MODULES ON SITES

03A4

Contents

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1. Introduction

03A5 In this document we work out basic notions of sheaves of modules on ringed topoi or ringed sites. We first work out some basic facts on abelian sheaves. After this we introduce ringed sites and ringed topoi. We work through some of the very basic notions on (pre)sheaves of $\mathcal{O}$-modules, analogous to the material on (pre)sheaves of $\mathcal{O}$-modules in the chapter on sheaves on spaces. Having done this, we duplicate much of the discussion in the chapter on sheaves of modules (see Modules, Section 1). Basic references are [Ser55], [DG67] and [AGV71].

2. Abelian presheaves

03A6 Let $\mathcal{C}$ be a category. Abelian presheaves were introduced in Sites, Sections 2 and 7 and discussed a bit more in Sites, Section 44. We will follow the convention of this last reference, in that we think of an abelian presheaf as a presheaf of sets endowed with addition rules on all sets of sections compatible with the restriction mappings. Recall that the category of abelian presheaves on $\mathcal{C}$ is denoted $PAb(\mathcal{C})$.

The category $PAb(\mathcal{C})$ is abelian as defined in Homology, Definition 5.1. Given a map of presheaves $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ the kernel of φ is the abelian presheaf $U \mapsto \mathrm{Ker}(\mathcal{G}_1(U) \to \mathcal{G}_2(U))$ and the cokernel of φ is the presheaf $U \mapsto \mathrm{Coker}(\mathcal{G}_1(U) \to \mathcal{G}_2(U))$. Since the category of abelian groups is abelian it follows that $\mathrm{Coim} = \mathrm{Im}$ because this holds over each U. A sequence of abelian presheaves

$$\mathcal{G}_1 \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_3$$

is exact if and only if $\mathcal{G}_1(U) \to \mathcal{G}_2(U) \to \mathcal{G}_3(U)$ is an exact sequence of abelian groups for all $U \in \mathrm{Ob}(\mathcal{C})$. We leave the verifications to the reader.

03CL **Lemma 2.1.** *Let $\mathcal{C}$ be a category.*

1. *All limits and colimits exist in $PAb(\mathcal{C})$.*
2. *All limits and colimits commute with taking sections over objects of $\mathcal{C}$.*

Proof. Let $\mathcal{I} \to PAb(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. We can simply define abelian presheaves L and C by the rules

$$L : U \longmapsto \lim_i \mathcal{F}_i(U)$$

and

$$C : U \longmapsto \mathrm{colim}_i \mathcal{F}_i(U).$$

It is clear that there are maps of abelian presheaves $L \to \mathcal{F}_i$ and $\mathcal{F}_i \to C$, by using the corresponding maps on groups of sections over each U. It is straightforward to check that L and C endowed with these maps are the limit and colimit of the diagram in $PAb(\mathcal{C})$. This proves (1) and (2). Details omitted. $\square$

3. Abelian sheaves

03CM Let $\mathcal{C}$ be a site. The category of abelian sheaves on $\mathcal{C}$ is denoted $Ab(\mathcal{C})$. It is the full subcategory of $PAb(\mathcal{C})$ consisting of those abelian presheaves whose underlying presheaves of sets are sheaves. Properties (α) – (ζ) of Sites, Section 44 hold, see Sites, Proposition 44.3. In particular the inclusion functor $Ab(\mathcal{C}) \to PAb(\mathcal{C})$ has a left adjoint, namely the sheafification functor $\mathcal{G} \mapsto \mathcal{G}^\#$.

We suggest the reader prove the lemma on a piece of scratch paper rather than reading the proof.

03CN **Lemma 3.1.** *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of abelian sheaves on $\mathcal{C}$.*

1. *The category $Ab(\mathcal{C})$ is an abelian category.*
2. *The kernel $\mathrm{Ker}(\varphi)$ of φ is the same as the kernel of φ as a morphism of presheaves.*
3. *The morphism φ is injective (Homology, Definition 5.3) if and only if φ is injective as a map of presheaves (Sites, Definition 3.1), if and only if φ is injective as a map of sheaves (Sites, Definition 11.1).*
4. *The cokernel $\mathrm{Coker}(\varphi)$ of φ is the sheafification of the cokernel of φ as a morphism of presheaves.*
5. *The morphism φ is surjective (Homology, Definition 5.3) if and only if φ is surjective as a map of sheaves (Sites, Definition 11.1).*
6. *A complex of abelian sheaves*
$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$
is exact at $\mathcal{G}$ if and only if for all $U \in \mathrm{Ob}(\mathcal{C})$ and all $s \in \mathcal{G}(U)$ mapping to zero in $\mathcal{H}(U)$ there exists a covering $\{U_i \to U\}_{i \in I}$ in $\mathcal{C}$ such that each $s|_{U_i}$ is in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

Proof. We claim that Homology, Lemma 7.4 applies to the categories $\mathcal{A} = Ab(\mathcal{C})$ and $\mathcal{B} = PAb(\mathcal{C})$, and the functors $a : \mathcal{A} \to \mathcal{B}$ (inclusion), and $b : \mathcal{B} \to \mathcal{A}$ (sheafification). Let us check the assumptions of Homology, Lemma 7.4. Assumption (1) is that $\mathcal{A}$, $\mathcal{B}$ are additive categories, a, b are additive functors, and a is right adjoint to b. The first two statements are clear and adjointness is Sites, Section 44 (ϵ). Assumption (2) says that $PAb(\mathcal{C})$ is abelian which we saw in Section 2 and that sheafification is left exact, which is Sites, Section 44 (ζ). The final assumption is that $ba \cong \mathrm{id}_\mathcal{A}$ which is Sites, Section 44 (δ). Hence Homology, Lemma 7.4 applies and we conclude that $Ab(\mathcal{C})$ is abelian.

In the proof of Homology, Lemma 7.4 it is shown that $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are equal to the sheafification of the kernel and cokernel of φ as a morphism of abelian presheaves. This proves (4). Since the kernel is a equalizer (i.e., a limit) and since sheafification commutes with finite limits, we conclude that (2) holds.

Statement (2) implies (3). Statement (4) implies (5) by our description of sheafification. The characterization of exactness in (6) follows from (2) and (5), and the fact that the sequence is exact if and only if $\mathrm{Im}(\mathcal{F} \to \mathcal{G}) = \mathrm{Ker}(\mathcal{G} \to \mathcal{H})$. $\square$

Another way to say part (6) of the lemma is that a sequence of abelian sheaves

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3$$

is exact if and only if the sheafification of $U \mapsto \mathrm{Im}(\mathcal{F}_1(U) \to \mathcal{F}_2(U))$ is equal to the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$.

03CO **Lemma 3.2.** *Let $\mathcal{C}$ be a site.*

(1) *All limits and colimits exist in $Ab(\mathcal{C})$.*
(2) *Limits are the same as the corresponding limits of abelian presheaves over $\mathcal{C}$ (i.e., commute with taking sections over objects of $\mathcal{C}$).*
(3) *Finite direct sums are the same as the corresponding finite direct sums in the category of abelian pre-sheaves over $\mathcal{C}$.*
(4) *A colimit is the sheafification of the corresponding colimit in the category of abelian presheaves.*
(5) *Filtered colimits are exact.*

Proof. By Lemma 2.1 limits and colimits of abelian presheaves exist, and are described by taking limits and colimits on the level of sections over objects.

Let $\mathcal{I} \to Ab(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. Let $\lim_i \mathcal{F}_i$ be the limit of the diagram as an abelian presheaf. By Sites, Lemma 10.1 this is an abelian sheaf. Then it is quite easy to see that $\lim_i \mathcal{F}_i$ is the limit of the diagram in $Ab(\mathcal{C})$. This proves limits exist and (2) holds.

By Categories, Lemma 24.5, and because sheafification is left adjoint to the inclusion functor we see that $\mathrm{colim}_i \mathcal{F}$ exists and is the sheafification of the colimit in $PAb(\mathcal{C})$. This proves colimits exist and (4) holds.

Finite direct sums are the same thing as finite products in any abelian category. Hence (3) follows from (2).

Proof of (5). The statement means that given a system $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ of exact sequences of abelian sheaves over a directed set I the sequence $0 \to \mathrm{colim}\, \mathcal{F}_i \to \mathrm{colim}\, \mathcal{G}_i \to \mathrm{colim}\, \mathcal{H}_i \to 0$ is exact as well. A formal argument using Homology, Lemma 5.8 and the definition of colimits shows that the sequence $\mathrm{colim}\, \mathcal{F}_i \to \mathrm{colim}\, \mathcal{G}_i \to \mathrm{colim}\, \mathcal{H}_i \to 0$ is exact. Note that $\mathrm{colim}\, \mathcal{F}_i \to \mathrm{colim}\, \mathcal{G}_i$ is the sheafification of the map of presheaf colimits which is injective as each of the maps $\mathcal{F}_i \to \mathcal{G}_i$ is injective. Since sheafification is exact we conclude. □

4. Free abelian presheaves

03CP In order to prepare notation for the following definition, let us agree to denote the free abelian group on a set S as[1] $\mathbf{Z}[S] = \bigoplus_{s \in S} \mathbf{Z}$. It is characterized by the property

$$\mathrm{Mor}_{Ab}(\mathbf{Z}[S], A) = \mathrm{Mor}_{Sets}(S, A)$$

In other words the construction $S \mapsto \mathbf{Z}[S]$ is a left adjoint to the forgetful functor $Ab \to Sets$.

03A7 **Definition 4.1.** Let $\mathcal{C}$ be a category. Let $\mathcal{G}$ be a presheaf of sets. The *free abelian presheaf* $\mathbf{Z}_\mathcal{G}$ on $\mathcal{G}$ is the abelian presheaf defined by the rule

$$U \longmapsto \mathbf{Z}[\mathcal{G}(U)].$$

[1]In other chapters the notation $\mathbf{Z}[S]$ sometimes indicates the polynomial ring over $\mathbf{Z}$ on S.

In the special case $\mathcal{G} = h_X$ of a representable presheaf associated to an object X of $\mathcal{C}$ we use the notation $\mathbf{Z}_X = \mathbf{Z}_{h_X}$. In other words

$$\mathbf{Z}_X(U) = \mathbf{Z}[\mathrm{Mor}_\mathcal{C}(U, X)].$$

This construction is clearly functorial in the presheaf $\mathcal{G}$. In fact it is adjoint to the forgetful functor $PAb(\mathcal{C}) \to PSh(\mathcal{C})$. Here is the precise statement.

03A8 **Lemma 4.2.** *Let $\mathcal{C}$ be a category. Let $\mathcal{G}$, $\mathcal{F}$ be a presheaves of sets. Let $\mathcal{A}$ be an abelian presheaf. Let U be an object of $\mathcal{C}$. Then we have*

$$\begin{aligned}
\mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) &= \mathcal{F}(U), \\
\mathrm{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_\mathcal{G}, \mathcal{A}) &= \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{G}, \mathcal{A}), \\
\mathrm{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_U, \mathcal{A}) &= \mathcal{A}(U).
\end{aligned}$$

All of these equalities are functorial.

Proof. Omitted. $\square$

03A9 **Lemma 4.3.** *Let $\mathcal{C}$ be a category. Let I be a set. For each $i \in I$ let $\mathcal{G}_i$ be a presheaf of sets. Then*

$$\mathbf{Z}_{\coprod_i \mathcal{G}_i} = \bigoplus\nolimits_{i \in I} \mathbf{Z}_{\mathcal{G}_i}$$

in $PAb(\mathcal{C})$.

Proof. Omitted. $\square$

5. Free abelian sheaves

03CQ Here is the notion of a free abelian sheaf on a sheaf of sets.

03AA **Definition 5.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets. The *free abelian sheaf* $\mathbf{Z}_\mathcal{G}^\#$ on $\mathcal{G}$ is the abelian sheaf $\mathbf{Z}_\mathcal{G}^\#$ which is the sheafification of the free abelian presheaf on $\mathcal{G}$. In the special case $\mathcal{G} = h_X$ of a representable presheaf associated to an object X of $\mathcal{C}$ we use the notation $\mathbf{Z}_X^\#$.

This construction is clearly functorial in the presheaf $\mathcal{G}$. In fact it provides an adjoint to the forgetful functor $Ab(\mathcal{C}) \to Sh(\mathcal{C})$. Here is the precise statement.

03AB **Lemma 5.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$, $\mathcal{F}$ be a sheaves of sets. Let $\mathcal{A}$ be an abelian sheaf. Let U be an object of $\mathcal{C}$. Then we have*

$$\begin{aligned}
\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) &= \mathcal{F}(U), \\
\mathrm{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}_\mathcal{G}^\#, \mathcal{A}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{A}), \\
\mathrm{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}_U^\#, \mathcal{A}) &= \mathcal{A}(U).
\end{aligned}$$

All of these equalities are functorial.

Proof. Omitted. $\square$

03AC **Lemma 5.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets. Then $\mathbf{Z}_\mathcal{G}^\# = (\mathbf{Z}_{\mathcal{G}^\#})^\#$.*

Proof. Omitted. $\square$

6. Ringed sites

04KQ In this chapter we mainly work with sheaves of modules on a ringed site. Hence we need to define this notion.

03AD **Definition 6.1.** Ringed sites.

(1) A *ringed site* is a pair $(\mathcal{C}, \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$. The sheaf $\mathcal{O}$ is called the *structure sheaf* of the ringed site.
(2) Let $(\mathcal{C}, \mathcal{O})$, $(\mathcal{C}', \mathcal{O}')$ be ringed sites. A *morphism of ringed sites*

$$(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{C}', \mathcal{O}')$$

is given by a morphism of sites $f : \mathcal{C} \to \mathcal{C}'$ (see Sites, Definition 14.1) together with a map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, which by adjunction is the same thing as a map of sheaves of rings $f^\sharp : \mathcal{O}' \to f_*\mathcal{O}$.
(3) Let $(f, f^\sharp) : (\mathcal{C}_1, \mathcal{O}_1) \to (\mathcal{C}_2, \mathcal{O}_2)$ and $(g, g^\sharp) : (\mathcal{C}_2, \mathcal{O}_2) \to (\mathcal{C}_3, \mathcal{O}_3)$ be morphisms of ringed sites. Then we define the *composition of morphisms of ringed sites* by the rule

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Here we use composition of morphisms of sites defined in Sites, Definition 14.5 and $f^\sharp \circ g^\sharp$ indicates the morphism of sheaves of rings

$$\mathcal{O}_3 \xrightarrow{g^\sharp} g_*\mathcal{O}_2 \xrightarrow{g_*f^\sharp} g_*f_*\mathcal{O}_1 = (g \circ f)_*\mathcal{O}_1$$

7. Ringed topoi

01D2 A ringed topos is just a ringed site, except that the notion of a morphism of ringed topoi is different from the notion of a morphism of ringed sites.

01D3 **Definition 7.1.** Ringed topoi.

(1) A *ringed topos* is a pair $(Sh(\mathcal{C}), \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$. The sheaf $\mathcal{O}$ is called the *structure sheaf* of the ringed topos.
(2) Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. A *morphism of ringed topoi*

$$(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}'), \mathcal{O}')$$

is given by a morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ (see Sites, Definition 15.1) together with a map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, which by adjunction is the same thing as a map of sheaves of rings $f^\sharp : \mathcal{O}' \to f_*\mathcal{O}$.
(3) Let $(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. Then we define the *composition of morphisms of ringed topoi* by the rule

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Here we use composition of morphisms of topoi defined in Sites, Definition 15.1 and $f^\sharp \circ g^\sharp$ indicates the morphism of sheaves of rings

$$\mathcal{O}_3 \xrightarrow{g^\sharp} g_*\mathcal{O}_2 \xrightarrow{g_*f^\sharp} g_*f_*\mathcal{O}_1 = (g \circ f)_*\mathcal{O}_1$$

Every morphism of ringed topoi is the composition of an equivalence of ringed topoi with a morphism of ringed topoi associated to a morphism of ringed sites. Here is the precise statement.

03CR **Lemma 7.2.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. There exists a factorization*

$$\begin{tikzcd}
(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[r, "(f,f^\sharp)"'] \arrow[d, "(g,g^\sharp)"'] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \arrow[d, "(e,e^\sharp)"] \\
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "(h,h^\sharp)"] & (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'})
\end{tikzcd}$$

where

1. *$g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ is an equivalence of topoi induced by a special cocontinuous functor $\mathcal{C} \to \mathcal{C}'$ (see Sites, Definition 29.2),*
2. *$e : Sh(\mathcal{D}) \to Sh(\mathcal{D}')$ is an equivalence of topoi induced by a special cocontinuous functor $\mathcal{D} \to \mathcal{D}'$ (see Sites, Definition 29.2),*
3. *$\mathcal{O}_{\mathcal{C}'} = g_*\mathcal{O}_\mathcal{C}$ and $g^\sharp$ is the obvious map,*
4. *$\mathcal{O}_{\mathcal{D}'} = e_*\mathcal{O}_\mathcal{D}$ and $e^\sharp$ is the obvious map,*
5. *the sites $\mathcal{C}'$ and $\mathcal{D}'$ have final objects and fibre products (i.e., all finite limits),*
6. *h is a morphism of sites induced by a continuous functor $u : \mathcal{D}' \to \mathcal{C}'$ which commutes with all finite limits (i.e., it satisfies the assumptions of Sites, Proposition 14.7), and*
7. *given any set of sheaves $\mathcal{F}_i$ (resp. $\mathcal{G}_j$) on $\mathcal{C}$ (resp. $\mathcal{D}$) we may assume each of these is a representable sheaf on $\mathcal{C}'$ (resp. $\mathcal{D}'$).*

Moreover, if $(f, f^\sharp)$ is an equivalence of ringed topoi, then we can choose the diagram such that $\mathcal{C}' = \mathcal{D}'$, $\mathcal{O}_{\mathcal{C}'} = \mathcal{O}_{\mathcal{D}'}$ and $(h, h^\sharp)$ is the identity.

Proof. This follows from Sites, Lemma 29.6, and Sites, Remarks 29.7 and 29.8. You just have to carry along the sheaves of rings. Some details omitted. $\square$

8. 2-morphisms of ringed topoi

04IB This is a brief section concerning the notion of a 2-morphism of ringed topoi.

04IC **Definition 8.1.** Let $f, g : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be two morphisms of ringed topoi. A *2-morphism from f to g* is given by a transformation of functors $t : f_* \to g_*$ such that

$$\begin{tikzcd}
& \mathcal{O}_\mathcal{D} \arrow[dl, "f^\sharp"'] \arrow[dr, "g^\sharp"] & \\
f_*\mathcal{O}_\mathcal{C} \arrow[rr, "t"] & & g_*\mathcal{O}_\mathcal{C}
\end{tikzcd}$$

is commutative.

