{F}) = \text{Inf}_y(\mathcal{G})$. These vector spaces are finite dimensional by Deformation Problems, Lemma 7.3 and the proof is complete. $\square$

0D4T **Lemma 14.11** (Strong formal effectiveness for polarized schemes). *Let (R_n) be an inverse system of rings with surjective transition maps whose kernels are locally nilpotent. Set $R = \lim R_n$. Set $S_n = \text{Spec}(R_n)$ and $S = \text{Spec}(R)$. Consider a commutative diagram*

$$\begin{array}{ccccccc} X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \dots \end{array}$$

of schemes with cartesian squares. Suppose given $(\mathcal{L}_n, \varphi_n)$ where each $\mathcal{L}_n$ is an invertible sheaf on X_n and $\varphi_n : i_n^ \mathcal{L}_{n+1} \rightarrow \mathcal{L}_n$ is an isomorphism. If*

- (1) $X_n \rightarrow S_n$ is proper, flat, of finite presentation, and
- (2) $\mathcal{L}_1$ is ample on X_1

then there exists a morphism of schemes $X \rightarrow S$ proper, flat, and of finite presentation and an ample invertible $\mathcal{O}_X$ -module $\mathcal{L}$ and isomorphisms $X_n \cong X \times_S S_n$ and $\mathcal{L}_n \cong \mathcal{L}|_{X_n}$ compatible with the morphisms i_n and φ_n .

Proof. Choose d_0 for $X_1 \rightarrow S_1$ and $\mathcal{L}_1$ as in More on Morphisms, Lemma 10.6. For any $n \geq 1$ set

$$A_n = R_n \oplus \bigoplus_{d \geq d_0} H^0(X_n, \mathcal{L}_n^{\otimes d})$$

By the lemma each A_n is a finitely presented graded R_n -algebra whose homogeneous parts $(A_n)_d$ are finite projective R_n -modules such that $X_n = \text{Proj}(A_n)$ and $\mathcal{L}_n = \mathcal{O}_{\text{Proj}(A_n)}(1)$. The lemma also guarantees that the maps

$$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \dots$$

induce isomorphisms $A_n = A_m \otimes_{R_m} R_n$ for $n \leq m$. We set

$$B = \bigoplus_{d \geq 0} B_d \quad \text{with} \quad B_d = \lim_n (A_n)_d$$

By More on Algebra, Lemma 13.4 we see that B_d is a finite projective R -module and that $B \otimes_R R_n = A_n$. Thus the scheme

$$X = \text{Proj}(B) \quad \text{and} \quad \mathcal{L} = \mathcal{O}_X(1)$$

is flat over S and $\mathcal{L}$ is a quasi-coherent $\mathcal{O}_X$ -module flat over S , see Divisors, Lemma 30.6. Because formation of Proj commutes with base change (Constructions, Lemma 11.6) we obtain canonical isomorphisms

$$X \times_S S_n = X_n \quad \text{and} \quad \mathcal{L}|_{X_n} \cong \mathcal{L}_n$$

compatible with the transition maps of the system. Thus we may think of $X_1 \subset X$ as a closed subscheme. Below we will show that B is of finite presentation over R . By Divisors, Lemmas 30.4 and 30.7 this implies that $X \rightarrow S$ is of finite presentation and proper and that $\mathcal{L} = \mathcal{O}_X(1)$ is of finite presentation as an $\mathcal{O}_X$ -module. Since the restriction of $\mathcal{L}$ to the base change $X_1 \rightarrow S_1$ is invertible, we see from More on Morphisms, Lemma 16.8 that $\mathcal{L}$ is invertible on an open neighbourhood of X_1 in X . Since $X \rightarrow S$ is closed and since $\text{Ker}(R \rightarrow R_1)$ is contained in the Jacobson radical (More on Algebra, Lemma 11.3) we see that any open neighbourhood of X_1 in X is equal to X . Thus $\mathcal{L}$ is invertible. Finally, the set of points in S where $\mathcal{L}$ is ample on the fibre is open in S (More on Morphisms, Lemma 50.3) and contains S_1 hence equals S . Thus $X \rightarrow S$ and $\mathcal{L}$ have all the properties required of them in the statement of the lemma.

We prove the claim above. Choose a presentation $A_1 = R_1[X_1, \dots, X_s]/(F_1, \dots, F_t)$ where X_i are variables having degrees d_i and F_j are homogeneous polynomials in X_i of degree e_j . Then we can choose a map

$$\Psi : R[X_1, \dots, X_s] \longrightarrow B$$

lifting the map $R_1[X_1, \dots, X_s] \rightarrow A_1$. Since each B_d is finite projective over R we conclude from Nakayama's lemma (Algebra, Lemma 20.1 using again that $\text{Ker}(R \rightarrow R_1)$ is contained in the Jacobson radical of R) that Ψ is surjective. Since $-\otimes_R R_1$ is right exact we can find $G_1, \dots, G_t \in \text{Ker}(\Psi)$ mapping to $F_1, \dots, F_t$ in $R_1[X_1, \dots, X_s]$. Observe that $\text{Ker}(\Psi)_d$ is a finite projective R -module for all $d \geq 0$ as the kernel of the surjection $R[X_1, \dots, X_s]_d \rightarrow B_d$ of finite projective R -modules. We conclude from Nakayama's lemma once more that $\text{Ker}(\Psi)$ is generated by $G_1, \dots, G_t$. $\square$

0D4U **Lemma 14.12.** *Consider the stack Polarized over the base scheme $\text{Spec}(\mathbf{Z})$. Then every formal object is effective.*

Proof. For definitions of the notions in the lemma, please see Artin's Axioms, Section 9. From the definitions we see the lemma follows immediately from the more general Lemma 14.11. $\square$

0D4V **Lemma 14.13.** *The stack in groupoids Polarized satisfies openness of versality over $\text{Spec}(\mathbf{Z})$. Similarly, after base change (Remark 14.5) openness of versality holds over any Noetherian base scheme S .*

Proof. This follows from Artin’s Axioms, Lemma 20.3 and Lemmas 14.7, 14.9, 14.8, and 14.11. For the “usual” proof of this fact, please see the discussion in the remark following this proof. $\square$

0D4W **Remark 14.14.** Lemma 14.13 can also be shown using an obstruction theory as in Artin’s Axioms, Lemma 22.2 (as in the second proof of Lemma 5.11). To do this one has to generalize the deformation and obstruction theory developed in Cotangent, Section 23 to the case of pairs of algebraic spaces and quasi-coherent modules. Another possibility is to use that the 1-morphism $\mathcal{Polarized} \rightarrow \mathcal{Spaces}'_{fp,flat,proper}$ is algebraic (Lemma 14.6) and the fact that we know openness of versality for the target (Lemma 13.9 and Remark 13.10).

0D4X **Theorem 14.15** (Algebraicity of the stack of polarized schemes). *The stack $\mathcal{Polarized}$ (Situation 14.1) is algebraic. In fact, for any algebraic space B the stack $B\text{-Polarized}$ (Remark 14.5) is algebraic.*

Proof. The absolute case follows from Artin’s Axioms, Lemma 17.1 and Lemmas 14.7, 14.9, 14.8, 14.12, and 14.13. The case over B follows from this, the description of $B\text{-Polarized}$ as a 2-fibre product in Remark 14.5, and the fact that algebraic stacks have 2-fibre products, see Algebraic Stacks, Lemma 14.3. $\square$

15. The stack of curves

0D4Y In this section we prove the stack of curves is algebraic. For a further discussion of moduli of curves, we refer the reader to Moduli of Curves, Section 1.