Pictorially we sometimes represent t as follows:

$$\begin{tikzcd}
(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow t"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})
\end{tikzcd}$$

As in Sites, Section 36 giving a 2-morphism $t : f_* \to g_*$ is equivalent to giving $t : g^{-1} \to f^{-1}$ (usually denoted by the same symbol) such that the diagram

$$\begin{tikzcd} f^{-1}\mathcal{O}_\mathcal{D} \arrow[dr, "f^\sharp"'] & & g^{-1}\mathcal{O}_\mathcal{D} \arrow[ll, "t"'] \arrow[dl, "g^\sharp"] \\ & \mathcal{O}_\mathcal{C} & \end{tikzcd}$$

is commutative. As in Sites, Section 36 the axioms of a strict 2-category hold with horizontal and vertical compositions defined as explained in loc. cit.

9. Presheaves of modules

03CS Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. At this point we have not yet defined a presheaf of $\mathcal{O}$-modules. Thus we do so right now.

03CT **Definition 9.1.** Let $\mathcal{C}$ be a category, and let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$.

(1) A *presheaf of $\mathcal{O}$-modules* is given by an abelian presheaf $\mathcal{F}$ together with a map of presheaves of sets

$$\mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F}$$

such that for every object U of $\mathcal{C}$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an $\mathcal{O}(U)$-module structure on the abelian group $\mathcal{F}(U)$.

(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules* is a morphism of abelian presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}\times\varphi"'] & \mathcal{F} \arrow[d, "\varphi"] \\ \mathcal{O} \times \mathcal{G} \arrow[r] & \mathcal{G} \end{tikzcd}$$

commutes.

(3) The set of $\mathcal{O}$-module morphisms as above is denoted $\mathrm{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G})$.

(4) The category of presheaves of $\mathcal{O}$-modules is denoted $\mathit{PMod}(\mathcal{O})$.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on the category $\mathcal{C}$. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition

$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$

We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the *restriction of $\mathcal{F}$*. We obtain the restriction functor

$$\mathit{PMod}(\mathcal{O}_2) \longrightarrow \mathit{PMod}(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule

$$U \longmapsto (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$

where $U \in \mathrm{Ob}(\mathcal{C})$, with obvious restriction mappings. The index p stands for "presheaf" and not "point". This presheaf is called the tensor product presheaf. We obtain the *change of rings* functor

$$\mathit{PMod}(\mathcal{O}_1) \longrightarrow \mathit{PMod}(\mathcal{O}_2)$$

03CU **Lemma 9.2.** *With $\mathcal{C}$, $\mathcal{O}_1 \to \mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors defined above are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. $\square$

10. Sheaves of modules

03CV

03CW **Definition 10.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$.

(1) A *sheaf of $\mathcal{O}$-modules* is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, see Definition 9.1, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.
(2) A *morphism of sheaves of $\mathcal{O}$-modules* is a morphism of presheaves of $\mathcal{O}$-modules.
(3) Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ the set of morphism of sheaves of $\mathcal{O}$-modules.
(4) The category of sheaves of $\mathcal{O}$-modules is denoted $\textit{Mod}(\mathcal{O})$.

This definition kind of makes sense even if $\mathcal{O}$ is just a presheaf of rings, although we do not know any examples where this is useful, and we will avoid using the terminology "sheaves of $\mathcal{O}$-modules" in case $\mathcal{O}$ is not a sheaf of rings.

11. Sheafification of presheaves of modules

03CX

03CY **Lemma 11.1.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{O}^\#$ be the sheafification of $\mathcal{O}$ as a presheaf of rings, see Sites, Section 44. Let $\mathcal{F}^\#$ be the sheafification of $\mathcal{F}$ as a presheaf of abelian groups. There exists a unique map of sheaves of sets*

$$\mathcal{O}^\# \times \mathcal{F}^\# \longrightarrow \mathcal{F}^\#$$

which makes the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{O}^\# \times \mathcal{F}^\# \arrow[r] & \mathcal{F}^\# \end{tikzcd}$$

commute and which makes $\mathcal{F}^\#$ into a sheaf of $\mathcal{O}^\#$-modules. In addition, if $\mathcal{G}$ is a sheaf of $\mathcal{O}^\#$-modules, then any morphism of presheaves of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ (into the restriction of $\mathcal{G}$ to a $\mathcal{O}$-module) factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ where $\mathcal{F}^\# \to \mathcal{G}$ is a morphism of $\mathcal{O}^\#$-modules.

Proof. Omitted. $\square$

This actually means that the functor $i : \textit{Mod}(\mathcal{O}^\#) \to \textit{PMod}(\mathcal{O})$ (combining restriction and including sheaves into presheaves) and the sheafification functor of the lemma ${}^\# : \textit{PMod}(\mathcal{O}) \to \textit{Mod}(\mathcal{O}^\#)$ are adjoint. In a formula

$$\mathrm{Mor}_{\textit{PMod}(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \mathrm{Mor}_{\textit{Mod}(\mathcal{O}^\#)}(\mathcal{F}^\#, \mathcal{G})$$

An important case happens when $\mathcal{O}$ is already a sheaf of rings. In this case the formula reads

$$\mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{F}^\#, \mathcal{G})$$

because $\mathcal{O} = \mathcal{O}^\#$ in this case.

03EI **Lemma 11.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$ The sheafification functor*

$$PMod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}^\#), \quad \mathcal{F} \longmapsto \mathcal{F}^\#$$

is exact.

Proof. This is true because it holds for sheafification $PAb(\mathcal{C}) \to Ab(\mathcal{C})$. See the discussion in Section 3. $\square$

Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on $\mathcal{C}$. In Section 9 we defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules then the restriction $\mathcal{F}_{\mathcal{O}_1}$ of $\mathcal{F}$ is clearly a sheaf of $\mathcal{O}_1$-modules. We obtain the restriction functor

$$Mod(\mathcal{O}_2) \longrightarrow Mod(\mathcal{O}_1)$$

On the other hand, given a sheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ the presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ is in general not a sheaf. Hence we define the *tensor product sheaf* $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$ by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})^\#$$

as the sheafification of our construction for presheaves. We obtain the *change of rings* functor

$$Mod(\mathcal{O}_1) \longrightarrow Mod(\mathcal{O}_2)$$

03CZ **Lemma 11.3.** *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from Lemma 9.2 and the fact that $\mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$ because $\mathcal{F}$ is a sheaf. $\square$

0930 **Lemma 11.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be an epimorphism of sheaves of rings. Let $\mathcal{G}_1, \mathcal{G}_2$ be $\mathcal{O}'$-modules. Then*

$$\mathrm{Hom}_{\mathcal{O}'}(\mathcal{G}_1, \mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{G}_1, \mathcal{G}_2).$$

In other words, the restriction functor $Mod(\mathcal{O}') \to Mod(\mathcal{O})$ is fully faithful.

Proof. This is the sheaf version of Algebra, Lemma 107.14 and is proved in exactly the same way. $\square$

12. Morphisms of topoi and sheaves of modules

03D0 All of this material is completely straightforward. We formulate everything in the case of morphisms of topoi, but of course the results also hold in the case of morphisms of sites.

03D1 **Lemma 12.1.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of sheaves of sets*

$$f_*\mathcal{O} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which turns $f_*\mathcal{F}$ into a sheaf of $f_*\mathcal{O}$-modules. This construction is functorial in $\mathcal{F}$.

Proof. Denote $\mu : \mathcal{O} \times \mathcal{F} \to \mathcal{F}$ the multiplication map. Recall that f_* (on sheaves of sets) is left exact and hence commutes with products. Hence $f_*\mu$ is a map as indicated. This proves the lemma. $\square$

03D2 **Lemma 12.2.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of sheaves of sets*

$$f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}$$

which turns $f^{-1}\mathcal{G}$ into a sheaf of $f^{-1}\mathcal{O}$-modules. This construction is functorial in $\mathcal{G}$.

Proof. Denote $\mu : \mathcal{O} \times \mathcal{G} \to \mathcal{G}$ the multiplication map. Recall that f^{-1} (on sheaves of sets) is exact and hence commutes with products. Hence $f^{-1}\mu$ is a map as indicated. This proves the lemma. $\square$

03D3 **Lemma 12.3.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a sheaf of $f^{-1}\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 12.2 and 12.1, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module by restriction via $\mathcal{O} \to f_*f^{-1}\mathcal{O}$.

Proof. First we note that we have

$$\mathrm{Mor}_{Ab(\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Ab(\mathcal{D})}(\mathcal{G}, f_*\mathcal{F}).$$

by Sites, Proposition 44.3. Suppose that $\alpha : f^{-1}\mathcal{G} \to \mathcal{F}$ and $\beta : \mathcal{G} \to f_*\mathcal{F}$ are morphisms of abelian sheaves which correspond via the formula above. We have to show that α is $f^{-1}\mathcal{O}$-linear if and only if β is $\mathcal{O}$-linear. For example, suppose α is $f^{-1}\mathcal{O}$-linear, then clearly $f_*\alpha$ is $f_*f^{-1}\mathcal{O}$-linear, and hence (as restriction is a functor) is $\mathcal{O}$-linear. Hence it suffices to prove that the adjunction map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ is $\mathcal{O}$-linear. Using that both f_* and f^{-1} commute with products (on sheaves of sets) this comes down to showing that

$$\begin{tikzcd} \mathcal{O} \times \mathcal{G} \arrow[r] \arrow[d] & f_*f^{-1}(\mathcal{O} \times \mathcal{G}) \arrow[d] \\ \mathcal{G} \arrow[r] & f_*f^{-1}\mathcal{G} \end{tikzcd}$$

is commutative. This holds because the adjunction mapping $\text{id}_{Sh(\mathcal{D})} \to f_*f^{-1}$ is a transformation of functors. We omit the proof of the implication β linear $\Rightarrow$ α linear. $\square$

03D4 **Lemma 12.4.** *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a sheaf of $f_*\mathcal{O}$-modules. Then*

$$\text{Mor}_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \text{Mor}_{Mod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 12.2 and 12.1, and we use the canonical map $f^{-1}f_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. Note that we have

$$\text{Mor}_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \text{Mor}_{Mod(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}})$$

by Lemma 11.3. Hence the result follows from Lemma 12.3. $\square$

13. Morphisms of ringed topoi and modules

03D5 We have now introduced enough notation so that we are able to define the pullback and pushforward of modules along a morphism of ringed topoi.

03D6 **Definition 13.1.** Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites.

(1) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{C}$-modules. We define the *pushforward* of $\mathcal{F}$ as the sheaf of $\mathcal{O}_\mathcal{D}$-modules which as a sheaf of abelian groups equals $f_*\mathcal{F}$ and with module structure given by the restriction via $f^\sharp : \mathcal{O}_\mathcal{D} \to f_*\mathcal{O}_\mathcal{C}$ of the module structure
$$f_*\mathcal{O}_\mathcal{C} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$
from Lemma 12.1.
(2) Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules. We define the *pullback* $f^*\mathcal{G}$ to be the sheaf of $\mathcal{O}_\mathcal{C}$-modules defined by the formula
$$f^*\mathcal{G} = \mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{G}$$
where the ring map $f^{-1}\mathcal{O}_\mathcal{D} \to \mathcal{O}_\mathcal{C}$ is $f^\sharp$, and where the module structure is given by Lemma 12.2.

Thus we have defined functors

$$\begin{array}{rcl} f_* : Mod(\mathcal{O}_\mathcal{C}) & \longrightarrow & Mod(\mathcal{O}_\mathcal{D}) \\ f^* : Mod(\mathcal{O}_\mathcal{D}) & \longrightarrow & Mod(\mathcal{O}_\mathcal{C}) \end{array}$$

The final result on these functors is that they are indeed adjoint as expected.

03D7 **Lemma 13.2.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{C}$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules. There is a canonical bijection*

$$\text{Hom}_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{G}, \mathcal{F}) = \text{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, f_*\mathcal{F}).$$

In other words: the functor f^* is the left adjoint to f_*.

Proof. This follows from the work we did before:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{\textit{Mod}(\mathcal{O}_\mathcal{C})}(\mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{\textit{Mod}(f^{-1}\mathcal{O}_\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}\mathcal{O}_\mathcal{D}}) \\ &= \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, f_*\mathcal{F}).\end{aligned}$$

Here we use Lemmas 11.3 and 12.3. $\square$

03D8 **Lemma 13.3.** *$(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. There are canonical isomorphisms of functors $(g \circ f)_* \cong g_* \circ f_*$ and $(g \circ f)^* \cong f^* \circ g^*$.*

Proof. This is clear from the definitions. $\square$

14. The abelian category of sheaves of modules

03D9 Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}$-modules, see Sheaves, Definition 10.1. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the sum of φ and ψ as morphisms of abelian sheaves. This is clearly again a map of $\mathcal{O}$-modules. It is also clear that composition of maps of $\mathcal{O}$-modules is bilinear with respect to this addition. Thus $\textit{Mod}(\mathcal{O})$ is a pre-additive category, see Homology, Definition 3.1.

We will denote 0 the sheaf of $\mathcal{O}$-modules which has constant value $\{0\}$ for all objects U of $\mathcal{C}$. Clearly this is both a final and an initial object of $\textit{Mod}(\mathcal{O})$. Given a morphism of $\mathcal{O}$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) φ is zero, (b) φ factors through 0, (c) φ is zero on sections over each object U.

Moreover, given a pair $\mathcal{F}$, $\mathcal{G}$ of sheaves of $\mathcal{O}$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps (i, j, p, q) as in Homology, Definition 3.5. Thus $\textit{Mod}(\mathcal{O})$ is an additive category, see Homology, Definition 3.8.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}$-modules. We may define $\mathrm{Ker}(\varphi)$ to be the kernel of φ as a map of abelian sheaves. By Section 3 this is the subsheaf of $\mathcal{F}$ with sections

$$\mathrm{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all objects U of $\mathcal{C}$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\mathrm{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$.

Similarly, we define $\mathrm{Coker}(\varphi)$ as the cokernel of φ as a map of abelian sheaves. There is a unique multiplication map

$$\mathcal{O} \times \mathrm{Coker}(\varphi) \longrightarrow \mathrm{Coker}(\varphi)$$

such that the map $\mathcal{G} \to \mathrm{Coker}(\varphi)$ becomes a morphism of $\mathcal{O}$-modules (verification omitted). The map $\mathcal{G} \to \mathrm{Coker}(\varphi)$ is surjective (as a map of sheaves of sets, see Section 3). To show that $\mathrm{Coker}(\varphi)$ is a cokernel in $\textit{Mod}(\mathcal{O})$, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}$-modules such that $\beta \circ \varphi$ is zero, then you get for every object U of $\mathcal{C}$ a map induced by β from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma 20.1) we obtain a canonical map $\mathrm{Coker}(\varphi) \to \mathcal{H}$ such that the original β is equal to the composition $\mathcal{G} \to \mathrm{Coker}(\varphi) \to \mathcal{H}$. The morphism $\mathrm{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

03DA **Lemma 14.1.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. The category $Mod(\mathcal{O})$ is an abelian category. The forgetful functor $Mod(\mathcal{O}) \to Ab(\mathcal{C})$ is exact, hence kernels, cokernels and exactness of $\mathcal{O}$-modules, correspond to the corresponding notions for abelian sheaves.*

Proof. Above we have seen that $Mod(\mathcal{O})$ is an additive category, with kernels and cokernels and that $Mod(\mathcal{O}) \to Ab(\mathcal{C})$ preserves kernels and cokernels. By Homology, Definition 5.1 we have to show that image and coimage agree. This is clear because it is true in $Ab(\mathcal{C})$. The lemma follows. $\square$

03DB **Lemma 14.2.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. All limits and colimits exist in $Mod(\mathcal{O})$ and the forgetful functor $Mod(\mathcal{O}) \to Ab(\mathcal{C})$ commutes with them. Moreover, filtered colimits are exact.*

Proof. The final statement follows from the first as filtered colimits are exact in $Ab(\mathcal{C})$ by Lemma 3.2. Let $\mathcal{I} \to Mod(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. Let $\lim_i \mathcal{F}_i$ be the limit of the diagram in $Ab(\mathcal{C})$. By the description of this limit in Lemma 3.2 we see immediately that there exists a multiplication

$$\mathcal{O} \times \lim_i \mathcal{F}_i \longrightarrow \lim_i \mathcal{F}_i$$

which turns $\lim_i \mathcal{F}_i$ into a sheaf of $\mathcal{O}$-modules. It is easy to see that this is the limit of the diagram in $Mod(\mathcal{C})$. Let $\text{colim}_i \mathcal{F}_i$ be the colimit of the diagram in $PAb(\mathcal{C})$. By the description of this colimit in the proof of Lemma 2.1 we see immediately that there exists a multiplication

$$\mathcal{O} \times \text{colim}_i \mathcal{F}_i \longrightarrow \text{colim}_i \mathcal{F}_i$$

which turns $\text{colim}_i \mathcal{F}_i$ into a presheaf of $\mathcal{O}$-modules. Applying sheafification we get a sheaf of $\mathcal{O}$-modules $(\text{colim}_i \mathcal{F}_i)^\#$, see Lemma 11.1. It is easy to see that $(\text{colim}_i \mathcal{F}_i)^\#$ is the colimit of the diagram in $Mod(\mathcal{O})$, and by Lemma 3.2 forgetting the $\mathcal{O}$-module structure is the colimit in $Ab(\mathcal{C})$. $\square$

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section 23. See Homology, Lemma 7.2 for a description of exactness properties in terms of short exact sequences.

03DC **Lemma 14.3.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi.*

1. *The functor f_* is left exact. In fact it commutes with all limits.*
2. *The functor f^* is right exact. In fact it commutes with all colimits.*

Proof. This is true because (f^*, f_*) is an adjoint pair of functors, see Lemma 13.2. See Categories, Section 24. $\square$

05V3 **Lemma 14.4.** *Let $\mathcal{C}$ be a site. If $\{p_i\}_{i \in I}$ is a conservative family of points, then we may check exactness of a sequence of abelian sheaves on the stalks at the points p_i, $i \in I$. If $\mathcal{C}$ has enough points, then exactness of a sequence of abelian sheaves may be checked on stalks.*

Proof. This is immediate from Sites, Lemma 38.2. $\square$

15. Exactness of pushforward

04BC Some technical lemmas concerning exactness properties of pushforward.

04DA **Lemma 15.1.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. The following are equivalent:*

1. *$f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective for all $\mathcal{F}$ in $Ab(\mathcal{C})$, and*
2. *$f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ reflects surjections.*

In this case the functor $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is faithful.

Proof. Assume (1). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of abelian sheaves on $\mathcal{C}$ such that f_*a is surjective. As f^{-1} is exact this implies that $f^{-1}f_*a : f^{-1}f_*\mathcal{F} \to f^{-1}f_*\mathcal{F}'$ is surjective. Combined with (1) this implies that a is surjective. This means that (2) holds.

Assume (2). Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. We have to show that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective. By (2) it suffices to show that $f_*f^{-1}f_*\mathcal{F} \to f_*\mathcal{F}$ is surjective. And this is true because there is a canonical map $f_*\mathcal{F} \to f_*f^{-1}f_*\mathcal{F}$ which is a one-sided inverse.

We omit the proof of the final assertion. $\square$

04DB **Lemma 15.2.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Assume at least one of the following properties holds*

1. *f_* transforms surjections of sheaves of sets into surjections,*
2. *f_* transforms surjections of abelian sheaves into surjections,*
3. *f_* commutes with coequalizers on sheaves of sets,*
4. *f_* commutes with pushouts on sheaves of sets,*

Then $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is exact.

Proof. Since $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is a right adjoint we already know that it transforms a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of abelian sheaves on $\mathcal{C}$ into an exact sequence

$$0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$$

see Categories, Sections 23 and 24 and Homology, Section 7. Hence it suffices to prove that the map $f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$ is surjective. If (1), (2) holds, then this is clear from the definitions. By Sites, Lemma 41.1 we see that either (3) or (4) formally implies (1), hence in these cases we are done also. $\square$

04BD **Lemma 15.3.** *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Assume u is almost cocontinuous. Then*

1. *$f_* : Ab(\mathcal{D}) \to Ab(\mathcal{C})$ is exact.*
2. *if $f^\sharp : f^{-1}\mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{D}$ is given so that f becomes a morphism of ringed sites, then $f_* : Mod(\mathcal{O}_\mathcal{D}) \to Mod(\mathcal{O}_\mathcal{C})$ is exact.*

Proof. Part (2) follows from part (1) by Lemma 14.2. Part (1) follows from Sites, Lemmas 42.6 and 41.1. $\square$

16. Exactness of lower shriek

04BE Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between sites. Assume that

(a) u is cocontinuous, and
(b) u is continuous.

Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated with u, see Sites, Lemma 21.1. Recall that $g^{-1} = u^p$, i.e., g^{-1} is given by the simple formula $(g^{-1}\mathcal{G})(U) = \mathcal{G}(u(U))$, see Sites, Lemma 21.5. We would like to show that $g^{-1} : Ab(\mathcal{D}) \to Ab(\mathcal{C})$ has a left adjoint $g_!$. By Sites, Lemma 21.5 the functor $g_!^{Sh} = (u_p\ )^\#$ is a left adjoint on sheaves of sets. Moreover, we know that $g_!^{Sh}\mathcal{F}$ is the sheaf associated to the presheaf

$$V \longmapsto \operatorname{colim}_{V \to u(U)} \mathcal{F}(U)$$

where the colimit is over $(\mathcal{I}_V^u)^{opp}$ and is taken in the category of sets. Hence the following definition is natural.

04BF **Definition 16.1.** With $u : \mathcal{C} \to \mathcal{D}$ satisfying (a), (b) above. For $\mathcal{F} \in PAb(\mathcal{C})$ we define $g_{p!}\mathcal{F}$ as the presheaf

$$V \longmapsto \operatorname{colim}_{V \to u(U)} \mathcal{F}(U)$$

with colimits over $(\mathcal{I}_V^u)^{opp}$ taken in Ab. For $\mathcal{F} \in PAb(\mathcal{C})$ we set $g_!\mathcal{F} = (g_{p!}\mathcal{F})^\#$.

The reason for being so explicit with this is that the functors $g_!^{Sh}$ and $g_!$ are different. Whenever we use both we have to be careful to make the distinction clear.

04BG **Lemma 16.2.** *The functor $g_{p!}$ is a left adjoint to the functor u^p. The functor $g_!$ is a left adjoint to the functor g^{-1}. In other words the formulas*

$$\begin{aligned} \operatorname{Mor}_{PAb(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) &= \operatorname{Mor}_{PAb(\mathcal{D})}(g_{p!}\mathcal{F}, \mathcal{G}), \\ \operatorname{Mor}_{Ab(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= \operatorname{Mor}_{Ab(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G}) \end{aligned}$$

hold bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. The second formula follows formally from the first, since if $\mathcal{F}$ and $\mathcal{G}$ are abelian sheaves then

$$\begin{aligned} \operatorname{Mor}_{Ab(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= \operatorname{Mor}_{PAb(\mathcal{D})}(g_{p!}\mathcal{F}, \mathcal{G}) \\ &= \operatorname{Mor}_{Ab(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G}) \end{aligned}$$

by the universal property of sheafification.

To prove the first formula, let $\mathcal{F}$, $\mathcal{G}$ be abelian presheaves. To prove the lemma we will construct maps from the group on the left to the group on the right and omit the verification that these are mutually inverse.

Note that there is a canonical map of abelian presheaves $\mathcal{F} \to u^p g_{p!}\mathcal{F}$ which on sections over U is the natural map $\mathcal{F}(U) \to \operatorname{colim}_{u(U) \to u(U')} \mathcal{F}(U')$, see Sites, Lemma 5.3. Given a map $\alpha : g_{p!}\mathcal{F} \to \mathcal{G}$ we get $u^p\alpha : u^p g_{p!}\mathcal{F} \to u^p\mathcal{G}$. which we can precompose by the map $\mathcal{F} \to u^p g_{p!}\mathcal{F}$.

Note that there is a canonical map of abelian presheaves $g_{p!}u^p\mathcal{G} \to \mathcal{G}$ which on sections over V is the natural map $\operatorname{colim}_{V \to u(U)} \mathcal{G}(u(U)) \to \mathcal{G}(V)$. It maps a section $s \in u(U)$ in the summand corresponding to $t : V \to u(U)$ to $t^*s \in \mathcal{G}(V)$. Hence, given a map $\beta : \mathcal{F} \to u^p\mathcal{G}$ we get a map $g_{p!}\beta : g_{p!}\mathcal{F} \to g_{p!}u^p\mathcal{G}$ which we can postcompose with the map $g_{p!}u^p\mathcal{G} \to \mathcal{G}$ above. $\square$

04BH **Lemma 16.3.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *fibre products and equalizers exist in $\mathcal{C}$ and u commutes with them.*

In this case the functor $g_! : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is exact.

Proof. Compare with Sites, Lemma 21.6. Assume (a), (b), and (c). We already know that $g_!$ is right exact as it is a left adjoint, see Categories, Lemma 24.6 and Homology, Section 7. We have $g_! = (g_{p!}\ )^\#$. We have to show that $g_!$ transforms injective maps of abelian sheaves into injective maps of abelian presheaves. Recall that sheafification of abelian presheaves is exact, see Lemma 3.2. Thus it suffices to show that $g_{p!}$ transforms injective maps of abelian presheaves into injective maps of abelian presheaves. To do this it suffices that colimits over the categories $(\mathcal{I}_V^u)^{opp}$ of Sites, Section 5 transform injective maps between diagrams into injections. This follows from Sites, Lemma 5.1 and Algebra, Lemma 8.10. $\square$

077I **Lemma 16.4.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *u is fully faithful.*

For $g_!, g^{-1}, g_*$ as above the canonical maps $\mathcal{F} \to g^{-1}g_!\mathcal{F}$ and $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ are isomorphisms for all abelian sheaves $\mathcal{F}$ on $\mathcal{C}$.

Proof. The map $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ is an isomorphism by Sites, Lemma 21.7 and the fact that pullback and pushforward of abelian sheaves agrees with pullback and pushforward on the underlying sheaves of sets.

Pick $U \in \text{Ob}(\mathcal{C})$. We will show that $g^{-1}g_!\mathcal{F}(U) = \mathcal{F}(U)$. First, note that $g^{-1}g_!\mathcal{F}(U) = g_!\mathcal{F}(u(U))$. Hence it suffices to show that $g_!\mathcal{F}(u(U)) = \mathcal{F}(U)$. We know that $g_!\mathcal{F}$ is the (abelian) sheaf associated to the presheaf $g_{p!}\mathcal{F}$ which is defined by the rule

$$V \longmapsto \text{colim}_{V \to u(U')} \mathcal{F}(U')$$

with colimit taken in Ab. If $V = u(U)$, then, as u is fully faithful this colimit is over $U \to U'$. Hence we conclude that $g_{p!}\mathcal{F}(u(U) = \mathcal{F}(U)$. Since u is cocontinuous and continuous any covering of $u(U)$ in $\mathcal{D}$ can be refined by a covering (!) $\{u(U_i) \to u(U)\}$ of $\mathcal{D}$ where $\{U_i \to U\}$ is a covering in $\mathcal{C}$. This implies that $(g_{p!}\mathcal{F})^+(u(U)) = \mathcal{F}(U)$ also, since in the colimit defining the value of $(g_{p!}\mathcal{F})^+$ on $u(U)$ we may restrict to the cofinal system of coverings $\{u(U_i) \to u(U)\}$ as above. Hence we see that $(g_{p!}\mathcal{F})^+(u(U)) = \mathcal{F}(U)$ for all objects U of $\mathcal{C}$ as well. Repeating this argument one more time gives the equality $(g_{p!}\mathcal{F})^\#(u(U)) = \mathcal{F}(U)$ for all objects U of $\mathcal{C}$. This produces the desired equality $g^{-1}g_!\mathcal{F} = \mathcal{F}$. $\square$

04BI **Remark 16.5.** In general the functor $g_!$ cannot be extended to categories of modules in case g is (part of) a morphism of ringed topoi. Namely, given any ring map $A \to B$ the functor $M \mapsto B \otimes_A M$ has a right adjoint (restriction) but not in general a left adjoint (because its existence would imply that $A \to B$ is flat). We will see in Section 19 below that it is possible to define $j_!$ on sheaves of modules in the case of a localization of sites. We will discuss this in greater generality in Section 41 below.

08P3 **Lemma 16.6.** *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to a continuous and cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$.*

1. *If u has a left adjoint w, then $g_!$ agrees with $g_!^{Sh}$ on underlying sheaves of sets and $g_!$ is exact.*
2. *If in addition w is cocontinuous, then $g_! = h^{-1}$ and $g^{-1} = h_*$ where $h : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to w.*

Proof. This Lemma is the analogue of Sites, Lemma 23.1. From Sites, Lemma 19.3 we see that the categories $\mathcal{I}_V^u$ have an initial object. Thus the underlying set of a colimit of a system of abelian groups over $(\mathcal{I}_V^u)^{opp}$ is the colimit of the underlying sets. Whence the agreement of $g_!^{Sh}$ and $g_!$ by our construction of $g_!$ in Definition 16.1. The exactness and (2) follow immediately from the corresponding statements of Sites, Lemma 23.1. □

17. Global types of modules

03DD

03DE **Definition 17.1.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules.

1. We say $\mathcal{F}$ is a *free $\mathcal{O}$-module* if $\mathcal{F}$ is isomorphic as an $\mathcal{O}$-module to a sheaf of the form $\bigoplus_{i \in I} \mathcal{O}$.
2. We say $\mathcal{F}$ is *finite free* if $\mathcal{F}$ is isomorphic as an $\mathcal{O}$-module to a sheaf of the form $\bigoplus_{i \in I} \mathcal{O}$ with a finite index set I.
3. We say $\mathcal{F}$ is *generated by global sections* if there exists a surjection
$$\bigoplus_{i \in I} \mathcal{O} \longrightarrow \mathcal{F}$$
from a free $\mathcal{O}$-module onto $\mathcal{F}$.
4. Given $r \geq 0$ we say $\mathcal{F}$ is *generated by r global sections* if there exists a surjection $\mathcal{O}^{\oplus r} \to \mathcal{F}$.
5. We say $\mathcal{F}$ is *generated by finitely many global sections* if it is generated by r global sections for some $r \geq 0$.
6. We say $\mathcal{F}$ has a *global presentation* if there exists an exact sequence
$$\bigoplus_{j \in J} \mathcal{O} \longrightarrow \bigoplus_{i \in I} \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$
of $\mathcal{O}$-modules.
7. We say $\mathcal{F}$ has a *global finite presentation* if there exists an exact sequence
$$\bigoplus_{j \in J} \mathcal{O} \longrightarrow \bigoplus_{i \in I} \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$
of $\mathcal{O}$-modules with I and J finite sets.

Note that for any set I the direct sum $\bigoplus_{i \in I} \mathcal{O}$ exists (Lemma 14.2) and is the sheafification of the presheaf $U \mapsto \bigoplus_{i \in I} \mathcal{O}(U)$. This module is called the *free $\mathcal{O}$-module on the set I*.

03DF **Lemma 17.2.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{D}$-module.*

1. *If $\mathcal{F}$ is free then $f^*\mathcal{F}$ is free.*
2. *If $\mathcal{F}$ is finite free then $f^*\mathcal{F}$ is finite free.*
3. *If $\mathcal{F}$ is generated by global sections then $f^*\mathcal{F}$ is generated by global sections.*
4. *Given $r \geq 0$ if $\mathcal{F}$ is generated by r global sections, then $f^*\mathcal{F}$ is generated by r global sections.*

(5) *If $\mathcal{F}$ is generated by finitely many global sections then $f^*\mathcal{F}$ is generated by finitely many global sections.*
(6) *If $\mathcal{F}$ has a global presentation then $f^*\mathcal{F}$ has a global presentation.*
(7) *If $\mathcal{F}$ has a finite global presentation then $f^*\mathcal{F}$ has a finite global presentation.*

Proof. This is true because f^* commutes with arbitrary colimits (Lemma 14.3) and $f^*\mathcal{O}_\mathcal{D} = \mathcal{O}_\mathcal{C}$. $\square$

18. Intrinsic properties of modules

03DG Let $\mathcal{P}$ be a property of sheaves of modules on ringed topoi. We say $\mathcal{P}$ is an *intrinsic property* if we have $\mathcal{P}(\mathcal{F}) \Leftrightarrow \mathcal{P}(f^*\mathcal{F})$ whenever $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ is an equivalence of ringed topoi. For example, the property of being free is intrinsic. Indeed, the free $\mathcal{O}$-module on the set I is characterized by the property that

$$\mathrm{Mor}_{Mod(\mathcal{O})}(\bigoplus\nolimits_{i\in I} \mathcal{O}, \mathcal{F}) = \prod\nolimits_{i\in I} \mathrm{Mor}_{Sh(\mathcal{C})}(\{*\}, \mathcal{F})$$

for a variable $\mathcal{F}$ in $Mod(\mathcal{O})$. Alternatively, we can also use Lemma 17.2 to see that being free is intrinsic. In fact, each of the properties defined in Definition 17.1 is intrinsic for the same reason. How will we go about defining other intrinsic properties of $\mathcal{O}$-modules?

The upshot of Lemma 7.2 is the following: Suppose you want to define an intrinsic property $\mathcal{P}$ of an $\mathcal{O}$-module on a topos. Then you can proceed as follows:

(1) Given any site $\mathcal{C}$, any sheaf of rings $\mathcal{O}$ on $\mathcal{C}$ and any $\mathcal{O}$-module $\mathcal{F}$ define the corresponding property $\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F})$.
(2) For any pair of sites $\mathcal{C}$, $\mathcal{C}'$, any special cocontinuous functor $u : \mathcal{C} \to \mathcal{C}'$, any sheaf of rings $\mathcal{O}$ on $\mathcal{C}$ any $\mathcal{O}$-module $\mathcal{F}$, show that
$$\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F}) \Leftrightarrow \mathcal{P}(\mathcal{C}', g_*\mathcal{O}, g_*\mathcal{F})$$
where $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ is the equivalence of topoi associated to u.

In this case, given any ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ and any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we simply say that $\mathcal{F}$ has property $\mathcal{P}$ if $\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F})$ is true. And Lemma 7.2 combined with (2) above guarantees that this is well defined.

Moreover, the same Lemma 7.2 also guarantees that if in addition

(3) For any morphism of ringed sites $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ such that f is given by a functor $u : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Sites, Proposition 14.7, and any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ we have
$$\mathcal{P}(\mathcal{D}, \mathcal{O}_\mathcal{D}, \mathcal{F}) \Rightarrow \mathcal{P}(\mathcal{C}, \mathcal{O}_\mathcal{C}, f^*\mathcal{F})$$

then it is true that $\mathcal{P}$ is preserved under pullback of modules w.r.t. arbitrary morphisms of ringed topoi.

We will use this method in the following sections to see that: locally free, locally generated by sections, locally generated by r sections, finite type, finite presentation, quasi-coherent, and coherent are intrinsic properties of modules.

Perhaps a more satisfying method would be to find an intrinsic definition of these notions, rather than the laborious process sketched here. On the other hand, in many geometric situations where we want to apply these definitions we are given a definite ringed site, and a definite sheaf of modules, and it is nice to have a definition already adapted to this language.

19. Localization of ringed sites

03DH Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. We explain the counterparts of the results in Sites, Section 25 in this setting.

Denote $\mathcal{O}_U = j_U^{-1}\mathcal{O}$ the restriction of $\mathcal{O}$ to the site $\mathcal{C}/U$. It is described by the simple rule $\mathcal{O}_U(V/U) = \mathcal{O}(V)$. With this notation the localization morphism j_U becomes a morphism of ringed topoi

$$(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

namely, we take $j_U^\sharp : j_U^{-1}\mathcal{O} \to \mathcal{O}_U$ the identity map. Moreover, we obtain the following descriptions for pushforward and pullback of modules.

04IX **Definition 19.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$.

(1) The ringed site $(\mathcal{C}/U, \mathcal{O}_U)$ is called the *localization of the ringed site* $(\mathcal{C}, \mathcal{O})$ *at the object* U.
(2) The morphism of ringed topoi $(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ is called the *localization morphism*.
(3) The functor $j_{U*} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ is called the *direct image functor*.
(4) For a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{C}$ the sheaf $j_U^*\mathcal{F}$ is called the *restriction of* $\mathcal{F}$ *to* $\mathcal{C}/U$. We will sometimes denote it by $\mathcal{F}|_{\mathcal{C}/U}$ or even $\mathcal{F}|_U$. It is described by the simple rule $j_U^*(\mathcal{F})(X/U) = \mathcal{F}(X)$.
(5) The left adjoint $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ of restriction is called *extension by zero*. It exists and is exact by Lemmas 19.2 and 19.3.

As in the topological case, see Sheaves, Section 31, the extension by zero $j_{U!}$ functor is different from extension by the empty set $j_{U!}$ defined on sheaves of sets. Here is the lemma defining extension by zero.

03DI **Lemma 19.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The restriction functor $j_U^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_U)$ has a left adjoint $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$. So*

$$\mathrm{Mor}_{Mod(\mathcal{O}_U)}(\mathcal{G}, j_U^*\mathcal{F}) = \mathrm{Mor}_{Mod(\mathcal{O})}(j_{U!}\mathcal{G}, \mathcal{F})$$

for $\mathcal{F} \in \mathrm{Ob}(Mod(\mathcal{O}))$ and $\mathcal{G} \in \mathrm{Ob}(Mod(\mathcal{O}_U))$. Moreover, the extension by zero $j_{U!}\mathcal{G}$ of $\mathcal{G}$ is the sheaf associated to the presheaf

$$V \longmapsto \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

with obvious restriction mappings and an obvious $\mathcal{O}$-module structure.