A curve in the Stacks project is a variety of dimension 1. However, when we speak of families of curves, we often allow the fibres to be reducible and/or nonreduced. In this section, the stack of curves will “parametrize proper schemes of dimension ≤ 1 ”. However, it turns out that in order to get the correct notion of a family we need to allow the total space of our family to be an algebraic space. This leads to the following definition.

0D4Z **Situation 15.1.** We define a category $\mathcal{Curves}$ as follows:

- (1) Objects are *families of curves*. More precisely, an object is a morphism $f : X \rightarrow S$ where the base S is a scheme, the *total space* X is an algebraic space, and f is flat, proper, of finite presentation, and has relative dimension ≤ 1 (Morphisms of Spaces, Definition 33.2).
- (2) A morphism $(X' \rightarrow S') \rightarrow (X \rightarrow S)$ between objects is given by a pair (f, g) where $f : X' \rightarrow X$ is a morphism of algebraic spaces and $g : S' \rightarrow S$ is a morphism of schemes which fit into a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad f \quad} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\quad g \quad} & S \end{array}$$

inducing an isomorphism $X' \rightarrow S' \times_S X$, in other words, the diagram is cartesian.

The forgetful functor

$$p : \mathcal{Curves} \longrightarrow \mathcal{Sch}_{fppf}, \quad (X \rightarrow S) \longmapsto S$$

is how we view $\mathcal{Curves}$ as a category over $\mathcal{Sch}_{fppf}$ (see Section 2 for notation).

It follows from Spaces over Fields, Lemma 9.3 and more generally More on Morphisms of Spaces, Lemma 43.6 that if S is the spectrum of a field, or an Artinian local ring, or a Noetherian complete local ring, then for any family of curves $X \rightarrow S$ the total space X is a scheme. On the other hand, there are families of curves over $\mathbf{A}_k^1$ where the total space is not a scheme, see Examples, Section 67.

It is clear that

$$0D50 \quad (15.1.1) \quad \text{Curves} \subset \mathcal{Spaces}'_{fp, flat, proper}$$

and that an object $X \rightarrow S$ of $\mathcal{Spaces}'_{fp, flat, proper}$ is in *Curves* if and only if $X \rightarrow S$ has relative dimension ≤ 1 . We will use this to verify Artin's axioms for *Curves*.

0D51 **Lemma 15.2.** *The category Curves is fibred in groupoids over Sch_{fppf} .*

Proof. Using the embedding (15.1.1), the description of the image, and the corresponding fact for $\mathcal{Spaces}'_{fp, flat, proper}$ (Lemma 13.1) this reduces to the following statement: Given a morphism

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

in $\mathcal{Spaces}'_{fp, flat, proper}$ (recall that this implies in particular the diagram is cartesian) if $X \rightarrow S$ has relative dimension ≤ 1 , then $X' \rightarrow S'$ has relative dimension ≤ 1 . This follows from Morphisms of Spaces, Lemma 34.3. $\square$

0D52 **Lemma 15.3.** *The category Curves is a stack in groupoids over Sch_{fppf} .*

Proof. Using the embedding (15.1.1), the description of the image, and the corresponding fact for $\mathcal{Spaces}'_{fp, flat, proper}$ (Lemma 13.3) this reduces to the following statement: Given an object $X \rightarrow S$ of $\mathcal{Spaces}'_{fp, flat, proper}$ and an fppf covering $\{S_i \rightarrow S\}_{i \in I}$ the following are equivalent:

- (1) $X \rightarrow S$ has relative dimension ≤ 1 , and
- (2) for each i the base change $X_i \rightarrow S_i$ has relative dimension ≤ 1 .

This follows from Morphisms of Spaces, Lemma 34.3. $\square$

0D53 **Lemma 15.4.** *The diagonal*

$$\Delta : \text{Curves} \longrightarrow \text{Curves} \times \text{Curves}$$

is representable by algebraic spaces.

Proof. This is immediate from the fully faithful embedding (15.1.1) and the corresponding fact for $\mathcal{Spaces}'_{fp, flat, proper}$ (Lemma 13.2). $\square$

0D54 **Remark 15.5.** Let B be an algebraic space over $\text{Spec}(\mathbf{Z})$. Let $B\text{-Curves}$ be the category consisting of pairs $(X \rightarrow S, h : S \rightarrow B)$ where $X \rightarrow S$ is an object of *Curves* and $h : S \rightarrow B$ is a morphism. A morphism $(X' \rightarrow S', h') \rightarrow (X \rightarrow S, h)$ in $B\text{-Curves}$ is a morphism (f, g) in *Curves* such that $h \circ g = h'$. In this situation the diagram

$$\begin{array}{ccc} B\text{-Curves} & \longrightarrow & \text{Curves} \\ \downarrow & & \downarrow \\ (Sch/B)_{fppf} & \longrightarrow & Sch_{fppf} \end{array}$$

is 2-fibre product square. This trivial remark will occasionally be useful to deduce results from the absolute case *Curves* to the case of families of curves over a given base algebraic space.

0D55 **Lemma 15.6.** *The stack $\text{Curves} \rightarrow \text{Sch}_{fppf}$ is limit preserving (Artin's Axioms, Definition 11.1).*

Proof. Using the embedding (15.1.1), the description of the image, and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 13.6) this reduces to the following statement: Let $T = \lim T_i$ be the limits of a directed inverse system of affine schemes. Let $i \in I$ and let $X_i \rightarrow T_i$ be an object of $\mathcal{S}paces'_{fp,flat,proper}$ over T_i . Assume that $T \times_{T_i} X_i \rightarrow T$ has relative dimension ≤ 1 . Then for some $i' \geq i$ the morphism $T_{i'} \times_{T_i} X_i \rightarrow T_i$ has relative dimension ≤ 1 . This follows from Limits of Spaces, Lemma 6.14. $\square$

0D56 **Lemma 15.7.** *Let*

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

be a pushout in the category of schemes where $T \rightarrow T'$ is a thickening and $T \rightarrow S$ is affine, see More on Morphisms, Lemma 14.3. Then the functor on fibre categories

$$\text{Curves}_{S'} \longrightarrow \text{Curves}_S \times_{\text{Curves}_T} \text{Curves}_{T'}$$

is an equivalence.