Proof. The $\mathcal{O}$-module structure on the presheaf is defined as follows. If $f \in \mathcal{O}(V)$ and $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$, then we define $f \cdot s = fs$ where $f \in \mathcal{O}_U(\varphi : V \to U) = \mathcal{O}(V)$ (because $\mathcal{O}_U$ is the restriction of $\mathcal{O}$ to $\mathcal{C}/U$).

Similarly, let $\alpha : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. In this case we can define a map from the presheaf of the lemma into $\mathcal{F}$ by mapping

$$\bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U) \longrightarrow \mathcal{F}(V)$$

by the rule that $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$ maps to $\alpha(s) \in \mathcal{F}(V)$. It is clear that this is $\mathcal{O}$-linear, and hence induces a morphism of $\mathcal{O}$-modules $\alpha' : j_{U!}\mathcal{G} \to \mathcal{F}$ by the properties of sheafification of modules (Lemma 11.1).

Conversely, let $\beta : j_{U!}\mathcal{G} \to \mathcal{F}$ by a map of $\mathcal{O}$-modules. Recall from Sites, Section 25 that there exists an extension by the empty set $j_{U!}^{Sh} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ on sheaves of sets which is left adjoint to j_U^{-1}. Moreover, $j_{U!}^{Sh}\mathcal{G}$ is the sheaf associated to the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

Hence there is a natural map $j_{U!}^{Sh}\mathcal{G} \to j_{U!}\mathcal{G}$ of sheaves of sets. Hence precomposing β by this map we get a map of sheaves of sets $j_{U!}^{Sh}\mathcal{G} \to \mathcal{F}$ which by adjunction corresponds to a map of sheaves of sets $\beta' : \mathcal{G} \to \mathcal{F}|_U$. We claim that β' is $\mathcal{O}_U$-linear. Namely, suppose that $\varphi : V \to U$ is an object of $\mathcal{C}/U$ and that $s, s' \in \mathcal{G}(\varphi : V \to U)$, and $f \in \mathcal{O}(V) = \mathcal{O}_U(\varphi : V \to U)$. Then by the discussion above we see that $\beta'(s + s')$, resp. $\beta'(fs)$ in $\mathcal{F}|_U(\varphi : V \to U)$ correspond to $\beta(s + s')$, resp. $\beta(fs)$ in $\mathcal{F}(V)$. Since β is a homomorphism we conclude.

To conclude the proof of the lemma we have to show that the constructions $\alpha \mapsto \alpha'$ and $\beta \mapsto \beta'$ are mutually inverse. We omit the verifications. □

Note that we have in the situation of Definition 19.1 we have

0G1V (19.2.1)
$$\mathrm{Hom}_\mathcal{O}(j_{U!}\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, j_U^*\mathcal{F}) = \mathcal{F}(U)$$

for every $\mathcal{O}$-module $\mathcal{F}$. Namely, the first equality holds by the adjointness of $j_{U!}$ and j_U^* and the second because $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, j_U^*\mathcal{F}) = j_U^*\mathcal{F}(U/U) = \mathcal{F}|_U(U/U) = \mathcal{F}(U)$.

03DJ **Lemma 19.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ is exact.*

Proof. Since $j_{U!}$ is a left adjoint to j_U^* we see that it is right exact (see Categories, Lemma 24.6 and Homology, Section 7). Hence it suffices to show that if $\mathcal{G}_1 \to \mathcal{G}_2$ is an injective map of $\mathcal{O}_U$-modules, then $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2$ is injective. The map on sections of presheaves over an object V (as in Lemma 19.2) is the map

$$\bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}_1(V \xrightarrow{\varphi} U) \longrightarrow \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}_2(V \xrightarrow{\varphi} U)$$

which is injective by assumption. Since sheafification is exact by Lemma 11.2 we conclude $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2$ is injective and we win. □

0E8G **Lemma 19.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. A complex of $\mathcal{O}_U$-modules $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3$ is exact if and only if $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2 \to j_{U!}\mathcal{G}_3$ is exact as a sequence of $\mathcal{O}$-modules.*

Proof. We already know that $j_{U!}$ is exact, see Lemma 19.3. Thus it suffices to show that $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ reflects injections and surjections.

For every $\mathcal{G}$ in $Mod(\mathcal{O}_U)$ we have the unit $\mathcal{G} \to j_U^* j_{U!}\mathcal{G}$ of the adjunction. We claim this map is an injection of sheaves. Namely, looking at the construction of Lemma 19.2 we see that this map is the sheafification of the rule sending the object V/U of $\mathcal{C}/U$ to the injective map

$$\mathcal{G}(V/U) \longrightarrow \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

given by the inclusion of the summand corresponding to the structure morphism $V \to U$. Since sheafification is exact the claim follows. Some details omitted.

If $\mathcal{G} \to \mathcal{G}'$ is a map of $\mathcal{O}_U$-modules with $j_{U!}\mathcal{G} \to j_{U!}\mathcal{G}'$ injective, then $j_U^*j_{U!}\mathcal{G} \to j_U^*j_{U!}\mathcal{G}'$ is injective (restriction is exact), hence $\mathcal{G} \to j_U^*j_{U!}\mathcal{G}'$ is injective, hence $\mathcal{G} \to \mathcal{G}'$ is injective. We conclude that $j_{U!}$ reflects injections.

Let $a : \mathcal{G} \to \mathcal{G}'$ be a map of $\mathcal{O}_U$-modules such that $j_{U!}\mathcal{G} \to j_{U!}\mathcal{G}'$ is surjective. Let $\mathcal{H}$ be the cokernel of a. Then $j_{U!}\mathcal{H} = 0$ as $j_{U!}$ is exact. By the above the map $\mathcal{H} \to j_U^*j_{U!}\mathcal{H}$ is injective. Hence $\mathcal{H} = 0$ as desired. □

04IY **Lemma 19.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then there exists a commutative diagram*

$$\begin{tikzcd}
(Sh(\mathcal{C}/V), \mathcal{O}_V) \arrow[rr, "{(j, j^\sharp)}"'] \arrow[rd, "{(j_V, j_V^\sharp)}"'] & & (Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[ld, "{(j_U, j_U^\sharp)}"] \\
& (Sh(\mathcal{C}), \mathcal{O}) &
\end{tikzcd}$$

of ringed topoi. Here $(j, j^\sharp)$ is the localization morphism associated to the object V/U of the ringed site $(\mathcal{C}/V, \mathcal{O}_V)$.

Proof. The only thing to check is that $j_V^\sharp = j^\sharp \circ j^{-1}(j_U^\sharp)$, since everything else follows directly from Sites, Lemma 25.8 and Sites, Equation (25.8.1). We omit the verification of the equality. □

08P4 **Remark 19.6.** In the situation of Lemma 19.2 the diagram

$$\begin{tikzcd}
Mod(\mathcal{O}_U) \arrow[r, "j_{U!}"'] \arrow[d, "forget"'] & Mod(\mathcal{O}_\mathcal{C}) \arrow[d, "forget"] \\
Ab(\mathcal{C}/U) \arrow[r, "j_{U!}^{Ab}"] & Ab(\mathcal{C})
\end{tikzcd}$$

commutes. This is clear from the explicit description of the functor $j_{U!}$ in the lemma.

03EJ **Remark 19.7.** Localization and presheaves of modules; see Sites, Remark 25.10. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let U be an object of $\mathcal{C}$. Strictly speaking the functors j_U^*, j_{U*} and $j_{U!}$ have not been defined for presheaves of $\mathcal{O}$-modules. But of course, we can think of a presheaf as a sheaf for the chaotic topology on $\mathcal{C}$ (see Sites, Examples 6.6). Hence we also obtain a functor

$$j_U^* : PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}_U)$$

and functors

$$j_{U*}, j_{U!} : PMod(\mathcal{O}_U) \longrightarrow PMod(\mathcal{O})$$

which are right, left adjoint to j_U^*. Inspecting the proof of Lemma 19.2 we see that $j_{U!}\mathcal{G}$ is the presheaf

$$V \longmapsto \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

In addition the functor $j_{U!}$ is exact (by Lemma 19.3 in the case of the discrete topologies). Moreover, if $\mathcal{C}$ is actually a site, and $\mathcal{O}$ is actually a sheaf of rings, then the diagram

$$\begin{tikzcd} Mod(\mathcal{O}_U) \arrow[r, "j_{U!}"'] \arrow[d, "forget"'] & Mod(\mathcal{O}) \\ PMod(\mathcal{O}_U) \arrow[r, "j_{U!}"] & PMod(\mathcal{O}) \arrow[u, "(\ )^\#"'] \end{tikzcd}$$

commutes.

0F6Z **Lemma 19.8.** *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Assume that every X in $\mathcal{C}$ has at most one morphism to U. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}/U$. The canonical maps $\mathcal{F} \to j_U^{-1} j_{U!}\mathcal{F}$ and $j_U^{-1} j_{U*}\mathcal{F} \to \mathcal{F}$ are isomorphisms.*

Proof. This is a special case of Lemma 16.4 because the assumption on U is equivalent to the fully faithfulness of the localization functor $\mathcal{C}/U \to \mathcal{C}$. $\square$

20. Localization of morphisms of ringed sites

04IZ This section is the analogue of Sites, Section 28.

04J0 **Lemma 20.1.** *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$ and set $U = u(V)$. Then there is a canonical map of sheaves of rings $(f')^\sharp$ such that the diagram of Sites, Lemma 28.1 is turned into a commutative diagram of ringed topoi*

$$\begin{tikzcd} (Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[r, "{(j_U, j_U^\sharp)}"'] \arrow[d, "{(f', (f')^\sharp)}"'] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d, "{(f, f^\sharp)}"] \\ (Sh(\mathcal{D}/V), \mathcal{O}'_V) \arrow[r, "{(j_V, j_V^\sharp)}"] & (Sh(\mathcal{D}), \mathcal{O}'). \end{tikzcd}$$

Moreover, in this situation we have $f'_* j_U^{-1} = j_V^{-1} f_*$ and $f'_* j_U^* = j_V^* f_*$.

Proof. Just take $(f')^\sharp$ to be

$$(f')^{-1}\mathcal{O}'_V = (f')^{-1} j_V^{-1}\mathcal{O}' = j_U^{-1} f^{-1}\mathcal{O}' \xrightarrow{j_U^{-1} f^\sharp} j_U^{-1}\mathcal{O} = \mathcal{O}_U$$

and everything else follows from Sites, Lemma 28.1. (Note that $j^{-1} = j^*$ on sheaves of modules if j is a localization morphism, hence the first equality of functors implies the second.) $\square$

04J1 **Lemma 20.2.** *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$, $U \in \mathrm{Ob}(\mathcal{C})$ and $c : U \to u(V)$ a morphism of $\mathcal{C}$. There exists a commutative diagram of ringed topoi*

$$\begin{tikzcd} (Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[r, "{(j_U, j_U^\sharp)}"'] \arrow[d, "{(f_c, f_c^\sharp)}"'] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d, "{(f, f^\sharp)}"] \\ (Sh(\mathcal{D}/V), \mathcal{O}'_V) \arrow[r, "{(j_V, j_V^\sharp)}"] & (Sh(\mathcal{D}), \mathcal{O}'). \end{tikzcd}$$

The morphism $(f_c, f_c^\sharp)$ is equal to the composition of the morphism

$$(f', (f')^\sharp) : (Sh(\mathcal{C}/u(V)), \mathcal{O}_{u(V)}) \longrightarrow (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$

of Lemma 20.1 and the morphism

$$(j, j^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}/u(V)), \mathcal{O}_{u(V)})$$

of Lemma 19.5. Given any morphisms $b : V' \to V$, $a : U' \to U$ and $c' : U' \to u(V')$ such that

$$\begin{tikzcd} U' \arrow[r, "c'"'] \arrow[d, "a"'] & u(V') \arrow[d, "u(b)"] \\ U \arrow[r, "c"] & u(V) \end{tikzcd}$$

commutes, then the following diagram of ringed topoi

$$\begin{tikzcd} (Sh(\mathcal{C}/U'), \mathcal{O}_{U'}) \arrow[r, "(j_{U'/U}, j^\sharp_{U'/U})"'] \arrow[d, "(f_{c'}, f^\sharp_{c'})"'] & (Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[d, "(f_c, f^\sharp_c)"] \\ (Sh(\mathcal{D}/V'), \mathcal{O}'_{V'}) \arrow[r, "(j_{V'/V}, j^\sharp_{V'/V})"] & (Sh(\mathcal{D}/V), \mathcal{O}'_{V'}) \end{tikzcd}$$

commutes.

Proof. On the level of morphisms of topoi this is Sites, Lemma 28.3. To check that the diagrams commute as morphisms of ringed topoi use Lemmas 19.5 and 20.1 exactly as in the proof of Sites, Lemma 28.3. □

21. Localization of ringed topoi

04ID This section is the analogue of Sites, Section 30 in the setting of ringed topoi.

04IE **Lemma 21.1.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F} \in Sh(\mathcal{C})$ be a sheaf. For a sheaf $\mathcal{H}$ on $\mathcal{C}$ denote $\mathcal{H}_\mathcal{F}$ the sheaf $\mathcal{H} \times \mathcal{F}$ seen as an object of the category $Sh(\mathcal{C})/\mathcal{F}$. The pair $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a ringed topos and there is a canonical morphism of ringed topoi*

$$(j_\mathcal{F}, j^\sharp_\mathcal{F}) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

which is a localization as in Section 19 such that

1. *the functor $j_\mathcal{F}^{-1}$ is the functor $\mathcal{H} \mapsto \mathcal{H}_\mathcal{F}$,*
2. *the functor $j_\mathcal{F}^*$ is the functor $\mathcal{H} \mapsto \mathcal{H}_\mathcal{F}$,*
3. *the functor $j_{\mathcal{F}!}$ on sheaves of sets is the forgetful functor $\mathcal{G}/\mathcal{F} \mapsto \mathcal{G}$,*
4. *the functor $j_{\mathcal{F}!}$ on sheaves of modules associates to the $\mathcal{O}_\mathcal{F}$-module $\varphi : \mathcal{G} \to \mathcal{F}$ the $\mathcal{O}$-module which is the sheafification of the presheaf*

$$V \longmapsto \bigoplus\nolimits_{s \in \mathcal{F}(V)} \{\sigma \in \mathcal{G}(V) \mid \varphi(\sigma) = s\}$$

for $V \in \mathrm{Ob}(\mathcal{C})$.

Proof. By Sites, Lemma 30.1 we see that $Sh(\mathcal{C})/\mathcal{F}$ is a topos and that (1) and (3) are true. In particular this shows that $j_\mathcal{F}^{-1}\mathcal{O} = \mathcal{O}_\mathcal{F}$ and shows that $\mathcal{O}_\mathcal{F}$ is a sheaf of rings. Thus we may choose the map $j^\sharp_\mathcal{F}$ to be the identity, in particular we see that (2) is true. Moreover, the proof of Sites, Lemma 30.1 shows that we may assume $\mathcal{C}$ is a site with all finite limits and a subcanonical topology and that $\mathcal{F} = h_U$ for some object U of $\mathcal{C}$. Then (4) follows from the description of $j_{\mathcal{F}!}$ in Lemma 19.2. Alternatively one could show directly that the functor described in (4) is a left adjoint to $j_\mathcal{F}^*$. □

04J2 **Definition 21.2.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F} \in Sh(\mathcal{C})$.

1. The ringed topos $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is called the *localization of the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ at $\mathcal{F}$.*

(2) The morphism of ringed topoi $(j_\mathcal{F}, j_\mathcal{F}^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C}), \mathcal{O})$ of Lemma 21.1 is called the *localization morphism*.

We continue the tradition, established in the chapter on sites, that we check the localization constructions on topoi are compatible with the constructions of localization on sites, whenever this makes sense.

04J3 **Lemma 21.3.** *With $(Sh(\mathcal{C}), \mathcal{O})$ and $\mathcal{F} \in Sh(\mathcal{C})$ as in Lemma 21.1. If $\mathcal{F} = h_U^\#$ for some object U of $\mathcal{C}$ then via the identification $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Sites, Lemma 25.4 we have*

(1) *canonically $\mathcal{O}_U = \mathcal{O}_\mathcal{F}$, and*
(2) *with these identifications we have $(j_\mathcal{F}, j_\mathcal{F}^\sharp) = (j_U, j_U^\sharp)$.*

Proof. The assertion for underlying topoi is Sites, Lemma 30.5. Note that $\mathcal{O}_U$ is the restriction of $\mathcal{O}$ which by Sites, Lemma 25.7 corresponds to $\mathcal{O} \times h_U^\#$ under the equivalence of Sites, Lemma 25.4. By definition of $\mathcal{O}_\mathcal{F}$ we get (1). What's left is to prove that $j_\mathcal{F}^\sharp = j_U^\sharp$ under this identification. We omit the verification. □

Localization is functorial in the following two ways: We can "relocalize" a localization (see Lemma 21.4) or we can given a morphism of ringed topoi, localize upstairs at the inverse image of a sheaf downstairs and get a commutative diagram of ringed topoi (see Lemma 22.1).

04J4 **Lemma 21.4.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. If $s : \mathcal{G} \to \mathcal{F}$ is a morphism of sheaves on $\mathcal{C}$ then there exists a natural commutative diagram of morphisms of ringed topoi*

$$\begin{tikzcd}
(Sh(\mathcal{C})/\mathcal{G}, \mathcal{O}_\mathcal{G}) \arrow[rr, "{(j,j^\sharp)}"'] \arrow[dr, "{(j_\mathcal{G}, j_\mathcal{G}^\sharp)}"'] & & (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \arrow[dl, "{(j_\mathcal{F}, j_\mathcal{F}^\sharp)}"] \\
& (Sh(\mathcal{C}), \mathcal{O}) &
\end{tikzcd}$$

where $(j, j^\sharp)$ is the localization morphism of the ringed topos $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ at the object $\mathcal{G}/\mathcal{F}$.

Proof. All assertions follow from Sites, Lemma 30.6 except the assertion that $j_\mathcal{G}^\sharp = j^\sharp \circ j^{-1}(j_\mathcal{F}^\sharp)$. We omit the verification. □

04J5 **Lemma 21.5.** *With $(Sh(\mathcal{C}), \mathcal{O})$, $s : \mathcal{G} \to \mathcal{F}$ as in Lemma 21.4. If there exist a morphism $f : V \to U$ of $\mathcal{C}$ such that $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\#$ and s is induced by f, then the diagrams of Lemma 19.5 and Lemma 21.4 agree via the identifications $(j_\mathcal{F}, j_\mathcal{F}^\sharp) = (j_U, j_U^\sharp)$ and $(j_\mathcal{G}, j_\mathcal{G}^\sharp) = (j_V, j_V^\sharp)$ of Lemma 21.3.*

Proof. All assertions follow from Sites, Lemma 30.7 except for the assertion that the two maps $j^\sharp$ agree. This holds since in both cases the map $j^\sharp$ is simply the identity. Some details omitted. □

22. Localization of morphisms of ringed topoi

04J6 This section is the analogue of Sites, Section 31.

04IF **Lemma 22.1.** *Let*
$$f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$$
be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then there exists a commutative diagram of ringed topoi
$$\begin{tikzcd}
(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \arrow[r, "{(j_\mathcal{F}, j_\mathcal{F}^\sharp)}"'] \arrow[d, "{(f', (f')^\sharp)}"'] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d, "{(f, f^\sharp)}"] \\
(Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G}) \arrow[r, "{(j_\mathcal{G}, j_\mathcal{G}^\sharp)}"] & (Sh(\mathcal{D}), \mathcal{O}')
\end{tikzcd}$$
We have $f'_*j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1}f_*$ and $f'_*j_\mathcal{F}^* = j_\mathcal{G}^*f_*$. Moreover, the morphism f' is characterized by the rule
$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} \mathcal{F}).$$

Proof. By Sites, Lemma 31.1 we have the diagram of underlying topoi, the equality $f'_*j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1}f_*$, and the description of $(f')^{-1}$. To define $(f')^\sharp$ we use the map
$$(f')^\sharp : \mathcal{O}'_\mathcal{G} = j_\mathcal{G}^{-1}\mathcal{O}' \xrightarrow{j_\mathcal{G}^{-1}f^\sharp} j_\mathcal{G}^{-1}f_*\mathcal{O} = f'_*j_\mathcal{F}^{-1}\mathcal{O} = f'_*\mathcal{O}_\mathcal{F}$$
or equivalently the map
$$(f')^\sharp : (f')^{-1}\mathcal{O}'_\mathcal{G} = (f')^{-1}j_\mathcal{G}^{-1}\mathcal{O}' = j_\mathcal{F}^{-1}f^{-1}\mathcal{O}' \xrightarrow{j_\mathcal{F}^{-1}f^\sharp} j_\mathcal{F}^{-1}\mathcal{O} = \mathcal{O}_\mathcal{F}.$$
We omit the verification that these two maps are indeed adjoint to each other. The second construction of $(f')^\sharp$ shows that the diagram commutes in the 2-category of ringed topoi (as the maps $j_\mathcal{F}^\sharp$ and $j_\mathcal{G}^\sharp$ are identities). Finally, the equality $f'_*j_\mathcal{F}^* = j_\mathcal{G}^*f_*$ follows from the equality $f'_*j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1}f_*$ and the fact that pullbacks of sheaves of modules and sheaves of sets agree, see Lemma 21.1. ☐

04J7 **Lemma 22.2.** *Let*
$$f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$$
be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. If f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ and $\mathcal{G} = h_V^\#$, then the commutative diagrams of Lemma 20.1 and Lemma 22.1 agree via the identifications of Lemma 21.3.

Proof. At the level of morphisms of topoi this is Sites, Lemma 31.2. This works also on the level of morphisms of ringed topoi since the formulas defining $(f')^\sharp$ in the proofs of Lemma 20.1 and Lemma 22.1 agree. ☐

04J8 **Lemma 22.3.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ a morphism of sheaves. There exists a commutative diagram of ringed topoi*
$$\begin{tikzcd}
(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \arrow[r, "{(j_\mathcal{F}, j_\mathcal{F}^\sharp)}"'] \arrow[d, "{(f_c, f_c^\sharp)}"'] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d, "{(f, f^\sharp)}"] \\
(Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G}) \arrow[r, "{(j_\mathcal{G}, j_\mathcal{G}^\sharp)}"] & (Sh(\mathcal{D}), \mathcal{O}').
\end{tikzcd}$$
The morphism $(f_s, f_s^\sharp)$ is equal to the composition of the morphism
$$(f', (f')^\sharp) : (Sh(\mathcal{C})/f^{-1}\mathcal{G}, \mathcal{O}_{f^{-1}\mathcal{G}}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$

of Lemma 22.1 and the morphism

$$(j, j^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C})/f^{-1}\mathcal{G}, \mathcal{O}_{f^{-1}\mathcal{G}})$$

of Lemma 21.4. Given any morphisms $b : \mathcal{G}' \to \mathcal{G}$, $a : \mathcal{F}' \to \mathcal{F}$, *and* $s' : \mathcal{F}' \to f^{-1}\mathcal{G}'$ *such that*

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "s'"'] \arrow[d, "a"'] & f^{-1}\mathcal{G}' \arrow[d, "f^{-1}b"] \\ \mathcal{F} \arrow[r, "s"] & f^{-1}\mathcal{G} \end{tikzcd}$$

commutes, then the following diagram of ringed topoi

$$\begin{tikzcd} (Sh(\mathcal{C})/\mathcal{F}', \mathcal{O}_{\mathcal{F}'}) \arrow[r, "(j_{\mathcal{F}'/\mathcal{F}}, j^\sharp_{\mathcal{F}'/\mathcal{F}})"'] \arrow[d, "(f_{s'}, f^\sharp_{s'})"'] & (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \arrow[d, "(f_s, f^\sharp_s)"] \\ (Sh(\mathcal{D})/\mathcal{G}', \mathcal{O}'_{\mathcal{G}'}) \arrow[r, "(j_{\mathcal{G}'/\mathcal{G}}, j^\sharp_{\mathcal{G}'/\mathcal{G}})"] & (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_{\mathcal{G}'}) \end{tikzcd}$$

commutes.

Proof. On the level of morphisms of topoi this is Sites, Lemma 31.3. To check that the diagrams commute as morphisms of ringed topoi use the commutative diagrams of Lemmas 21.4 and 22.1. ☐

04J9 **Lemma 22.4.** *Let* $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$, $s : \mathcal{F} \to f^{-1}\mathcal{G}$ *be as in Lemma 22.3. If* f *is given by a continuous functor* $u : \mathcal{D} \to \mathcal{C}$ *and* $\mathcal{G} = h_V^\#$, $\mathcal{F} = h_U^\#$ *and* s *comes from a morphism* $c : U \to u(V)$, *then the commutative diagrams of Lemma 20.2 and Lemma 22.3 agree via the identifications of Lemma 21.3.*

Proof. This is formal using Lemmas 21.5 and 22.2. ☐

23. Local types of modules

03DK According to our general strategy explained in Section 18 we first define the local types for sheaves of modules on a ringed site, and then we immediately show that these types are intrinsic, hence make sense for sheaves of modules on ringed topoi.

03DL **Definition 23.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. We will freely use the notions defined in Definition 17.1.

(1) We say $\mathcal{F}$ is *locally free* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is a free $\mathcal{O}_{U_i}$-module.
(2) We say $\mathcal{F}$ is *finite locally free* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is a finite free $\mathcal{O}_{U_i}$-module.
(3) We say $\mathcal{F}$ is *locally generated by sections* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by global sections.
(4) Given $r \geq 0$ we sat $\mathcal{F}$ is *locally generated by r sections* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by r global sections.
(5) We say $\mathcal{F}$ is *of finite type* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by finitely many global sections.

(6) We say $\mathcal{F}$ is *quasi-coherent* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module which has a global presentation.
(7) We say $\mathcal{F}$ is *of finite presentation* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module which has a finite global presentation.
(8) We say $\mathcal{F}$ is *coherent* if and only if $\mathcal{F}$ is of finite type, and for every object U of $\mathcal{C}$ and any $s_1, \ldots, s_n \in \mathcal{F}(U)$ the kernel of the map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type on $(\mathcal{C}/U, \mathcal{O}_U)$.

03DM **Lemma 23.2.** *Any of the properties (1) – (8) of Definition 23.1 is intrinsic (see discussion in Section 18).*

Proof. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functor. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the equivalence of topoi associated to u. Set $\mathcal{O}' = g_*\mathcal{O}$, and let $g^\sharp : \mathcal{O}' \to g_*\mathcal{O}$ be the identity. Finally, set $\mathcal{F}' = g_*\mathcal{F}$. Let $\mathcal{P}_l$ be one of the properties (1) – (7) listed in Definition 23.1. (We will discuss the coherent case at the end of the proof.) Let $\mathcal{P}_g$ denote the corresponding property listed in Definition 17.1. We have already seen that $\mathcal{P}_g$ is intrinsic. We have to show that $\mathcal{P}_l(\mathcal{C}, \mathcal{O}, \mathcal{F})$ holds if and only if $\mathcal{P}_l(\mathcal{D}, \mathcal{O}', \mathcal{F}')$ holds.

Assume that $\mathcal{F}$ has $\mathcal{P}_l$. Let V be an object of $\mathcal{D}$. One of the properties of a special cocontinuous functor is that there exists a covering $\{u(U_i) \to V\}_{i \in I}$ in the site $\mathcal{D}$. By assumption, for each i there exists a covering $\{U_{ij} \to U_i\}_{j \in J_i}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{U_{ij}}$ is $\mathcal{P}_g$. By Sites, Lemma 29.3 we have commutative diagrams of ringed topoi

$$\begin{tikzcd}
(Sh(\mathcal{C}/U_{ij}), \mathcal{O}_{U_{ij}}) \arrow[r] \arrow[d] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d] \\
(Sh(\mathcal{D}/u(U_{ij})), \mathcal{O}'_{u(U_{ij})}) \arrow[r] & (Sh(\mathcal{D}), \mathcal{O}')
\end{tikzcd}$$

where the vertical arrows are equivalences. Hence we conclude that $\mathcal{F}'|_{u(U_{ij})}$ has property $\mathcal{P}_g$ also. And moreover, $\{u(U_{ij}) \to V\}_{i \in I, j \in J_i}$ is a covering of the site $\mathcal{D}$. Hence $\mathcal{F}'$ has property $\mathcal{P}_l$.

Assume that $\mathcal{F}'$ has $\mathcal{P}_l$. Let U be an object of $\mathcal{C}$. By assumption, there exists a covering $\{V_i \to u(U)\}_{i \in I}$ such that $\mathcal{F}'|_{V_i}$ has property $\mathcal{P}_g$. Because u is cocontinuous we can refine this covering by a family $\{u(U_j) \to u(U)\}_{j \in J}$ where $\{U_j \to U\}_{j \in J}$ is a covering in $\mathcal{C}$. Say the refinement is given by $\alpha : J \to I$ and $u(U_j) \to V_{\alpha(j)}$. Restricting is transitive, i.e., $(\mathcal{F}'|_{V_{\alpha(j)}})|_{u(U_j)} = \mathcal{F}'|_{u(U_j)}$. Hence by Lemma 17.2 we see that $\mathcal{F}'|_{u(U_j)}$ has property $\mathcal{P}_g$. Hence the diagram

$$\begin{tikzcd}
(Sh(\mathcal{C}/U_j), \mathcal{O}_{U_j}) \arrow[r] \arrow[d] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d] \\
(Sh(\mathcal{D}/u(U_j)), \mathcal{O}'_{u(U_j)}) \arrow[r] & (Sh(\mathcal{D}), \mathcal{O}')
\end{tikzcd}$$

where the vertical arrows are equivalences shows that $\mathcal{F}|_{U_j}$ has property $\mathcal{P}_g$ also. Thus $\mathcal{F}$ has property $\mathcal{P}_l$ as desired.

Finally, we prove the lemma in case $\mathcal{P}_l = \textit{coherent}$[2]. Assume $\mathcal{F}$ is coherent. This implies that $\mathcal{F}$ is of finite type and hence $\mathcal{F}'$ is of finite type also by the first part of the proof. Let V be an object of $\mathcal{D}$ and let $s_1, \ldots, s_n \in \mathcal{F}'(V)$. We have to show that the kernel $\mathcal{K}'$ of $\bigoplus_{j=1,\ldots,n} \mathcal{O}_V \to \mathcal{F}'|_V$ is of finite type on $\mathcal{D}/V$. This means we have to show that for any V'/V there exists a covering $\{V_i' \to V'\}$ such that $\mathcal{F}'|_{V_i'}$ is generated by finitely many sections. Replacing V by V' (and restricting the sections s_j to V') we reduce to the case where $V' = V$. Since u is a special cocontinuous functor, there exists a covering $\{u(U_i) \to V\}_{i \in I}$ in the site $\mathcal{D}$. Using the isomorphism of topoi $\mathit{Sh}(\mathcal{C}/U_i) = \mathit{Sh}(\mathcal{D}/u(U_i))$ we see that $\mathcal{K}'|_{u(U_i)}$ corresponds to the kernel $\mathcal{K}_i$ of a map $\bigoplus_{j=1,\ldots,n} \mathcal{O}_{U_i} \to \mathcal{F}|_{U_i}$. Since $\mathcal{F}$ is coherent we see that $\mathcal{K}_i$ is of finite type. Hence we conclude (by the first part of the proof again) that $\mathcal{K}|_{u(U_i)}$ is of finite type. Thus there exist coverings $\{V_{il} \to u(U_i)\}$ such that $\mathcal{K}|_{V_{il}}$ is generated by finitely many global sections. Since $\{V_{il} \to V\}$ is a covering of $\mathcal{D}$ we conclude that $\mathcal{K}$ is of finite type as desired.

Assume $\mathcal{F}'$ is coherent. This implies that $\mathcal{F}'$ is of finite type and hence $\mathcal{F}$ is of finite type also by the first part of the proof. Let U be an object of $\mathcal{C}$, and let $s_1, \ldots, s_n \in \mathcal{F}(U)$. We have to show that the kernel $\mathcal{K}$ of $\bigoplus_{j=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type on $\mathcal{C}/U$. Using the isomorphism of topoi $\mathit{Sh}(\mathcal{C}/U) = \mathit{Sh}(\mathcal{D}/u(U))$ we see that $\mathcal{K}|_U$ corresponds to the kernel $\mathcal{K}'$ of a map $\bigoplus_{j=1,\ldots,n} \mathcal{O}_{u(U)} \to \mathcal{F}'|_{u(U)}$. As $\mathcal{F}'$ is coherent, we see that $\mathcal{K}'$ is of finite type. Hence, by the first part of the proof again, we conclude that $\mathcal{K}$ is of finite type. $\square$

Hence from now on we may refer to the properties of $\mathcal{O}$-modules defined in Definition 23.1 without specifying a site.

03DN **Lemma 23.3.** *Let $(\mathit{Sh}(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume that the site $\mathcal{C}$ has a final object X. Then*

1. *The following are equivalent*
   (a) *$\mathcal{F}$ is locally free,*
   (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a locally free $\mathcal{O}_{X_i}$-module, and*
   (c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a free $\mathcal{O}_{X_i}$-module.*
2. *The following are equivalent*
   (a) *$\mathcal{F}$ is finite locally free,*
   (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a finite locally free $\mathcal{O}_{X_i}$-module, and*
   (c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a finite free $\mathcal{O}_{X_i}$-module.*
3. *The following are equivalent*
   (a) *$\mathcal{F}$ is locally generated by sections,*
   (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module locally generated by sections, and*
   (c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module globally generated by sections.*
4. *Given $r \geq 0$, the following are equivalent*

[2]The mechanics of this are a bit awkward, and we suggest the reader skip this part of the proof.

(a) *$\mathcal{F}$ is locally generated by r sections,*
(b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module locally generated by r sections, and*
(c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module globally generated by r sections.*

(5) *The following are equivalent*
(a) *$\mathcal{F}$ is of finite type,*
(b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module of finite type, and*
(c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module globally generated by finitely many sections.*

(6) *The following are equivalent*
(a) *$\mathcal{F}$ is quasi-coherent,*
(b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a quasi-coherent $\mathcal{O}_{X_i}$-module, and*
(c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module which has a global presentation.*

(7) *The following are equivalent*
(a) *$\mathcal{F}$ is of finite presentation,*
(b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module of finite presentation, and*
(c) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module has a finite global presentation.*

(8) *The following are equivalent*
(a) *$\mathcal{F}$ is coherent, and*
(b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a coherent $\mathcal{O}_{X_i}$-module.*

Proof. In each case we have (a) $\Rightarrow$ (b). In each of the cases (1) - (6) condition (b) implies condition (c) by axiom (2) of a site (see Sites, Definition 6.2) and the definition of the local types of modules. Suppose $\{X_i \to X\}$ is a covering. Then for every object U of $\mathcal{C}$ we get an induced covering $\{X_i \times_X U \to U\}$. Moreover, the global property for $\mathcal{F}|_{\mathcal{C}/X_i}$ in part (c) implies the corresponding global property for $\mathcal{F}|_{\mathcal{C}/X_i \times_X U}$ by Lemma 17.2, hence the sheaf has property (a) by definition. We omit the proof of (b) $\Rightarrow$ (a) in case (7). $\square$

03DO **Lemma 23.4.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{D}$-module.*

(1) *If $\mathcal{F}$ is locally free then $f^*\mathcal{F}$ is locally free.*
(2) *If $\mathcal{F}$ is finite locally free then $f^*\mathcal{F}$ is finite locally free.*
(3) *If $\mathcal{F}$ is locally generated by sections then $f^*\mathcal{F}$ is locally generated by sections.*
(4) *If $\mathcal{F}$ is locally generated by r sections then $f^*\mathcal{F}$ is locally generated by r sections.*
(5) *If $\mathcal{F}$ is of finite type then $f^*\mathcal{F}$ is of finite type.*
(6) *If $\mathcal{F}$ is quasi-coherent then $f^*\mathcal{F}$ is quasi-coherent.*
(7) *If $\mathcal{F}$ is of finite presentation then $f^*\mathcal{F}$ is of finite presentation.*

Proof. According to the discussion in Section 18 we need only check preservation under pullback for a morphism of ringed sites $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ such that

f is given by a left exact, continuous functor $u : \mathcal{D} \to \mathcal{C}$ between sites which have all finite limits. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules which has one of the properties (1) – (6) of Definition 23.1. We know $\mathcal{D}$ has a final object Y and $X = u(Y)$ is a final object for $\mathcal{C}$. By assumption we have a covering $\{Y_i \to Y\}$ such that $\mathcal{G}|_{\mathcal{D}/Y_i}$ has the corresponding global property. Set $X_i = u(Y_i)$ so that $\{X_i \to X\}$ is a covering in $\mathcal{C}$. We get a commutative diagram of morphisms ringed sites

$$\begin{tikzcd} (\mathcal{C}/X_i, \mathcal{O}_\mathcal{C}|_{X_i}) \arrow[r] \arrow[d] & (\mathcal{C}, \mathcal{O}_\mathcal{C}) \arrow[d] \\ (\mathcal{D}/Y_i, \mathcal{O}_\mathcal{D}|_{Y_i}) \arrow[r] & (\mathcal{D}, \mathcal{O}_\mathcal{D}) \end{tikzcd}$$

by Sites, Lemma 28.2. Hence by Lemma 17.2 that $f^*\mathcal{G}|_{X_i}$ has the corresponding global property. Hence we conclude that $\mathcal{G}$ has the local property we started out with by Lemma 23.3. □

24. Basic results on local types of modules

082S Basic lemmas related to the definitions made above.