Proof. Using the embedding (15.1.1), the description of the image, and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 13.7) this reduces to the following statement: given a morphism $X' \rightarrow S'$ of an algebraic space to S' which is of finite presentation, flat, proper then $X' \rightarrow S'$ has relative dimension ≤ 1 if and only if $S \times_{S'} X' \rightarrow S$ and $T' \times_{S'} X' \rightarrow T'$ have relative dimension ≤ 1 . One implication follows from the fact that having relative dimension ≤ 1 is preserved under base change (Morphisms of Spaces, Lemma 34.3). The other follows from the fact that having relative dimension ≤ 1 is checked on the fibres and that the fibres of $X' \rightarrow S'$ (over points of the scheme S') are the same as the fibres of $S \times_{S'} X' \rightarrow S$ since $S \rightarrow S'$ is a thickening by More on Morphisms, Lemma 14.3. $\square$

0D57 **Lemma 15.8.** *Let k be a field and let $x = (X \rightarrow \text{Spec}(k))$ be an object of $\mathcal{X} = \text{Curves}$ over $\text{Spec}(k)$.*

- (1) *If k is of finite type over $\mathbf{Z}$, then the vector spaces $T\mathcal{F}_{\mathcal{X},k,x}$ and $\text{Inf}(\mathcal{F}_{\mathcal{X},k,x})$ (see Artin's Axioms, Section 8) are finite dimensional, and*
- (2) *in general the vector spaces $T_x(k)$ and $\text{Inf}_x(k)$ (see Artin's Axioms, Section 21) are finite dimensional.*

Proof. This is immediate from the fully faithful embedding (15.1.1) and the corresponding fact for $\mathcal{S}paces'_{fp,flat,proper}$ (Lemma 13.8). $\square$

0D58 **Lemma 15.9.** *Consider the stack Curves over the base scheme $\text{Spec}(\mathbf{Z})$. Then every formal object is effective.*

Proof. For definitions of the notions in the lemma, please see Artin's Axioms, Section 9. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian complete local ring. Let $(X_n \rightarrow \mathrm{Spec}(A/\mathfrak{m}^n))$ be a formal object of *Curves* over A . By More on Morphisms of Spaces, Lemma 43.5 there exists a projective morphism $X \rightarrow \mathrm{Spec}(A)$ and a compatible system of isomorphisms $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n) \cong X_n$. By More on Morphisms, Lemma 12.4 we see that $X \rightarrow \mathrm{Spec}(A)$ is flat. By More on Morphisms, Lemma 30.6 we see that $X \rightarrow \mathrm{Spec}(A)$ has relative dimension ≤ 1 . This proves the lemma. $\square$

0D59 **Lemma 15.10.** *The stack in groupoids $\mathcal{X} = \mathrm{Curves}$ satisfies openness of versality over $\mathrm{Spec}(\mathbf{Z})$. Similarly, after base change (Remark 15.5) openness of versality holds over any Noetherian base scheme S .*

Proof. This is immediate from the fully faithful embedding (15.1.1) and the corresponding fact for $\mathrm{Spaces}'_{fp,flat,proper}$ (Lemma 13.9). $\square$

0D5A **Theorem 15.11** (Algebraicity of the stack of curves). *The stack Curves (Situation 15.1) is algebraic. In fact, for any algebraic space B the stack $B\text{-Curves}$ (Remark 15.5) is algebraic.*

See [dJHS11, Proposition 3.3, page 8] and [Smy13, Appendix B by Jack Hall, Theorem B.1].

Proof. The absolute case follows from Artin's Axioms, Lemma 17.1 and Lemmas 15.4, 15.7, 15.6, 15.9, and 15.10. The case over B follows from this, the description of $B\text{-Curves}$ as a 2-fibre product in Remark 15.5, and the fact that algebraic stacks have 2-fibre products, see Algebraic Stacks, Lemma 14.3. $\square$

0D5B **Lemma 15.12.** *The 1-morphism (15.1.1)*

$$\mathrm{Curves} \longrightarrow \mathrm{Spaces}'_{fp,flat,proper}$$

is representable by open and closed immersions.

Proof. Since (15.1.1) is a fully faithful embedding of categories it suffices to show the following: given an object $X \rightarrow S$ of $\mathrm{Spaces}'_{fp,flat,proper}$ there exists an open and closed subscheme $U \subset S$ such that a morphism $S' \rightarrow S$ factors through U if and only if the base change $X' \rightarrow S'$ of $X \rightarrow S$ has relative dimension ≤ 1 . This follows immediately from More on Morphisms of Spaces, Lemma 31.5. $\square$

0D5C **Remark 15.13.** Consider the 2-fibre product

$$\begin{array}{ccc} \mathrm{Curves} \times_{\mathrm{Spaces}'_{fp,flat,proper}} \mathrm{Polarized} & \longrightarrow & \mathrm{Polarized} \\ \downarrow & & \downarrow \\ \mathrm{Curves} & \longrightarrow & \mathrm{Spaces}'_{fp,flat,proper} \end{array}$$

This fibre product parametrized polarized curves, i.e., families of curves endowed with a relatively ample invertible sheaf. It turns out that the left vertical arrow

$$\mathrm{PolarizedCurves} \longrightarrow \mathrm{Curves}$$

is algebraic, smooth, and surjective. Namely, this 1-morphism is algebraic (as base change of the arrow in Lemma 14.6), every point is in the image, and there are no obstructions to deforming invertible sheaves on curves (see proof of Lemma 15.9). This gives another approach to the algebraicity of *Curves*. Namely, by Lemma 15.12 we see that *PolarizedCurves* is an open and closed substack of the algebraic stack *Polarized* and any stack in groupoids which is the target of a smooth algebraic morphism from an algebraic stack is an algebraic stack.

16. Moduli of complexes on a proper morphism

0DLB The title and the material of this section are taken from [Lie06a]. Let S be a scheme and let $f : X \rightarrow B$ be a proper, flat, finitely presented morphism of algebraic spaces. We will prove that there is an algebraic stack

$$\mathcal{Complexes}_{X/B}$$

parametrizing “families” of objects of D_{Coh}^b of the fibres with vanishing negative self-exts. More precisely a family is given by a relatively perfect object of the derived category of the total space; this somewhat technical notion is studied in More on Morphisms of Spaces, Section 52.

Already if X is a proper algebraic space over a field k we obtain a very interesting algebraic stack. Namely, there is an embedding

$$\mathcal{Coh}_{X/k} \longrightarrow \mathcal{Complexes}_{X/k}$$

since for any $\mathcal{O}$ -module $\mathcal{F}$ (on any ringed topos) we have $\mathrm{Ext}_{\mathcal{O}}^i(\mathcal{F}, \mathcal{F}) = 0$ for $i < 0$. Although this certainly shows our stack is nonempty, the true motivation for the study of $\mathcal{Complexes}_{X/k}$ is that there are often objects of the derived category $D_{Coh}^b(\mathcal{O}_X)$ with vanishing negative self-exts and nonvanishing cohomology sheaves in more than one degree. For example, X could be derived equivalent to another proper algebraic space Y over k , i.e., we have a k -linear equivalence

$$F : D_{Coh}^b(\mathcal{O}_Y) \longrightarrow D_{Coh}^b(\mathcal{O}_X)$$

There are cases where this happens and F is not given by an isomorphism between X and Y ; for example in the case of an abelian variety and its dual. In this situation F induces an isomorphism of algebraic stacks

$$\mathcal{Complexes}_{Y/k} \longrightarrow \mathcal{Complexes}_{X/k}$$

(insert future reference here) and in particular the stack of coherent sheaves on Y maps into the stack of complexes on X . Turning this around, if we can understand well enough the geometry of $\mathcal{Complexes}_{X/k}$, then we can try to use this to study all possible derived equivalent Y .