082T **Lemma 24.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\theta : \mathcal{G} \to \mathcal{F}$ be a surjective $\mathcal{O}$-module map with $\mathcal{F}$ of finite presentation and $\mathcal{G}$ of finite type. Then* $\mathrm{Ker}(\theta)$ *is of finite type.*

Proof. Omitted. Hint: See Modules, Lemma 11.3. □

0GZN **Lemma 24.2.** *Let $\mathcal{C}$ be a category viewed as a site with the chaotic topology, see Sites, Example 6.6. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$ and let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if for all $U \to V$ in $\mathcal{C}$ the canonical map*

$$\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \longrightarrow \mathcal{F}(U)$$

is an isomorphism.

Proof. Assume $\mathcal{F}$ is quasi-coherent and let $U \to V$ be a morphism of $\mathcal{C}$. Since every covering of V is given by an isomorphism we conclude from Definition 23.1 that there exists a presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_V \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_V \longrightarrow \mathcal{F}|_{\mathcal{C}/V} \longrightarrow 0$$

Since the topology on $\mathcal{C}$ is chaotic, taking sections over any object of $\mathcal{C}$ is exact. We conclude that we obtain a presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}(V) \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}(V) \longrightarrow \mathcal{F}(V) \longrightarrow 0$$

of $\mathcal{F}(V)$ as an $\mathcal{O}(V)$-module and similarly for $\mathcal{F}(U)$. This easily shows that the displayed map in the statement of the lemma is an isomorphism.

Assume the displayed map in the statement of the lemma is an isomorphism for every morphism $U \to V$ in $\mathcal{C}$. Fix V and choose a presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}(V) \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}(V) \longrightarrow \mathcal{F}(V) \longrightarrow 0$$

of $\mathcal{F}(V)$ as an $\mathcal{O}(V)$-module. Then the assumption on $\mathcal{F}$ exactly means that the corresponding sequence

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_V \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_V \longrightarrow \mathcal{F}|_{\mathcal{C}/V} \longrightarrow 0$$

is exact and we conclude that $\mathcal{F}$ is quasi-coherent. $\square$

0GZP **Lemma 24.3.** *Let $\mathcal{C}$ be a category viewed as a site with the chaotic topology, see Sites, Example 6.6. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Assume for all $U \to V$ in $\mathcal{C}$ the restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$ is a flat ring map. Then the category of quasi-coherent $\mathcal{O}$-modules is a weak Serre subcategory of $Mod(\mathcal{O})$.*

Proof. We will check the definition of a weak Serre subcategory, see Homology, Definition 10.1. To do this we will use the characterization of quasi-coherent modules given in Lemma 24.2. Consider an exact sequence

$$\mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \mathcal{F}_4$$

in $Mod(\mathcal{O})$ with $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_3$, and $\mathcal{F}_4$ quasi-coherent. Let $U \to V$ be a morphism of $\mathcal{C}$ and consider the commutative diagram

$$\begin{tikzcd}
\mathcal{F}_0(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{F}_1(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{F}_2(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{F}_3(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{F}_4(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \arrow[d] \\
\mathcal{F}_0(U) \arrow[r] & \mathcal{F}_1(U) \arrow[r] & \mathcal{F}_2(U) \arrow[r] & \mathcal{F}_3(U) \arrow[r] & \mathcal{F}_4(U)
\end{tikzcd}$$

By assumption the vertical arrows with indices 0, 1, 3, 4 are isomorphisms. Since the topology on $\mathcal{C}$ is chaotic taking sections over an object of $\mathcal{C}$ is exact and hence the lower row is exact. Since $\mathcal{O}(V) \to \mathcal{O}(U)$ is flat also the upper row is exact. Thus we conclude that the middle arrow is an isomorphism by the 5 lemma (Homology, Lemma 5.20). $\square$

25. Closed immersions of ringed topoi

08M2 When do we declare a morphism of ringed topoi $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ to be a closed immersion? By analogy with the discussion in Modules, Section 13 it seems natural to assume at least:

(1) The functor i is a closed immersion of topoi (Sites, Definition 43.7).
(2) The associated map $\mathcal{O}' \to i_*\mathcal{O}$ is surjective.

These conditions already imply a number of pleasing results which we discuss in this section. However, it seems prudent to not actually define the notion of a closed immersion of ringed topoi as there are many different definitions we could use.

08M3 **Lemma 25.1.** *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Assume i is a closed immersion of topoi and $i^\sharp : \mathcal{O}' \to i_*\mathcal{O}$ is surjective. Denote $\mathcal{I} \subset \mathcal{O}'$ the kernel of $i^\sharp$. The functor*

$$i_* : Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}')$$

is exact, fully faithful, with essential image those $\mathcal{O}'$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. By Lemma 15.2 and Sites, Lemma 43.8 we see that i_* is exact. From the fact that i_* is fully faithful on sheaves of sets, and the fact that $i^\sharp$ is surjective it follows that i_* is fully faithful as a functor $Mod(\mathcal{O}) \to Mod(\mathcal{O}')$. Namely, suppose that $\alpha : i_*\mathcal{F}_1 \to i_*\mathcal{F}_2$ is an $\mathcal{O}'$-module map. By the fully faithfulness of i_* we obtain a map $\beta : \mathcal{F}_1 \to \mathcal{F}_2$ of sheaves of sets. To prove β is a map of modules we have to show that

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F}_1 \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[d] \\ \mathcal{O} \times \mathcal{F}_2 \arrow[r] & \mathcal{F}_2 \end{tikzcd}$$

commutes. It suffices to prove commutativity after applying i_*. Consider

$$\begin{tikzcd} \mathcal{O}' \times i_*\mathcal{F}_1 \arrow[r] \arrow[d] & i_*\mathcal{O} \times i_*\mathcal{F}_1 \arrow[r] \arrow[d] & i_*\mathcal{F}_1 \arrow[d] \\ \mathcal{O}' \times i_*\mathcal{F}_2 \arrow[r] & i_*\mathcal{O} \times i_*\mathcal{F}_2 \arrow[r] & i_*\mathcal{F}_2 \end{tikzcd}$$

We know the outer rectangle commutes. Since $i^\sharp$ is surjective we conclude.

To finish the proof we have to prove the statement on the essential image of i_*. It is clear that $i_*\mathcal{F}$ is annihilated by $\mathcal{I}$ for any $\mathcal{O}$-module $\mathcal{F}$. Conversely, let $\mathcal{G}$ be a $\mathcal{O}'$-module with $\mathcal{I}\mathcal{G} = 0$. By definition of a closed subtopos there exists a subsheaf $\mathcal{U}$ of the final object of $\mathcal{D}$ such that the essential image of i_* on sheaves of sets is the class of sheaves of sets $\mathcal{H}$ such that $\mathcal{H} \times \mathcal{U} \to \mathcal{U}$ is an isomorphism. In particular, $i_*\mathcal{O} \times \mathcal{U} = \mathcal{U}$. This implies that $\mathcal{I} \times \mathcal{U} = \mathcal{O} \times \mathcal{U}$. Hence our module $\mathcal{G}$ satisfies $\mathcal{G} \times \mathcal{U} = \{0\} \times \mathcal{U} = \mathcal{U}$ (because the zero module is isomorphic to the final object of sheaves of sets). Thus there exists a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ with $i_*\mathcal{F} = \mathcal{G}$. Since i_* is fully faithful on sheaves of sets, we see that in order to define the addition $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$ and the multiplication $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ it suffices to use the addition

$$\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$$

(given to us as $\mathcal{G}$ is a $\mathcal{O}'$-module) and the multiplication

$$i_*\mathcal{O} \times \mathcal{G} \to \mathcal{G}$$

which is given to us as we have the multiplication by $\mathcal{O}'$ which annihilates $\mathcal{I}$ by assumption and $i_*\mathcal{O} = \mathcal{O}'/\mathcal{I}$. By construction $\mathcal{G}$ is isomorphic to the pushforward of the $\mathcal{O}$-module $\mathcal{F}$ so constructed. □

26. Tensor product

03EK In Sections 9 and 11 we defined the change of rings functor by a tensor product construction. To be sure this construction makes sense also to define the tensor product of presheaves of $\mathcal{O}$-modules. To be precise, suppose $\mathcal{C}$ is a category, $\mathcal{O}$ is a presheaf of rings, and $\mathcal{F}$, $\mathcal{G}$ are presheaves of $\mathcal{O}$-modules. In this case we define $\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$ to be the presheaf

$$U \longmapsto (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$$

If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings and $\mathcal{F}$, $\mathcal{G}$ are sheaves of $\mathcal{O}$-modules then we define

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} = (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})^\#$$

to be the sheaf of $\mathcal{O}$-modules associated to the presheaf $\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$.

Here are some formulas which we will use below without further mention:

$$(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}) \otimes_{p,\mathcal{O}} \mathcal{H} = \mathcal{F} \otimes_{p,\mathcal{O}} (\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{H}),$$

and similarly for sheaves. If $\mathcal{O}_1 \to \mathcal{O}_2$ is a map of presheaves of rings, then

$$(\mathcal{F} \otimes_{p,\mathcal{O}_1} \mathcal{G}) \otimes_{p,\mathcal{O}_1} \mathcal{O}_2 = (\mathcal{F} \otimes_{p,\mathcal{O}_1} \mathcal{O}_2) \otimes_{p,\mathcal{O}_2} (\mathcal{G} \otimes_{p,\mathcal{O}_1} \mathcal{O}_2),$$

and similarly for sheaves. These follow from their algebraic counterparts and sheafification.

0GMW **Lemma 26.1.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$, $\mathcal{G}$ be presheaves of $\mathcal{O}$-modules. Then $\mathcal{F}^\# \otimes_{\mathcal{O}^\#} \mathcal{G}^\#$ is equal to $(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})^\#$.*

Proof. Omitted. Hint: use the characterization of tensor product in terms of bilinear maps below and use the universal property of sheafification. □

Let $\mathcal{C}$ be a site, let $\mathcal{O}$ be a sheaf of rings and let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be sheaves of $\mathcal{O}$-modules. In this case we define

$$\text{Bilin}_\mathcal{O}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) = \{\varphi \in \text{Mor}_{Sh(\mathcal{C})}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) \mid \varphi \text{ is } \mathcal{O}\text{-bilinear}\}.$$

With this definition we have

$$\text{Hom}_\mathcal{O}(\mathcal{F} \otimes_\mathcal{O} \mathcal{G}, \mathcal{H}) = \text{Bilin}_\mathcal{O}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

In other words $\mathcal{F} \otimes_\mathcal{O} \mathcal{G}$ represents the functor which associates to $\mathcal{H}$ the set of bilinear maps $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$. In particular, since the notion of a bilinear map makes sense for a pair of modules on a ringed topos, we see that the tensor product of sheaves of modules is intrinsic to the topos (compare the discussion in Section 18). In fact we have the following.

03EL **Lemma 26.2.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_\mathcal{D}$-modules. Then $f^*(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_\mathcal{C}} f^*\mathcal{G}$ functorially in $\mathcal{F}$, $\mathcal{G}$.*

Proof. For a sheaf $\mathcal{H}$ of $\mathcal{O}_\mathcal{C}$ modules we have

$$\begin{aligned}
\text{Hom}_{\mathcal{O}_\mathcal{C}}(f^*(\mathcal{F} \otimes_\mathcal{O} \mathcal{G}), \mathcal{H}) &= \text{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{F} \otimes_\mathcal{O} \mathcal{G}, f_*\mathcal{H}) \\
&= \text{Bilin}_{\mathcal{O}_\mathcal{D}}(\mathcal{F} \times \mathcal{G}, f_*\mathcal{H}) \\
&= \text{Bilin}_{f^{-1}\mathcal{O}_\mathcal{D}}(f^{-1}\mathcal{F} \times f^{-1}\mathcal{G}, \mathcal{H}) \\
&= \text{Hom}_{f^{-1}\mathcal{O}_\mathcal{D}}(f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{G}, \mathcal{H}) \\
&= \text{Hom}_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F} \otimes_{f^*\mathcal{O}_\mathcal{D}} f^*\mathcal{G}, \mathcal{H})
\end{aligned}$$

The interesting "=" in this sequence of equalities is the third equality. It follows from the definition and adjointness of f_* and f^{-1} (as discussed in previous sections) in a straightforward manner. □

03L6 **Lemma 26.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}$-modules.*

(1) *If $\mathcal{F}$, $\mathcal{G}$ are locally free, so is $\mathcal{F} \otimes_\mathcal{O} \mathcal{G}$.*
(2) *If $\mathcal{F}$, $\mathcal{G}$ are finite locally free, so is $\mathcal{F} \otimes_\mathcal{O} \mathcal{G}$.*
(3) *If $\mathcal{F}$, $\mathcal{G}$ are locally generated by sections, so is $\mathcal{F} \otimes_\mathcal{O} \mathcal{G}$.*
(4) *If $\mathcal{F}$, $\mathcal{G}$ are of finite type, so is $\mathcal{F} \otimes_\mathcal{O} \mathcal{G}$.*
(5) *If $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent, so is $\mathcal{F} \otimes_\mathcal{O} \mathcal{G}$.*
(6) *If $\mathcal{F}$, $\mathcal{G}$ are of finite presentation, so is $\mathcal{F} \otimes_\mathcal{O} \mathcal{G}$.*
(7) *If $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is coherent, then $\mathcal{F} \otimes_\mathcal{O} \mathcal{G}$ is coherent.*
(8) *If $\mathcal{F}$, $\mathcal{G}$ are coherent, so is $\mathcal{F} \otimes_\mathcal{O} \mathcal{G}$.*

Proof. Omitted. Hint: Compare with Sheaves of Modules, Lemma 16.6. □

27. Internal Hom

04TT Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$, $\mathcal{G}$ be presheaves of $\mathcal{O}$-modules. Consider the rule

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

For $\varphi : V \to U$ in $\mathcal{C}$ we define a restriction mapping

$$\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U) \longrightarrow \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{F}|_V, \mathcal{G}|_V)$$

by restricting via the relocalization morphism $j : \mathcal{C}/V \to \mathcal{C}/U$, see Sites, Lemma 25.8. Hence this defines a presheaf $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by f on $\mathcal{F}|_U$ or postcomposing with multiplication by f on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a presheaf of $\mathcal{O}$-modules. There is a canonical "evaluation" morphism

$$\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

03EM **Lemma 27.1.** *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, $\mathcal{F}$ is a presheaf of $\mathcal{O}$-modules, and $\mathcal{G}$ is a sheaf of $\mathcal{O}$-modules, then $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is a sheaf of $\mathcal{O}$-modules.*

Proof. Omitted. Hints: Note first that $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^\#, \mathcal{G})$, which reduces the question to the case where both $\mathcal{F}$ and $\mathcal{G}$ are sheaves. The result for sheaves of sets is Sites, Lemma 26.1. $\square$

0E8H **Lemma 27.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}$-modules. Then formation of $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ commutes with restriction to U for $U \in \mathrm{Ob}(\mathcal{C})$.*

Proof. Immediate from the definition. $\square$

0GMX **Remark 27.3.** Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_\mathcal{D}$-modules. There is a canonical map

$$f^* \mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G})$$

Namely, this map is adjoint to the map

$$\mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{F}, \mathcal{G}) \longrightarrow f_* \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G})$$

defined as follows. Say f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. For sections over $V \in \mathrm{Ob}(\mathcal{D})$ we use the map

$$\begin{aligned}
\Gamma(V, \mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{F}, \mathcal{G})) &= \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{F}|_V, \mathcal{G}|_V) \\
&\longrightarrow \mathrm{Hom}_{\mathcal{O}_{u(V)}}(f^*\mathcal{F}|_{u(V)}, \mathcal{G}|_{u(V)}) \\
&= \Gamma(u(V), \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G})) \\
&= \Gamma(V, f_* \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G}))
\end{aligned}$$

where for the arrow we use pullback by the morphism $(\mathcal{C}/u(V), \mathcal{O}_{u(V)}) \to (\mathcal{D}/V, \mathcal{O}_V)$ induced by f.

In the situation of Lemma 27.1 the "evaluation" morphism factors through the tensor product of sheaves of modules

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

03EN **Lemma 27.4.** *Internal hom and (co)limits. Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings.*

(1) *For any presheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*
$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$$
commutes with arbitrary limits.

(2) *For any presheaf of $\mathcal{O}$-modules $\mathcal{G}$ the functor*
$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O})^{opp}, \quad \mathcal{F} \longmapsto \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$$
commutes with arbitrary colimits, in a formula
$$\mathcal{H}om_\mathcal{O}(\mathrm{colim}_i \mathcal{F}_i, \mathcal{G}) = \lim_i \mathcal{H}om_\mathcal{O}(\mathcal{F}_i, \mathcal{G}).$$

Suppose that $\mathcal{C}$ is a site, and $\mathcal{O}$ is a sheaf of rings.

(3) *For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*
$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$$
commutes with arbitrary limits.

(4) *For any sheaf of $\mathcal{O}$-modules $\mathcal{G}$ the functor*
$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O})^{opp}, \quad \mathcal{F} \longmapsto \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$$
commutes with arbitrary colimits, in a formula
$$\mathcal{H}om_\mathcal{O}(\mathrm{colim}_i \mathcal{F}_i, \mathcal{G}) = \lim_i \mathcal{H}om_\mathcal{O}(\mathcal{F}_i, \mathcal{G}).$$

Proof. Let $\mathcal{I} \to PMod(\mathcal{O})$, $i \mapsto \mathcal{G}_i$ be a diagram. Let U be an object of the category $\mathcal{C}$. As j_U^* is both a left and a right adjoint we see that $\lim_i j_U^* \mathcal{G}_i = j_U^* \lim_i \mathcal{G}_i$. Hence we have
$$\begin{aligned} \mathcal{H}om_\mathcal{O}(\mathcal{F}, \lim_i \mathcal{G}_i)(U) &= \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \lim_i \mathcal{G}_i|_U) \\ &= \lim_i \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}_i|_U) \\ &= \lim_i \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}_i)(U) \end{aligned}$$
by definition of a limit. This proves (1). Part (2) is proved in exactly the same way. Part (3) follows from (1) because the limit of a diagram of sheaves is the same as the limit in the category of presheaves. Finally, (4) follow because, in the formula we have
$$\mathrm{Mor}_{Mod(\mathcal{O})}(\mathrm{colim}_i \mathcal{F}_i, \mathcal{G}) = \mathrm{Mor}_{PMod(\mathcal{O})}(\mathrm{colim}_i^{PSh} \mathcal{F}_i, \mathcal{G})$$
as the colimit $\mathrm{colim}_i \mathcal{F}_i$ is the sheafification of the colimit $\mathrm{colim}_i^{PSh} \mathcal{F}_i$ in $PMod(\mathcal{O})$. Hence (4) follows from (2) (by the remark on limits above again). $\square$

0GMY **Lemma 27.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules.*

(1) *If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is an exact sequence of $\mathcal{O}$-modules, then*
$$0 \to \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_\mathcal{O}(\mathcal{F}_1, \mathcal{G}) \to \mathcal{H}om_\mathcal{O}(\mathcal{F}_2, \mathcal{G})$$
is exact.