0DLC **Lemma 16.1.** *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f is proper, flat, and of finite presentation. Let $K, E \in D(\mathcal{O}_X)$. Assume K is pseudo-coherent and E is Y -perfect (More on Morphisms of Spaces, Definition 52.1). For a field k and a morphism $y : \mathrm{Spec}(k) \rightarrow Y$ denote K_y, E_y the pullback to the fibre X_y .*

(1) *There is an open $W \subset Y$ characterized by the property*

$$y \in |W| \Leftrightarrow \mathrm{Ext}_{\mathcal{O}_{X_y}}^i(K_y, E_y) = 0 \text{ for } i < 0.$$

(2) *For any morphism $V \rightarrow Y$ factoring through W we have*

$$\mathrm{Ext}_{\mathcal{O}_{X_V}}^i(K_V, E_V) = 0 \text{ for } i < 0$$

where X_V is the base change of X and K_V and E_V are the derived pullbacks of K and E to X_V .

(3) *The functor $V \mapsto \mathrm{Hom}_{\mathcal{O}_{X_V}}(K_V, E_V)$ is a sheaf on $(\mathrm{Spaces}/W)_{fppf}$ representable by an algebraic space affine and of finite presentation over W .*

Proof. For any morphism $V \rightarrow Y$ the complex K_V is pseudo-coherent (Cohomology on Sites, Lemma 45.3) and E_V is V -perfect (More on Morphisms of Spaces, Lemma 52.6). Another observation is that given $y : \text{Spec}(k) \rightarrow Y$ and a field extension k'/k with $y' : \text{Spec}(k') \rightarrow Y$ the induced morphism, we have

$$\text{Ext}_{\mathcal{O}_{X_{y'}}}^i(K_{y'}, E_{y'}) = \text{Ext}_{\mathcal{O}_{X_y}}^i(K_y, E_y) \otimes_k k'$$

by Derived Categories of Schemes, Lemma 22.6. Thus the vanishing in (1) is really a property of the induced point $y \in |Y|$. We will use these two observations without further mention in the proof.

Assume first Y is an affine scheme. Then we may apply More on Morphisms of Spaces, Lemma 52.11 and find a pseudo-coherent $L \in D(\mathcal{O}_Y)$ which “universally computes” $Rf_* R\mathcal{H}om(K, E)$ in the sense described in that lemma. Unwinding the definitions, we obtain for a point $y \in Y$ the equality

$$\text{Ext}_{\kappa(y)}^i(L \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \kappa(y), \kappa(y)) = \text{Ext}_{\mathcal{O}_{X_y}}^i(K_y, E_y)$$

We conclude that

$$H^i(L \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \kappa(y)) = 0 \text{ for } i > 0 \Leftrightarrow \text{Ext}_{\mathcal{O}_{X_y}}^i(K_y, E_y) = 0 \text{ for } i < 0.$$

By Derived Categories of Schemes, Lemma 31.1 the set W of $y \in Y$ where this happens defines an open of Y . This open W then satisfies the requirement in (1) for all morphisms from spectra of fields, by the “universality” of L .

Let’s go back to Y a general algebraic space. Choose an étale covering $\{V_i \rightarrow Y\}$ by affine schemes V_i . Then we see that the subset $W \subset |Y|$ pulls back to the corresponding subset $W_i \subset |V_i|$ for X_{V_i} , K_{V_i} , E_{V_i} . By the previous paragraph we find that W_i is open, hence W is open. This proves (1) in general. Moreover, parts (2) and (3) are entirely formulated in terms of the category Spaces/W and the restrictions X_W , K_W , E_W . This reduces us to the case $W = Y$.

Assume $W = Y$. We claim that for any algebraic space V over Y we have $Rf_{V,*} R\mathcal{H}om(K_V, E_V)$ has vanishing cohomology sheaves in degrees < 0 . This will prove (2) because

$$\text{Ext}_{\mathcal{O}_{X_V}}^i(K_V, E_V) = H^i(X_V, R\mathcal{H}om(K_V, E_V)) = H^i(V, Rf_{V,*} R\mathcal{H}om(K_V, E_V))$$

by Cohomology on Sites, Lemmas 35.1 and 20.5 and the vanishing of the cohomology sheaves implies the cohomology group H^i is zero for $i < 0$ by Derived Categories, Lemma 16.1.

To prove the claim, we may work étale locally on V . In particular, we may assume Y is affine and $W = Y$. Let $L \in D(\mathcal{O}_Y)$ be as in the second paragraph of the proof. For an algebraic space V over Y denote L_V the derived pullback of L to V . (An important feature we will use is that L “works” for all algebraic spaces V over Y and not just affine V .) As $W = Y$ we have $H^i(L) = 0$ for $i > 0$ (use More on Algebra, Lemma 75.6 to go from fibres to stalks). Hence $H^i(L_V) = 0$ for $i > 0$. The property defining L is that

$$Rf_{V,*} R\mathcal{H}om(K_V, E_V) = R\mathcal{H}om(L_V, \mathcal{O}_V)$$

Since L_V sits in degrees ≤ 0 , we conclude that $R\mathcal{H}om(L_V, \mathcal{O}_V)$ sits in degrees ≥ 0 thereby proving the claim. This finishes the proof of (2).

Assume $W = Y$ but make no assumptions on the algebraic space Y . Since we have (2), we see from *Simplicial Spaces*, Lemma 35.1 that the functor F given by $F(V) = \mathrm{Hom}_{\mathcal{O}_{X_V}}(K_V, E_V)$ is a sheaf⁴ on $(\mathrm{Spaces}/Y)_{fppf}$. Thus to prove that F is an algebraic space and that $F \rightarrow Y$ is affine and of finite presentation, we may work étale locally on Y ; see *Bootstrap*, Lemma 11.2 and *Morphisms of Spaces*, Lemmas 20.3 and 28.4. We conclude that it suffices to prove F is an affine algebraic space of finite presentation over Y when Y is an affine scheme. In this case we go back to our pseudo-coherent complex $L \in D(\mathcal{O}_Y)$. Since $H^i(L) = 0$ for $i > 0$, we can represent L by a complex of the form

$$\dots \rightarrow \mathcal{O}_Y^{\oplus m_1} \rightarrow \mathcal{O}_Y^{\oplus m_0} \rightarrow 0 \rightarrow \dots$$

with the last term in degree 0, see *More on Algebra*, Lemma 64.5. Combining the two displayed formulas earlier in the proof we find that

$$F(V) = \mathrm{Ker}(\mathrm{Hom}_V(\mathcal{O}_V^{\oplus m_0}, \mathcal{O}_V) \rightarrow \mathrm{Hom}_V(\mathcal{O}_V^{\oplus m_1}, \mathcal{O}_V))$$

In other words, there is a fibre product diagram

$$\begin{array}{ccc} F & \longrightarrow & Y \\ \downarrow & & \downarrow 0 \\ \mathbf{A}_Y^{m_0} & \longrightarrow & \mathbf{A}_Y^{m_1} \end{array}$$

which proves what we want. $\square$

ODLD Lemma 16.2. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over S . Assume f is proper, flat, and of finite presentation. Let $E \in D(\mathcal{O}_X)$. Assume*

- (1) *E is S -perfect (*More on Morphisms of Spaces*, Definition 52.1), and*
- (2) *for every point $s \in S$ we have*

$$\mathrm{Ext}_{\mathcal{O}_{X_s}}^i(E_s, E_s) = 0 \quad \text{for } i < 0$$

where E_s is the pullback to the fibre X_s .

Then

- (a) *(1) and (2) are preserved by arbitrary base change $V \rightarrow Y$,*
- (b) *$\mathrm{Ext}_{\mathcal{O}_{X_V}}^i(E_V, E_V) = 0$ for $i < 0$ and all V over Y ,*
- (c) *$V \mapsto \mathrm{Hom}_{\mathcal{O}_{X_V}}(E_V, E_V)$ is representable by an algebraic space affine and of finite presentation over Y .*

Here X_V is the base change of X and E_V is the derived pullback of E to X_V .

Proof. Immediate consequence of Lemma 16.1. $\square$

ODLE Situation 16.3. *Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . Assume f is proper, flat, and of finite presentation. We denote $\mathrm{Complexes}_{X/B}$ the category whose objects are triples (T, g, E) where*

- (1) *T is a scheme over S ,*
- (2) *$g : T \rightarrow B$ is a morphism over S , and setting $X_T = T \times_{g,B} X$*
- (3) *E is an object of $D(\mathcal{O}_{X_T})$ satisfying conditions (1) and (2) of Lemma 16.2.*

⁴To check the sheaf property for a covering $\{V_i \rightarrow V\}_{i \in I}$ first consider the Čech fppf hypercovering $a : V_\bullet \rightarrow V$ with $V_n = \coprod_{i_0 \dots i_n} V_{i_0} \times_V \dots \times_V V_{i_n}$ and then set $U_\bullet = V_\bullet \times_{a,V} X_V$. Then $U_\bullet \rightarrow X_V$ is an fppf hypercovering to which we may apply *Simplicial Spaces*, Lemma 35.1.

A morphism $(T, g, E) \rightarrow (T', g', E')$ is given by a pair (h, φ) where

- (1) $h : T \rightarrow T'$ is a morphism of schemes over B (i.e., $g' \circ h = g$), and
- (2) $\varphi : L(h')^* E' \rightarrow E$ is an isomorphism of $D(\mathcal{O}_{X_T})$ where $h' : X_T \rightarrow X_{T'}$ is the base change of h .

Thus $\text{Complexes}_{X/B}$ is a category and the rule

$$p : \text{Complexes}_{X/B} \longrightarrow (\text{Sch}/S)_{fppf}, \quad (T, g, E) \longmapsto T$$

is a functor. For a scheme T over S we denote $\text{Complexes}_{X/B, T}$ the fibre category of p over T . These fibre categories are groupoids.

0DLF **Lemma 16.4.** *In Situation 16.3 the functor $p : \text{Complexes}_{X/B} \longrightarrow (\text{Sch}/S)_{fppf}$ is fibred in groupoids.*

Proof. We show that p is fibred in groupoids by checking conditions (1) and (2) of Categories, Definition 35.1. Given an object (T', g', E') of $\text{Complexes}_{X/B}$ and a morphism $h : T \rightarrow T'$ of schemes over S we can set $g = h \circ g'$ and $E = L(h')^* E'$ where $h' : X_T \rightarrow X_{T'}$ is the base change of h . Then it is clear that we obtain a morphism $(T, g, E) \rightarrow (T', g', E')$ of $\text{Complexes}_{X/B}$ lying over h . This proves (1). For (2) suppose we are given morphisms

$$(h_1, \varphi_1) : (T_1, g_1, E_1) \rightarrow (T, g, E) \quad \text{and} \quad (h_2, \varphi_2) : (T_2, g_2, E_2) \rightarrow (T, g, E)$$

of $\text{Complexes}_{X/B}$ and a morphism $h : T_1 \rightarrow T_2$ such that $h_2 \circ h = h_1$. Then we can let φ be the composition

$$L(h')^* E_2 \xrightarrow{L(h')^* \varphi_2^{-1}} L(h')^* L(h_2)^* E = L(h_1)^* E \xrightarrow{\varphi_1} E_1$$

to obtain the morphism $(h, \varphi) : (T_1, g_1, E_1) \rightarrow (T_2, g_2, E_2)$ that witnesses the truth of condition (2). $\square$

0DLG **Lemma 16.5.** *In Situation 16.3. Denote $\mathcal{X} = \text{Complexes}_{X/B}$. Then $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.*

Proof. Consider two objects $x = (T, g, E)$ and $y = (T, g', E')$ of $\mathcal{X}$ over a scheme T . We have to show that $\text{Isom}_{\mathcal{X}}(x, y)$ is an algebraic space over T , see Algebraic Stacks, Lemma 10.11. If for $h : T' \rightarrow T$ the restrictions $x|_{T'}$ and $y|_{T'}$ are isomorphic in the fibre category $\mathcal{X}_{T'}$, then $g \circ h = g' \circ h$. Hence there is a transformation of presheaves

$$\text{Isom}_{\mathcal{X}}(x, y) \longrightarrow \text{Equalizer}(g, g')$$

Since the diagonal of B is representable (by schemes) this equalizer is a scheme. Thus we may replace T by this equalizer and E and E' by their pullbacks. Thus we may assume $g = g'$.

Assume $g = g'$. After replacing B by T and X by X_T we arrive at the following problem. Given $E, E' \in D(\mathcal{O}_X)$ satisfying conditions (1), (2) of Lemma 16.2 we have to show that $\text{Isom}(E, E')$ is an algebraic space. Here $\text{Isom}(E, E')$ is the functor

$$(\text{Sch}/B)^{opp} \rightarrow \text{Sets}, \quad T \mapsto \{\varphi : E_T \rightarrow E'_T \text{ isomorphism in } D(\mathcal{O}_{X_T})\}$$

where E_T and E'_T are the derived pullbacks of E and E' to X_T . Now, let $W \subset B$, resp. $W' \subset B$ be the open subspace of B associated to E, E' , resp. to E', E by Lemma 16.1. Clearly, if there exists an isomorphism $E_T \rightarrow E'_T$ as in the definition of $\text{Isom}(E, E')$, then we see that $T \rightarrow B$ factors into both W and W' (because we

have condition (1) for E and E' and we'll obviously have $E_t \cong E'_t$ so no nonzero maps $E_t[i] \rightarrow E_t$ or $E'_t[i] \rightarrow E_t$ over the fibre X_t for $i > 0$. Thus we may replace B by the open $W \cap W'$. In this case the functor $H = \mathcal{H}om(E, E')$

$$(Sch/B)^{opp} \rightarrow Sets, \quad T \mapsto \text{Hom}_{\mathcal{O}_{X_T}}(E_T, E'_T)$$

is an algebraic space affine and of finite presentation over B by Lemma 16.1. The same is true for $H' = \mathcal{H}om(E', E)$, $I = \mathcal{H}om(E, E)$, and $I' = \mathcal{H}om(E', E')$. Therefore we can repeat the argument of the proof of Proposition 4.3 to see that

$$Isom(E, E') = (H' \times_B H) \times_{c, I \times_B I', \sigma} B$$

for some morphisms c and σ . Thus $Isom(E, E')$ is an algebraic space. $\square$

0DLH Lemma 16.6. *In Situation 16.3 the functor $p : \text{Complexes}_{X/B} \rightarrow (Sch/S)_{fppf}$ is a stack in groupoids.*

Proof. To prove that $\text{Complexes}_{X/B}$ is a stack in groupoids, we have to show that the presheaves $Isom$ are sheaves and that descent data are effective. The statement on $Isom$ follows from Lemma 16.5, see Algebraic Stacks, Lemma 10.11. Let us prove the statement on descent data.