(2) *If $0 \to \mathcal{G} \to \mathcal{G}_1 \to \mathcal{G}_2$ is an exact sequence of $\mathcal{O}$-modules, then*
$$0 \to \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}_1) \to \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}_2)$$
is exact.

Proof. Follows from Lemma 27.4 and Homology, Lemma 7.2. $\square$

03EO **Lemma 27.6.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.*

(1) *Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be presheaves of $\mathcal{O}$-modules. There is a canonical isomorphism*
$$\mathcal{H}om_\mathcal{O}(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{H}om_\mathcal{O}(\mathcal{G}, \mathcal{H}))$$
which is functorial in all three entries (sheaf Hom in all three spots). In particular,
$$\mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}, \mathcal{H}) = \mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{F}, \mathcal{H}om_\mathcal{O}(\mathcal{G}, \mathcal{H}))$$
(2) *Suppose that $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are sheaves of $\mathcal{O}$-modules. There is a canonical isomorphism*
$$\mathcal{H}om_\mathcal{O}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{H}om_\mathcal{O}(\mathcal{G}, \mathcal{H}))$$
which is functorial in all three entries (sheaf Hom in all three spots). In particular,
$$\mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) = \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{F}, \mathcal{H}om_\mathcal{O}(\mathcal{G}, \mathcal{H}))$$

Proof. This is the analogue of Algebra, Lemma 12.8. The proof is the same, and is omitted. □

03EP **Lemma 27.7.** *Tensor product and colimits. Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings.*

(1) *For any presheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*
$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$$
commutes with arbitrary colimits.
(2) *Suppose that $\mathcal{C}$ is a site, and $\mathcal{O}$ is a sheaf of rings. For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*
$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$$
commutes with arbitrary colimits.

Proof. This is because tensor product is adjoint to internal hom according to Lemma 27.6. See Categories, Lemma 24.5. □

0932 **Lemma 27.8.** *Let $\mathcal{C}$ be a category, resp. a site Let $\mathcal{O} \to \mathcal{O}'$ be a map of presheaves, resp. sheaves of rings. Then*
$$\mathrm{Hom}_\mathcal{O}(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}'}(\mathcal{G}, \mathcal{H}om_\mathcal{O}(\mathcal{O}', \mathcal{F}))$$
for any $\mathcal{O}'$-module $\mathcal{G}$ and $\mathcal{O}$-module $\mathcal{F}$.

Proof. This is the analogue of Algebra, Lemma 14.4. The proof is the same, and is omitted. □

0E8I **Lemma 27.9.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. For $\mathcal{G}$ in $Mod(\mathcal{O}_U)$ and $\mathcal{F}$ in $Mod(\mathcal{O})$ we have $j_{U!}\mathcal{G} \otimes_\mathcal{O} \mathcal{F} = j_{U!}(\mathcal{G} \otimes_{\mathcal{O}_U} \mathcal{F}|_U)$.*

Proof. Let $\mathcal{H}$ be an object of $Mod(\mathcal{O})$. Then
$$\begin{aligned}
\mathrm{Hom}_\mathcal{O}(j_{U!}(\mathcal{G} \otimes_{\mathcal{O}_U} \mathcal{F}|_U), \mathcal{H}) &= \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G} \otimes_{\mathcal{O}_U} \mathcal{F}|_U, \mathcal{H}|_U) \\
&= \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{H}|_U)) \\
&= \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{H})|_U) \\
&= \mathrm{Hom}_\mathcal{O}(j_{U!}\mathcal{G}, \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{H})) \\
&= \mathrm{Hom}_\mathcal{O}(j_{U!}\mathcal{G} \otimes_\mathcal{O} \mathcal{F}, \mathcal{H})
\end{aligned}$$

The first equality because $j_{U!}$ is a left adjoint to restriction of modules. The second by Lemma 27.6. The third by Lemma 27.2. The fourth because $j_{U!}$ is a left adjoint to restriction of modules. The fifth by Lemma 27.6. The lemma follows from this and the Yoneda lemma. $\square$

0EYY **Remark 27.10.** Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{C}$ and consider the localization morphism $j : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$. See Sites, Definition 30.4. We claim that (a) $j_!\mathbf{Z} = \mathbf{Z}_{\mathcal{F}}^\#$ and (b) $j_!(j^{-1}\mathcal{H}) = j_!\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{H}$ for any abelian sheaf $\mathcal{H}$ on $\mathcal{C}$. Let $\mathcal{G}$ be an abelian on $\mathcal{C}$. Part (a) follows from the Yoneda lemma because

$$\mathrm{Hom}(j_!\mathbf{Z}, \mathcal{G}) = \mathrm{Hom}(\mathbf{Z}, j^{-1}\mathcal{G}) = \mathrm{Hom}(\mathbf{Z}_{\mathcal{F}}^\#, \mathcal{G})$$

where the second equality holds because both sides of the equality evaluate to the set of maps from $\mathcal{F} \to \mathcal{G}$ viewed as an abelian group. For (b) we use the Yoneda lemma and

$$\begin{aligned}
\mathrm{Hom}(j_!(j^{-1}\mathcal{H}), \mathcal{G}) &= \mathrm{Hom}(j^{-1}\mathcal{H}, j^{-1}\mathcal{G}) \\
&= \mathrm{Hom}(\mathbf{Z}, \mathcal{H}om(j^{-1}\mathcal{H}, j^{-1}\mathcal{G})) \\
&= \mathrm{Hom}(\mathbf{Z}, j^{-1}\,\mathcal{H}om(\mathcal{H}, \mathcal{G})) \\
&= \mathrm{Hom}(j_!\mathbf{Z}, \mathcal{H}om(\mathcal{H}, \mathcal{G})) \\
&= \mathrm{Hom}(j_!\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{H}, \mathcal{G})
\end{aligned}$$

Here we use adjunction, the fact that taking $\mathcal{H}om$ commutes with localization, and Lemma 27.6.

0GMZ **Lemma 27.11.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module of finite presentation. Let $\mathcal{G} = \mathrm{colim}_{\lambda \in \Lambda} \mathcal{G}_\lambda$ be a filtered colimit of $\mathcal{O}$-modules. Then the canonical map*

$$\mathrm{colim}_\lambda \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_\lambda) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

is an isomorphism.

Proof. It suffices to show the arrow is an isomorphism after restriction to U for all U in $\mathcal{C}$. Both taking colimits of sheaves of modules and taking internal hom commute with restriction to U. See for example Lemmas 14.3 and 27.2. Fix U. Given a covering $\{U_i \to U\}_{i \in I}$, then it suffices to prove the restriction to each U_i is an isomorphism. Hence we may assume $\mathcal{F}$ has a global presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O} \to \mathcal{F} \to 0$$

The functor $\mathcal{H}om_{\mathcal{O}}(-,-)$ commutes with finite direct sums in either variable and $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}, -)$ is the identity functor. By this and by Lemma 27.5 we obtain an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{G} \to \bigoplus\nolimits_{j=1,\ldots,m} \mathcal{G}$$

Since filtered colimits are exact in $Mod(\mathcal{O})$ by Lemma 14.2 also the top row in the following commutative diagram is exact

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{colim}_\lambda \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_\lambda) \arrow[r] \arrow[d] & \mathrm{colim}_\lambda \bigoplus_{i=1,\ldots,n} \mathcal{G}_\lambda \arrow[r] \arrow[d] & \mathrm{colim}_\lambda \bigoplus_{j=1,\ldots,m} \mathcal{G}_\lambda \arrow[d] \\
0 \arrow[r] & \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \arrow[r] & \bigoplus_{i=1,\ldots,n} \mathcal{G} \arrow[r] & \bigoplus_{j=1,\ldots,m} \mathcal{G}
\end{tikzcd}$$

Since the right two vertical arrows are isomorphisms we conclude. □

0GN0 **Lemma 27.12.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{G} = \text{colim}_{\lambda \in \Lambda} \mathcal{G}_\lambda$ be a filtered colimit of $\mathcal{O}$-modules. Let $\mathcal{F}$ be an $\mathcal{O}$-module of finite presentation. Then we have*

$$\text{colim}_\lambda \text{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G}_\lambda) = \text{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G}).$$

if the hypotheses of Sites, Lemma 17.8 part (4) are satisfied for the site $\mathcal{C}$; please see Sites, Remark 17.9.

Proof. Set $\mathcal{H} = \mathcal{H}om_\mathcal{O}(\mathcal{F}, \text{colim}\, \mathcal{G}_\lambda)$ and $\mathcal{H}_\lambda = \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}_\lambda)$. Recall that

$$\text{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G}) = \Gamma(\mathcal{C}, \mathcal{H}) \quad \text{and} \quad \text{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G}_\lambda) = \Gamma(\mathcal{C}, \mathcal{H}_\lambda)$$

by construction. By Lemma 27.11 we have $\mathcal{H} = \text{colim}\, \mathcal{H}_\lambda$. Thus the lemma follows from Sites, Lemma 17.8. □

28. Flat modules

03EQ We can define flat modules exactly as in the case of modules over rings.

03ER **Definition 28.1.** Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.

(1) A presheaf $\mathcal{F}$ of $\mathcal{O}$-modules is called *flat* if the functor

$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F}$$

is exact.

(2) A map $\mathcal{O} \to \mathcal{O}'$ of presheaves of rings is called *flat* if $\mathcal{O}'$ is flat as a presheaf of $\mathcal{O}$-modules.

(3) If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings and $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then we say $\mathcal{F}$ is *flat* if the functor

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_\mathcal{O} \mathcal{F}$$

is exact.

(4) A map $\mathcal{O} \to \mathcal{O}'$ of sheaves of rings on a site is called *flat* if $\mathcal{O}'$ is flat as a sheaf of $\mathcal{O}$-modules.

The notion of a flat module or flat ring map is intrinsic (Section 18).

03ES **Lemma 28.2.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. If each $\mathcal{F}(U)$ is a flat $\mathcal{O}(U)$-module, then $\mathcal{F}$ is flat.*

Proof. This is immediate from the definitions. □

03ET **Lemma 28.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}^\#$ is a flat $\mathcal{O}^\#$-module.*

Proof. Omitted. (Hint: Sheafification is exact.) □

0GN1 **Lemma 28.4.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Assume that every object U of $\mathcal{C}$ has a covering $\{U_i \to U\}_{i \in I}$ such that $\mathcal{F}(U_i)$ is a flat $\mathcal{O}(U_i)$-module. Then $\mathcal{F}^\#$ is a flat $\mathcal{O}^\#$-module.*

Proof. Let $\mathcal{G} \subset \mathcal{G}'$ be an inclusion of $\mathcal{O}^\#$-modules. We have to show that

$$\mathcal{G} \otimes_{\mathcal{O}^\#} \mathcal{F}^\# \longrightarrow \mathcal{G}' \otimes_{\mathcal{O}^\#} \mathcal{F}^\#$$

is injective. By Lemma 26.1 the source of this arrow is the sheafification of the presheaf $\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F}$ and similarly for the target. If U is an object of $\mathcal{C}$ such that $\mathcal{F}(U)$ is a flat $\mathcal{O}(U)$-module, then

$$(\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F})(U) = \mathcal{G}(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) \longrightarrow \mathcal{G}'(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) = (\mathcal{G}' \otimes_{p,\mathcal{O}} \mathcal{F})(U)$$

is injective. Hence we reduce to showing: given a map of presheaves $f : \mathcal{H} \to \mathcal{H}'$ on $\mathcal{C}$ such that every U in $\mathcal{C}$ has a covering $\{U_i \to U\}_{i \in I}$ with $\mathcal{H}(U_i) \to \mathcal{H}'(U_i)$ injective, then $f^\#$ is injective. This we leave to the reader as an exercise. ☐

03EU **Lemma 28.5.** *Colimits and tensor product.*

1. *A filtered colimit of flat presheaves of modules is flat. A direct sum of flat presheaves of modules is flat.*
2. *A filtered colimit of flat sheaves of modules is flat. A direct sum of flat sheaves of modules is flat.*

Proof. Part (1) follows from Lemma 27.7 and Algebra, Lemma 8.8 by looking at sections over objects. To see part (2), use Lemma 27.7 and the fact that a filtered colimit of exact complexes is an exact complex (this uses that sheafification is exact and commutes with colimits). Some details omitted. ☐

0E8J **Lemma 28.6.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}|_U$ is a flat $\mathcal{O}_U$-module.*

Proof. Let $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3$ be an exact complex of $\mathcal{O}_U$-modules. Since $j_{U!}$ is exact (Lemma 19.3) and $\mathcal{F}$ is flat as an $\mathcal{O}$-modules then we see that the complex made up of the modules

$$j_{U!}(\mathcal{G}_i \otimes_{\mathcal{O}_U} \mathcal{F}|_U) = j_{U!}\mathcal{G}_i \otimes_{\mathcal{O}} \mathcal{F}$$

(Lemma 27.9) is exact. We conclude that $\mathcal{G}_1 \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{G}_2 \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{G}_3 \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ is exact by Lemma 19.4. ☐

03EV **Lemma 28.7.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let U be an object of $\mathcal{C}$. Consider the functor $j_U : \mathcal{C}/U \to \mathcal{C}$.*

1. *The presheaf of $\mathcal{O}$-modules $j_{U!}\mathcal{O}_U$ (see Remark 19.7) is flat.*
2. *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, $j_{U!}\mathcal{O}_U$ is a flat sheaf of $\mathcal{O}$-modules.*

Proof. Proof of (1). By the discussion in Remark 19.7 we see that

$$j_{U!}\mathcal{O}_U(V) = \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{O}(V)$$

which is a flat $\mathcal{O}(V)$-module. Hence (1) follows from Lemma 28.2. Then (2) follows as $j_{U!}\mathcal{O}_U = (j_{U!}\mathcal{O}_U)^\#$ (the first $j_{U!}$ on sheaves, the second on presheaves) and Lemma 28.3. ☐

03EW **Lemma 28.8.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.*

1. *Any presheaf of $\mathcal{O}$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
2. *Any presheaf of $\mathcal{O}$-modules is a quotient of a flat presheaf of $\mathcal{O}$-modules.*
3. *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, then any sheaf of $\mathcal{O}$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
4. *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, then any sheaf of $\mathcal{O}$-modules is a quotient of a flat sheaf of $\mathcal{O}$-modules.*

Proof. Proof of (1). For every object U of $\mathcal{C}$ and every $s \in \mathcal{F}(U)$ we get a morphism $j_{U!}\mathcal{O}_U \to \mathcal{F}$, namely the adjoint to the morphism $\mathcal{O}_U \to \mathcal{F}|_U$, $1 \mapsto s$. Clearly the map

$$\bigoplus\nolimits_{(U,s)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{F}$$

is surjective. The source is flat by combining Lemmas 28.5 and 28.7 which proves (2). The sheaf case follows from this either by sheafifying or repeating the same argument. □

03EX **Lemma 28.9.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let*

$$0 \to \mathcal{F}'' \to \mathcal{F}' \to \mathcal{F} \to 0$$

be a short exact sequence of presheaves of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules.

(1) *If $\mathcal{F}$ is a flat presheaf of modules, then the sequence*
$$0 \to \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G} \to 0$$
is exact.

(2) *If $\mathcal{C}$ is a site, $\mathcal{O}$, $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$, and $\mathcal{G}$ are sheaves, and $\mathcal{F}$ is flat as a sheaf of modules, then the sequence*
$$0 \to \mathcal{F}'' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to 0$$
is exact.

Proof. Choose a flat presheaf of $\mathcal{O}$-modules $\mathcal{G}'$ which surjects onto $\mathcal{G}$. This is possible by Lemma 28.8. Let $\mathcal{G}'' = \mathrm{Ker}(\mathcal{G}' \to \mathcal{G})$. The lemma follows by applying the snake lemma to the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G} & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G} & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns. The middle row is exact because tensoring with the flat module $\mathcal{G}'$ is exact. The proof in the case of sheaves is exactly the same. □

03EY **Lemma 28.10.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let*

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

be a short exact sequence of presheaves of $\mathcal{O}$-modules.

(1) *If $\mathcal{F}_2$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_1$.*
(2) *If $\mathcal{F}_1$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_2$.*

If $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings then the same result holds in Mod($\mathcal{O}$).

Proof. Let $\mathcal{G}^\bullet$ be an arbitrary exact complex of presheaves of $\mathcal{O}$-modules. Assume that $\mathcal{F}_0$ is flat. By Lemma 28.9 we see that

$$0 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_0 \to 0$$

is a short exact sequence of complexes of presheaves of $\mathcal{O}$-modules. Hence (1) and (2) follow from the snake lemma. The case of sheaves of modules is proved in the same way. $\square$

03EZ **Lemma 28.11.** *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let*

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{Q} \to 0$$

be an exact complex of presheaves of $\mathcal{O}$-modules. If $\mathcal{Q}$ and all $\mathcal{F}_i$ are flat $\mathcal{O}$-modules, then for any presheaf $\mathcal{G}$ of $\mathcal{O}$-modules the complex

$$\ldots \to \mathcal{F}_2 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}_1 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}_0 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{Q} \otimes_{p,\mathcal{O}} \mathcal{G} \to 0$$

is exact also. If $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings then the same result holds $Mod(\mathcal{O})$.

Proof. Follows from Lemma 28.9 by splitting the complex into short exact sequences and using Lemma 28.10 to prove inductively that $\mathrm{Im}(\mathcal{F}_{i+1} \to \mathcal{F}_i)$ is flat. $\square$

0G6Q **Lemma 28.12.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $\mathcal{G}$ and $\mathcal{F}$ are flat $\mathcal{O}$-modules, then $\mathcal{G} \otimes_\mathcal{O} \mathcal{F}$ is a flat $\mathcal{O}$-module.*

Proof. This is true because

$$(\mathcal{G} \otimes_\mathcal{O} \mathcal{F}) \otimes_\mathcal{O} \mathcal{H} = \mathcal{G} \otimes_\mathcal{O} (\mathcal{F} \otimes_\mathcal{O} \mathcal{H})$$

and a composition of exact functors is exact. $\square$

05V4 **Lemma 28.13.** *Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on a site $\mathcal{C}$. If $\mathcal{G}$ is a flat $\mathcal{O}_1$-module, then $\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2$ is a flat $\mathcal{O}_2$-module.*

Proof. This is true because

$$(\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2) \otimes_{\mathcal{O}_2} \mathcal{H} = \mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{F}$$

(as sheaves of abelian groups for example). $\square$

The following lemma is the analogue of the equational criterion of flatness (Algebra, Lemma 39.11).