Suppose that $\{a_i : T_i \rightarrow T\}$ is an fppf covering of schemes over S . Let (ξ_i, φ_{ij}) be a descent datum for $\{T_i \rightarrow T\}$ with values in $\text{Complexes}_{X/B}$. For each i we can write $\xi_i = (T_i, g_i, E_i)$. Denote $\text{pr}_0 : T_i \times_T T_j \rightarrow T_i$ and $\text{pr}_1 : T_i \times_T T_j \rightarrow T_j$ the projections. The condition that $\xi_i|_{T_i \times_T T_j} \cong \xi_j|_{T_i \times_T T_j}$ implies in particular that $g_i \circ \text{pr}_0 = g_j \circ \text{pr}_1$. Thus there exists a unique morphism $g : T \rightarrow B$ such that $g_i = g \circ a_i$, see Descent on Spaces, Lemma 7.2. Denote $X_T = T \times_{g, B} X$. Set $X_i = X_{T_i} = T_i \times_{g_i, B} X = T_i \times_{a_i, T} X_T$ and

$$X_{ij} = X_{T_i} \times_{X_T} X_{T_j} = X_i \times_{X_T} X_j$$

with projections pr_i and pr_j to X_i and X_j . Observe that the pullback of (T_i, g_i, E_i) by $\text{pr}_0 : T_i \times_T T_j \rightarrow T_i$ is given by $(T_i \times_T T_j, g_i \circ \text{pr}_0, L\text{pr}_i^* E_i)$. Hence a descent datum for $\{T_i \rightarrow T\}$ in $\text{Complexes}_{X/B}$ is given by the objects $(T_i, g \circ a_i, E_i)$ and for each pair i, j an isomorphism in $D\mathcal{O}_{X_{ij}}$

$$\varphi_{ij} : L\text{pr}_i^* E_i \rightarrow L\text{pr}_j^* E_j$$

satisfying the cocycle condition over the pullback of X to $T_i \times_T T_j \times_T T_k$. Using the vanishing of negative Exts provided by (b) of Lemma 16.2, we may apply Simplicial Spaces, Lemma 35.2 to obtain descent⁵ for these complexes. In other words, we find there exists an object E in $D_{QCoh}(\mathcal{O}_{X_T})$ restricting to E_i on X_{T_i} compatible with φ_{ij} . Recall that being T -perfect signifies being pseudo-coherent and having locally finite tor dimension over $f^{-1}\mathcal{O}_T$. Thus E is T -perfect by an application of More on Morphisms of Spaces, Lemmas 54.1 and 54.2. Finally, we have to check condition (2) from Lemma 16.2 for E . This immediately follows from the description of the open W in Lemma 16.1 and the fact that (2) holds for E_i on X_{T_i}/T_i . $\square$

0DLI Remark 16.7. In Situation 16.3 the rule $(T, g, E) \mapsto (T, g)$ defines a 1-morphism

$$\text{Complexes}_{X/B} \rightarrow \mathcal{S}_B$$

⁵To check this, first consider the Čech fppf hypercovering $a : T_\bullet \rightarrow T$ with $T_n = \coprod_{i_0, \dots, i_n} T_{i_0} \times_T \dots \times_T T_{i_n}$ and then set $U_\bullet = T_\bullet \times_{a, T} X_T$. Then $U_\bullet \rightarrow X_T$ is an fppf hypercovering to which we may apply Simplicial Spaces, Lemma 35.2.

of stacks in groupoids (see Lemma 16.6, Algebraic Stacks, Section 7, and Examples of Stacks, Section 10). Let $B' \rightarrow B$ be a morphism of algebraic spaces over S . Let $\mathcal{S}_{B'} \rightarrow \mathcal{S}_B$ be the associated 1-morphism of stacks fibred in sets. Set $X' = X \times_B B'$. We obtain a stack in groupoids $\text{Complexes}_{X'/B'} \rightarrow (\text{Sch}/S)_{fppf}$ associated to the base change $f' : X' \rightarrow B'$. In this situation the diagram

$$\begin{array}{ccc} \text{Complexes}_{X'/B'} & \longrightarrow & \text{Complexes}_{X/B} \\ \downarrow & & \downarrow \\ \mathcal{S}_{B'} & \longrightarrow & \mathcal{S}_B \end{array} \quad \begin{array}{c} \text{or in} \\ \text{another} \\ \text{notation} \end{array} \quad \begin{array}{ccc} \text{Complexes}_{X'/B'} & \longrightarrow & \text{Complexes}_{X/B} \\ \downarrow & & \downarrow \\ \text{Sch}/B' & \longrightarrow & \text{Sch}/B \end{array}$$

is 2-fibre product square. This trivial remark will occasionally be useful to change the base algebraic space.

0DLJ **Lemma 16.8.** *In Situation 16.3 assume that $B \rightarrow S$ is locally of finite presentation. Then $p : \text{Complexes}_{X/B} \rightarrow (\text{Sch}/S)_{fppf}$ is limit preserving (Artin's Axioms, Definition 11.1).*

Proof. Write $B(T)$ for the discrete category whose objects are the S -morphisms $T \rightarrow B$. Let $T = \lim T_i$ be a filtered limit of affine schemes over S . Assigning to an object (T, h, E) of $\text{Complexes}_{X/B, T}$ the object h of $B(T)$ gives us a commutative diagram of fibre categories

$$\begin{array}{ccc} \text{colim } \text{Complexes}_{X/B, T_i} & \longrightarrow & \text{Complexes}_{X/B, T} \\ \downarrow & & \downarrow \\ \text{colim } B(T_i) & \longrightarrow & B(T) \end{array}$$

We have to show the top horizontal arrow is an equivalence. Since we have assume that B is locally of finite presentation over S we see from Limits of Spaces, Remark 3.11 that the bottom horizontal arrow is an equivalence. This means that we may assume $T = \lim T_i$ be a filtered limit of affine schemes over B . Denote $g_i : T_i \rightarrow B$ and $g : T \rightarrow B$ the corresponding morphisms. Set $X_i = T_i \times_{g_i, B} X$ and $X_T = T \times_{g, B} X$. Observe that $X_T = \text{colim } X_i$. By More on Morphisms of Spaces, Lemma 52.9 the category of T -perfect objects of $D(\mathcal{O}_{X_T})$ is the colimit of the categories of T_i -perfect objects of $D(\mathcal{O}_{X_{T_i}})$. Thus all we have to prove is that given an T_i -perfect object E_i of $D(\mathcal{O}_{X_{T_i}})$ such that the derived pullback E of E_i to X_T satisfies condition (2) of Lemma 16.2, then after increasing i we have that E_i satisfies condition (2) of Lemma 16.2. Let $W \subset |T_i|$ be the open constructed in Lemma 16.1 for E_i and E_i . By assumption on E we find that $T \rightarrow T_i$ factors through W . Hence there is an $i' \geq i$ such that $T_{i'} \rightarrow T_i$ factors through W , see Limits, Lemma 4.10 Then i' works by construction of W . $\square$

0DLK **Lemma 16.9.** *In Situation 16.3. Let*

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout in the category of schemes over S where $Z \rightarrow Z'$ is a finite order thickening and $Z \rightarrow Y$ is affine, see More on Morphisms, Lemma 14.3. Then the

functor on fibre categories

$$\text{Complexes}_{X/B, Y'} \longrightarrow \text{Complexes}_{X/B, Y} \times_{\text{Complexes}_{X/B, Z}} \text{Complexes}_{X/B, Z'}$$

is an equivalence.

Proof. Observe that the corresponding map

$$B(Y') \longrightarrow B(Y) \times_{B(Z)} B(Z')$$

is a bijection, see Pushouts of Spaces, Lemma 6.1. Thus using the commutative diagram

$$\begin{array}{ccc} \text{Complexes}_{X/B, Y'} & \longrightarrow & \text{Complexes}_{X/B, Y} \times_{\text{Complexes}_{X/B, Z}} \text{Complexes}_{X/B, Z'} \\ \downarrow & & \downarrow \\ B(Y') & \longrightarrow & B(Y) \times_{B(Z)} B(Z') \end{array}$$

we see that we may assume that Y' is a scheme over B' . By Remark 16.7 we may replace B by Y' and X by $X \times_B Y'$. Thus we may assume $B = Y'$.

Assume $B = Y'$. We first prove fully faithfulness of our functor. To do this, let ξ_1, ξ_2 be two objects of $\text{Complexes}_{X/B}$ over Y' . Then we have to show that

$$\text{Isom}(\xi_1, \xi_2)(Y') \longrightarrow \text{Isom}(\xi_1, \xi_2)(Y) \times_{\text{Isom}(\xi_1, \xi_2)(Z)} \text{Isom}(\xi_1, \xi_2)(Z')$$

is bijective. However, we already know that $\text{Isom}(\xi_1, \xi_2)$ is an algebraic space over $B = Y'$. Thus this bijectivity follows from Artin's Axioms, Lemma 4.1 (or the aforementioned Pushouts of Spaces, Lemma 6.1).

Essential surjectivity. Let $(E_Y, E_{Z'}, \alpha)$ be a triple, where $E_Y \in D(\mathcal{O}_Y)$ and $E_{Z'} \in D(\mathcal{O}_{X_{Z'}})$ are objects such that $(Y, Y \rightarrow B, E_Y)$ is an object of $\text{Complexes}_{X/B}$ over Y , such that $(Z', Z' \rightarrow B, E_{Z'})$ is an object of $\text{Complexes}_{X/B}$ over Z' , and $\alpha : L(X_Z \rightarrow X_Y)^* E_Y \rightarrow L(X_Z \rightarrow X_{Z'})^* E_{Z'}$ is an isomorphism in $D(\mathcal{O}_{Z'})$. That is to say

$$((Y, Y \rightarrow B, E_Y), (Z', Z' \rightarrow B, E_{Z'}), \alpha)$$

is an object of the target of the arrow of our lemma. Observe that the diagram

$$\begin{array}{ccc} X_Z & \longrightarrow & X_{Z'} \\ \downarrow & & \downarrow \\ X_Y & \longrightarrow & X_{Y'} \end{array}$$

is a pushout with $X_Z \rightarrow X_Y$ affine and $X_Z \rightarrow X_{Z'}$ a thickening (see Pushouts of Spaces, Lemma 6.7). Hence by Pushouts of Spaces, Lemma 8.1 we find an object $E_{Y'} \in D(\mathcal{O}_{X_{Y'}})$ together with isomorphisms $L(X_Y \rightarrow X_{Y'})^* E_{Y'} \rightarrow E_Y$ and $L(X_{Z'} \rightarrow X_{Y'})^* E_{Y'} \rightarrow E_{Z'}$ compatible with α . Clearly, if we show that $E_{Y'}$ is Y' -perfect, then we are done, because property (2) of Lemma 16.2 is a property on points (and Y and Y' have the same points). This follows from More on Morphisms of Spaces, Lemma 54.4. $\square$

ODLL Lemma 16.10. *In Situation 16.3 assume that S is a locally Noetherian scheme and $B \rightarrow S$ is locally of finite presentation. Let k be a finite type field over S and let $x_0 = (\text{Spec}(k), g_0, E_0)$ be an object of $\mathcal{X} = \text{Complexes}_{X/B}$ over k . Then the spaces $T\mathcal{F}_{\mathcal{X}, k, x_0}$ and $\text{Inf}(\mathcal{F}_{\mathcal{X}, k, x_0})$ (Artin's Axioms, Section 8) are finite dimensional.*

Proof. Observe that by Lemma 16.9 our stack in groupoids $\mathcal{X}$ satisfies property (RS*) defined in Artin's Axioms, Section 18. In particular $\mathcal{X}$ satisfies (RS). Hence all associated predeformation categories are deformation categories (Artin's Axioms, Lemma 6.1) and the statement makes sense.

In this paragraph we show that we can reduce to the case $B = \text{Spec}(k)$. Set $X_0 = \text{Spec}(k) \times_{g_0, B} X$ and denote $\mathcal{X}_0 = \text{Complexes}_{X_0/k}$. In Remark 16.7 we have seen that $\mathcal{X}_0$ is the 2-fibre product of $\mathcal{X}$ with $\text{Spec}(k)$ over B as categories fibred in groupoids over $(\text{Sch}/S)_{fppf}$. Thus by Artin's Axioms, Lemma 8.2 we reduce to proving that B , $\text{Spec}(k)$, and $\mathcal{X}_0$ have finite dimensional tangent spaces and infinitesimal automorphism spaces. The tangent space of B and $\text{Spec}(k)$ are finite dimensional by Artin's Axioms, Lemma 8.1 and of course these have vanishing Inf. Thus it suffices to deal with $\mathcal{X}_0$.

Let $k[\epsilon]$ be the dual numbers over k . Let $\text{Spec}(k[\epsilon]) \rightarrow B$ be the composition of $g_0 : \text{Spec}(k) \rightarrow B$ and the morphism $\text{Spec}(k[\epsilon]) \rightarrow \text{Spec}(k)$ coming from the inclusion $k \rightarrow k[\epsilon]$. Set $X_0 = \text{Spec}(k) \times_B X$ and $X_\epsilon = \text{Spec}(k[\epsilon]) \times_B X$. Observe that X_ϵ is a first order thickening of X_0 flat over the first order thickening $\text{Spec}(k) \rightarrow \text{Spec}(k[\epsilon])$. Observe that X_0 and X_ϵ give rise to canonically equivalent small étale topoi, see More on Morphisms of Spaces, Section 9. By More on Morphisms of Spaces, Lemma 54.4 we see that $T\mathcal{F}_{\mathcal{X}_0, k, x_0}$ is the set of isomorphism classes of lifts of E_0 to X_ϵ in the sense of Deformation Theory, Lemma 16.7. We conclude that

$$T\mathcal{F}_{\mathcal{X}_0, k, x_0} = \text{Ext}_{\mathcal{O}_{X_0}}^1(E_0, E_0)$$

Here we have used the identification $\epsilon k[\epsilon] \cong k$ of $k[\epsilon]$ -modules. Using Deformation Theory, Lemma 16.7 once more we see that there is a surjection

$$\text{Inf}(\mathcal{F}_{\mathcal{X}, k, x_0}) \leftarrow \text{Ext}_{\mathcal{O}_{X_0}}^0(E_0, E_0)$$

of k -vector spaces. As E_0 is pseudo-coherent it lies in $D_{\text{Coh}}^-(\mathcal{O}_{X_0})$ by Derived Categories of Spaces, Lemma 13.7. Since E_0 locally has finite tor dimension and X_0 is quasi-compact we see $E_0 \in D_{\text{Coh}}^b(\mathcal{O}_{X_0})$. Thus the Exts above are finite dimensional k -vector spaces by Derived Categories of Spaces, Lemma 8.4. $\square$

0DLM **Lemma 16.11.** *In Situation 16.3 assume $B = S$ is locally Noetherian. Then strong formal effectiveness in the sense of Artin's Axioms, Remark 20.2 holds for $p : \text{Complexes}_{X/S} \rightarrow (\text{Sch}/S)_{fppf}$.*

Proof. Let (R_n) be an inverse system of S -algebras with surjective transition maps whose kernels are locally nilpotent. Set $R = \lim R_n$. Let (ξ_n) be a system of objects of $\text{Complexes}_{X/B}$ lying over $(\text{Spec}(R_n))$. We have to show (ξ_n) is effective, i.e., there exists an object ξ of $\text{Complexes}_{X/B}$ lying over $\text{Spec}(R)$.

Write $X_R = \text{Spec}(R) \times_S X$ and $X_n = \text{Spec}(R_n) \times_S X$. Of course X_n is the base change of X_R by $R \rightarrow R_n$. Since $S = B$, we see that ξ_n corresponds simply to an R_n -perfect object $E_n \in D(\mathcal{O}_{X_n})$ satisfying condition (2) of Lemma 16.2. In particular E_n is pseudo-coherent. The isomorphisms $\xi_{n+1}|_{\text{Spec}(R_n)} \cong \xi_n$ correspond to isomorphisms $L(X_n \rightarrow X_{n+1})^* E_{n+1} \rightarrow E_n$. Therefore by Flatness on Spaces, Theorem 13.6 we find a pseudo-coherent object E of $D(\mathcal{O}_{X_R})$ with E_n equal to the derived pullback of E for all n compatible with the transition isomorphisms.

Observe that $(R, \text{Ker}(R \rightarrow R_1))$ is a henselian pair, see More on Algebra, Lemma 11.3. In particular, $\text{Ker}(R \rightarrow R_1)$ is contained in the Jacobson radical of R . Then we may apply More on Morphisms of Spaces, Lemma 54.5 to see that E is R -perfect.

Finally, we have to check condition (2) of Lemma 16.2. By Lemma 16.1 the set of points t of $\text{Spec}(R)$ where the negative self-exts of E_t vanish is an open. Since this condition is true in $V(\text{Ker}(R \rightarrow R_1))$ and since $\text{Ker}(R \rightarrow R_1)$ is contained in the Jacobson radical of R we conclude it holds for all points. $\square$

0DLN **Theorem 16.12** (Algebraicity of moduli of complexes on a proper morphism). *Let S be a scheme. Let $f : X \rightarrow B$ be morphism of algebraic spaces over S . Assume that f is proper, flat, and of finite presentation. Then $\text{Complexes}_{X/B}$ is an algebraic stack over S .* [Lie06a]

Proof. Set $\mathcal{X} = \text{Complexes}_{X/B}$. We have seen that $\mathcal{X}$ is a stack in groupoids over $(\text{Sch}/S)_{fppf}$ with diagonal representable by algebraic spaces (Lemmas 16.6 and 16.5). Hence it suffices to find a scheme W and a surjective and smooth morphism $W \rightarrow \mathcal{X}$.

Let B' be a scheme and let $B' \rightarrow B$ be a surjective étale morphism. Set $X' = B' \times_B X$ and denote $f' : X' \rightarrow B'$ the projection. Then $\mathcal{X}' = \text{Complexes}_{X'/B'}$ is equal to the 2-fibre product of $\mathcal{X}$ with the category fibred in sets associated to B' over the category fibred in sets associated to B (Remark 16.7). By the material in Algebraic Stacks, Section 10 the morphism $\mathcal{X}' \rightarrow \mathcal{X}$ is surjective and étale. Hence it suffices to prove the result for $\mathcal{X}'$. In other words, we may assume B is a scheme.

Assume B is a scheme. In this case we may replace S by B , see Algebraic Stacks, Section 19. Thus we may assume $S = B$.

Assume $S = B$. Choose an affine open covering $S = \bigcup U_i$. Denote $\mathcal{X}_i$ the restriction of $\mathcal{X}$ to $(\text{Sch}/U_i)_{fppf}$. If we can find schemes W_i over U_i and surjective smooth morphisms $W_i \rightarrow \mathcal{X}_i$, then we set $W = \coprod W_i$ and we obtain a surjective smooth morphism $W \rightarrow \mathcal{X}$. Thus we may assume $S = B$ is affine.

Assume $S = B$ is affine, say $S = \text{Spec}(\Lambda)$. Write $\Lambda = \text{colim } \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \rightarrow \text{Spec}(\Lambda_i)$ which is proper, flat, of finite presentation and whose base change to Λ is X . See Limits of Spaces, Lemmas 7.1, 6.12, and 6.13. If we show that $\text{Complexes}_{X_i/\text{Spec}(\Lambda_i)}$ is an algebraic stack, then it follows by base change (Remark 16.7 and Algebraic Stacks, Section 19) that $\mathcal{X}$ is an algebraic stack. Thus we may assume that Λ is a finite type $\mathbf{Z}$ -algebra.

Assume $S = B = \text{Spec}(\Lambda)$ is affine of finite type over $\mathbf{Z}$. In this case we will verify conditions (1), (2), (3), (4), and (5) of Artin's Axioms, Lemma 17.1 to conclude that $\mathcal{X}$ is an algebraic stack. Note that Λ is a G-ring, see More on Algebra, Proposition 50.12. Hence all local rings of S are G-rings. Thus (5) holds. To check (2) we have to verify axioms [-1], [0], [1], [2], and [3] of Artin's Axioms, Section 14. We omit the verification of [-1] and axioms [0], [1], [2], [3] correspond respectively to Lemmas 16.6, 16.8, 16.9, 16.10. Condition (3) follows from Lemma 16.11. Condition (1) is Lemma 16.5.

It remains to show condition (4) which is openness of versality. To see this we will use Artin's Axioms, Lemma 20.3. We have already seen that $\mathcal{X}$ has diagonal representable by algebraic spaces, has (RS*), and is limit preserving (see lemmas used

above). Hence we only need to see that $\mathcal{X}$ satisfies the strong formal effectiveness formulated in Artin's Axioms, Lemma 20.3. This follows from Lemma 16.11 and the proof is complete. $\square$

17. Other chapters

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