08FC **Lemma 28.14.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module. The following are equivalent*

(1) *$\mathcal{F}$ is a flat $\mathcal{O}$-module.*
(2) *Let U be an object of $\mathcal{C}$ and let*

$$\mathcal{O}_U \xrightarrow{(f_1, \ldots, f_n)} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} \mathcal{F}|_U$$

be a complex of $\mathcal{O}_U$-modules. Then there exists a covering $\{U_i \to U\}$ and for each i a factorization

$$\mathcal{O}_{U_i}^{\oplus n} \xrightarrow{B_i} \mathcal{O}_{U_i}^{\oplus l_i} \xrightarrow{(t_{i1}, \ldots, t_{il_i})} \mathcal{F}|_{U_i}$$

of $(s_1, \ldots, s_n)|_{U_i}$ such that $B_i \circ (f_1, \ldots, f_n)|_{U_i} = 0$.
(3) *Let U be an object of $\mathcal{C}$ and let*

$$\mathcal{O}_U^{\oplus m} \xrightarrow{A} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} \mathcal{F}|_U$$

be a complex of $\mathcal{O}_U$-modules. Then there exists a covering $\{U_i \to U\}$ and for each i a factorization

$$\mathcal{O}_{U_i}^{\oplus n} \xrightarrow{B_i} \mathcal{O}_{U_i}^{\oplus l_i} \xrightarrow{(t_{i1}, \ldots, t_{il_i})} \mathcal{F}|_{U_i}$$

of $(s_1, \ldots, s_n)|_{U_i}$ *such that* $B_i \circ A|_{U_i} = 0$.

Proof. Assume (1). Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals generated by $f_1, \ldots, f_n$. Then $\sum f_j \otimes s_j$ is a section of $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ which maps to zero in $\mathcal{F}|_U$. As $\mathcal{F}|_U$ is flat (Lemma 28.6) the map $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{F}|_U$ is injective. Since $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ is the sheaf associated to the presheaf tensor product, we see there exists a covering $\{U_i \to U\}$ such that $\sum f_j|_{U_i} \otimes s_j|_{U_i}$ is zero in $\mathcal{I}(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{F}(U_i)$. Unwinding the definitions using Algebra, Lemma 107.10 we find $t_{i1}, \ldots, t_{il_i} \in \mathcal{F}(U_i)$ and $a_{ijk} \in \mathcal{O}(U_i)$ such that $\sum_j a_{ijk} f_j|_{U_i} = 0$ and $s_j|_{U_i} = \sum_k a_{ijk} t_{ik}$. Thus (2) holds.

Assume (2). Let U, n, m, A and $s_1, \ldots, s_n$ as in (3) be given. Observe that A has m columns. We will prove the assertion of (3) is true by induction on m. For the base case $m = 0$ we can use the factorization through the zero sheaf (in other words $l_i = 0$). Let $(f_1, \ldots, f_n)$ be the last column of A and apply (2). This gives new diagrams

$$\mathcal{O}_{U_i}^{\oplus m} \xrightarrow{B_i \circ A|_{U_i}} \mathcal{O}_{U_i}^{\oplus l_i} \xrightarrow{(t_{i1}, \ldots, t_{il_i})} \mathcal{F}|_{U_i}$$

but the first column of $A_i = B_i \circ A|_{U_i}$ is zero. Hence we can apply the induction hypothesis to U_i, l_i, $m - 1$, the matrix consisting of the first $m - 1$ columns of A_i, and $t_{i1}, \ldots, t_{il_i}$ to get coverings $\{U_{ij} \to U_j\}$ and factorizations

$$\mathcal{O}_{U_{ij}}^{\oplus l_i} \xrightarrow{C_{ij}} \mathcal{O}_{U_{ij}}^{\oplus k_{ij}} \xrightarrow{(v_{ij1}, \ldots, v_{ijk_{ij}})} \mathcal{F}|_{U_{ij}}$$

of $(t_{i1}, \ldots, t_{il_i})|_{U_{ij}}$ such that $C_i \circ B_i|_{U_{ij}} \circ A|_{U_{ij}} = 0$. Then $\{U_{ij} \to U\}$ is a covering and we get the desired factorizations using $B_{ij} = C_i \circ B_i|_{U_{ij}}$ and v_{ija}. In this way we see that (2) implies (3).

Assume (3). Let $\mathcal{G} \to \mathcal{H}$ be an injective homomorphism of $\mathcal{O}$-modules. We have to show that $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{H} \otimes_{\mathcal{O}} \mathcal{F}$ is injective. Let U be an object of $\mathcal{C}$ and let $s \in (\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F})(U)$ be a section which maps to zero in $\mathcal{H} \otimes_{\mathcal{O}} \mathcal{F}$. We have to show that s is zero. Since $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$ is a sheaf, it suffices to find a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that $s|_{U_i}$ is zero for all $i \in I$. Hence we may always replace U by the members of a covering. In particular, since $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$ is the sheafification of $\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F}$ we may assume that s is the image of $s' \in \mathcal{G}(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U)$. Arguing similarly for $\mathcal{H} \otimes_{\mathcal{O}} \mathcal{F}$ we may assume that s' maps to zero in $\mathcal{H}(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U)$. Write $\mathcal{F}(U) = \operatorname{colim} M_\alpha$ as a filtered colimit of finitely presented $\mathcal{O}(U)$-modules M_α (Algebra, Lemma 11.3). Since tensor product commutes with filtered colimits (Algebra, Lemma 12.9) we can choose an α such that s' comes from some $s'' \in \mathcal{G}(U) \otimes_{\mathcal{O}(U)} M_\alpha$ and such that s'' maps to zero in $\mathcal{H}(U) \otimes_{\mathcal{O}(U)} M_\alpha$. Fix α and s''. Choose a presentation

$$\mathcal{O}(U)^{\oplus m} \xrightarrow{A} \mathcal{O}(U)^{\oplus n} \to M_\alpha \to 0$$

We apply (3) to the corresponding complex of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{\oplus m} \xrightarrow{A} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} \mathcal{F}|_U$$

After replacing U by the members of the covering U_i we find that the map

$$M_\alpha \to \mathcal{F}(U)$$

factors through a free module $\mathcal{O}(U)^{\oplus l}$ for some l. Since $\mathcal{G}(U) \to \mathcal{H}(U)$ is injective we conclude that

$$\mathcal{G}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U)^{\oplus l} \to \mathcal{H}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U)^{\oplus l}$$

is injective too. Hence as s'' maps to zero in the module on the right, it also maps to zero in the module on the left, i.e., s is zero as desired. $\square$

08M4 **Lemma 28.15.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}' \to \mathcal{O}$ be a surjection of sheaves of rings whose kernel $\mathcal{I}$ is an ideal of square zero. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = \mathcal{F}'/\mathcal{I}\mathcal{F}'$. The following are equivalent*

1. *$\mathcal{F}'$ is a flat $\mathcal{O}'$-module, and*
2. *$\mathcal{F}$ is a flat $\mathcal{O}$-module and $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{F}'$ is injective.*

Proof. If (1) holds, then $\mathcal{F} = \mathcal{F}' \otimes_{\mathcal{O}'} \mathcal{O}$ is flat over $\mathcal{O}$ by Lemma 28.13 and we see the map $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{F}'$ is injective by applying $- \otimes_{\mathcal{O}'} \mathcal{F}'$ to the exact sequence $0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$, see Lemma 28.9. Assume (2). In the rest of the proof we will use without further mention that $\mathcal{K} \otimes_{\mathcal{O}'} \mathcal{F}' = \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}$ for any $\mathcal{O}'$-module $\mathcal{K}$ annihilated by $\mathcal{I}$. Let $\alpha : \mathcal{G}' \to \mathcal{H}'$ be an injective map of $\mathcal{O}'$-modules. Let $\mathcal{G} \subset \mathcal{G}'$, resp. $\mathcal{H} \subset \mathcal{H}'$ be the subsheaf of sections annihilated by $\mathcal{I}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{G} \otimes_{\mathcal{O}'} \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{G}' \otimes_{\mathcal{O}'} \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{G}'/\mathcal{G} \otimes_{\mathcal{O}'} \mathcal{F}' \arrow[r] \arrow[d] & 0 \\
\mathcal{H} \otimes_{\mathcal{O}'} \mathcal{F}' \arrow[r] & \mathcal{H}' \otimes_{\mathcal{O}'} \mathcal{F}' \arrow[r] & \mathcal{H}'/\mathcal{H} \otimes_{\mathcal{O}'} \mathcal{F}' \arrow[r] & 0
\end{tikzcd}$$

Note that $\mathcal{G}'/\mathcal{G}$ and $\mathcal{H}'/\mathcal{H}$ are annihilated by $\mathcal{I}$ and that $\mathcal{G}'/\mathcal{G} \to \mathcal{H}'/\mathcal{H}$ is injective. Thus the right vertical arrow is injective as $\mathcal{F}$ is flat over $\mathcal{O}$. The same is true for the left vertical arrow. Hence the middle vertical arrow is injective and $\mathcal{F}'$ is flat. $\square$

0GLY **Lemma 28.16.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a flat homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals such that the induced map $\mathcal{O}/\mathcal{I} \to \mathcal{O}'/\mathcal{I}\mathcal{O}'$ is an isomorphism. For any $\mathcal{O}$-module $\mathcal{F}$ annihilated by $\mathcal{I}^n$ for some $n \geq 0$ the map $id \otimes 1 : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}'$ is an isomorphism.*

Proof. Omitted. Hint: See More on Algebra, Lemma 89.2. $\square$

29. Duals

0FNX Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The category of $\mathcal{O}$-modules endowed with the tensor product constructed in Section 26 is a symmetric monoidal category. For an $\mathcal{O}$-module $\mathcal{F}$ the following are equivalent

1. $\mathcal{F}$ has a left dual in the monoidal category of $\mathcal{O}$-modules,
2. for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}|_{U_i}$-module, and
3. $\mathcal{F}$ is of finite presentation and flat as an $\mathcal{O}$-module.

This is proved in Example 29.1 and Lemmas 29.2 and 29.3 of this section.

0FNY **Example 29.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module such that for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}|_{U_i}$-module. Then the map

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{F})$$

is an isomorphism. Namely, this is a local question, it is true if $\mathcal{F}$ is finite free, and it holds for any summand of a module for which it is true (details omitted). Denote

$$\eta : \mathcal{O} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O})$$

the map sending 1 to the section corresponding to $\mathrm{id}_\mathcal{F}$ under the isomorphism above. Denote

$$\epsilon : \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{O}) \otimes_\mathcal{O} \mathcal{F} \longrightarrow \mathcal{O}$$

the evaluation map. Then we see that $\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{O}), \eta, \epsilon$ is a left dual for $\mathcal{F}$ as in Categories, Definition 43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_\mathcal{F}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{O})}$.

0FNZ **Lemma 29.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a $\mathcal{O}$-module. Let $\mathcal{G}, \eta, \epsilon$ be a left dual of $\mathcal{F}$ in the monoidal category of $\mathcal{O}$-modules, see Categories, Definition 43.5. Then*

1. *for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}|_{U_i}$-module,*
2. *the map $e : \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{O}) \to \mathcal{G}$ sending a local section λ to $(\lambda \otimes 1)(\eta)$ is an isomorphism,*
3. *we have $\epsilon(f, g) = e^{-1}(g)(f)$ for local sections f and g of $\mathcal{F}$ and $\mathcal{G}$.*

Proof. The assumptions mean that

$$\mathcal{F} \xrightarrow{\eta \otimes 1} \mathcal{F} \otimes_\mathcal{O} \mathcal{G} \otimes_\mathcal{O} \mathcal{F} \xrightarrow{1 \otimes \epsilon} \mathcal{F} \quad \text{and} \quad \mathcal{G} \xrightarrow{1 \otimes \eta} \mathcal{G} \otimes_\mathcal{O} \mathcal{F} \otimes_\mathcal{O} \mathcal{G} \xrightarrow{\epsilon \otimes 1} \mathcal{G}$$

are the identity map. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering of U, we can find a finite number of sections $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ of $\mathcal{F}$ and $\mathcal{G}$ over U such that $\eta(1) = \sum f_i g_i$. Denote

$$\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$$

the map sending the ith basis vector to f_i. Then we can factor the map $\eta|_U$ over a map $\tilde{\eta} : \mathcal{O}_U \to \mathcal{O}_U^{\oplus n} \otimes_{\mathcal{O}_U} \mathcal{G}|_U$. We obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{F}|_U \arrow[r, "\eta \otimes 1"'] \arrow[dr, "\tilde{\eta} \otimes 1"'] & \mathcal{F}|_U \otimes \mathcal{G}|_U \otimes \mathcal{F}|_U \arrow[r, "1 \otimes \epsilon"'] & \mathcal{F}|_U \\
& \mathcal{O}_U^{\oplus n} \otimes \mathcal{G}|_U \otimes \mathcal{F}|_U \arrow[r, "1 \otimes \epsilon"] \arrow[u] & \mathcal{O}_U^{\oplus n} \arrow[u]
\end{tikzcd}$$

This shows that the identity on $\mathcal{F}|_U$ factors through a finite free $\mathcal{O}_U$-module. This proves (1). Part (2) follows from Categories, Lemma 43.6 and its proof. Part (3) follows from the first equality of the proof. You can also deduce (2) and (3) from the uniqueness of left duals (Categories, Remark 43.7) and the construction of the left dual in Example 29.1. □

08FD **Lemma 29.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be locally of finite presentation and flat. Then given an object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}_{U_i}$-module.*

Proof. Choose an object U of $\mathcal{C}$. After replacing U by the members of a covering, we may assume there exists a presentation

$$\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

By Lemma 28.14 we may, after replacing U by the members of a covering, assume there exists a factorization

$$\mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus n_1} \to \mathcal{F}|_U$$

such that the composition $\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus n_r}$ is zero. This means that the surjection $\mathcal{O}_U^{\oplus n_r} \to \mathcal{F}|_U$ has a section and we win. □

30. Towards constructible modules

0933 Recall that a quasi-compact object of a site is roughly an object such that every covering of it can be refined by a finite covering (the actual definition is slightly more involved, see Sites, Section 17). It turns out that if every object of a site has a covering by quasi-compact objects, then the modules $j_!\mathcal{O}_U$ with U quasi-compact form a particularly nice set of generators for the category of all modules.

0934 **Lemma 30.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Then the sequence*

$$\bigoplus j_{U_i \times_U U_j!}\mathcal{O}_{U_i \times_U U_j} \to \bigoplus j_{U_i!}\mathcal{O}_{U_i} \to j_!\mathcal{O}_U \to 0$$

is exact.

Proof. For any $\mathcal{O}$-module $\mathcal{F}$ the functor $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ turns our sequence into the exact sequence $0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_i \times_U U_j)$, see (19.2.1). The lemma follows from this and Homology, Lemma 5.8. $\square$

0G1W **Lemma 30.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be covering of $\mathcal{C}$. If U is quasi-compact, then there exist a finite subset $I' \subset I$ such that the sequence*

$$\bigoplus\nolimits_{i,i' \in I'} j_{U_i \times_U U_{i'}!}\mathcal{O}_{U_i \times_U U_{i'}} \to \bigoplus\nolimits_{i \in I'} j_{U_i!}\mathcal{O}_{U_i} \to j_!\mathcal{O}_U \to 0$$

is exact.

Proof. This lemma is immediate from Lemma 30.1 if U satisfies condition (3) of Sites, Lemma 17.2. We urge the reader to skip the proof in the general case. By definition there exists a covering $\mathcal{V} = \{V_j \to U\}_{j \in J}$ and a morphism $\mathcal{V} \to \mathcal{U}$ of families of maps with fixed target given by $\mathrm{id} : U \to U$, $\alpha : J \to I$, and $f_j : V_j \to U_{\alpha(j)}$ (see Sites, Definition 8.1) such that the image $I' \subset I$ of α is finite. By Homology, Lemma 5.8 it suffices to show that for any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ turns the sequence of the lemma into an exact sequence. By (19.2.1) we obtain the usual sequence

$$0 \to \mathcal{F}(U) \to \prod\nolimits_{i \in I'} \mathcal{F}(U_i) \to \prod\nolimits_{i,i' \in I'} \mathcal{F}(U_i \times_U U_{i'})$$

This is an exact sequence by Sites, Lemma 8.6 applied to the family of maps $\{U_i \to U\}_{i \in I'}$ which is refined by the covering $\mathcal{V}$. $\square$

0935 **Lemma 30.3.** *Let $\mathcal{C}$ be a site. Let W be a quasi-compact object of $\mathcal{C}$.*

1. *The functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}(W)$ commutes with coproducts.*
2. *Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. The functor $Mod(\mathcal{O}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}(W)$ commutes with direct sums.*

Proof. Proof of (1). Taking sections over W commutes with filtered colimits with injective transition maps by Sites, Lemma 17.7. If $\mathcal{F}_i$ is a family of sheaves of sets indexed by a set I. Then $\coprod \mathcal{F}_i$ is the filtered colimit over the partially ordered set of finite subsets $E \subset I$ of the coproducts $\mathcal{F}_E = \coprod_{i \in E} \mathcal{F}_i$. Since the transition maps are injective we conclude.

Proof of (2). Let $\mathcal{F}_i$ be a family of sheaves of $\mathcal{O}$-modules indexed by a set I. Then $\bigoplus \mathcal{F}_i$ is the filtered colimit over the partially ordered set of finite subsets $E \subset I$ of the direct sums $\mathcal{F}_E = \bigoplus_{i \in E} \mathcal{F}_i$. A filtered colimit of abelian sheaves can be computed in the category of sheaves of sets. Moreover, for $E \subset E'$ the transition map $\mathcal{F}_E \to \mathcal{F}_{E'}$ is injective (as sheafification is exact and the injectivity is clear on underlying presheaves). Hence it suffices to show the result for a finite index set by Sites, Lemma 17.7. The finite case is dealt with in Lemma 3.2 (it holds over any object of $\mathcal{C}$). □

0936 **Lemma 30.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be a quasi-compact object of $\mathcal{C}$. Then the functor $\text{Hom}_{\mathcal{O}}(j_!\mathcal{O}_U, -)$ commutes with direct sums.*

Proof. This is true because $\text{Hom}_{\mathcal{O}}(j_!\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$ by (19.2.1) and because the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ commutes with direct sums by Lemma 30.3. □

In order to state the sharpest possible results in the following we introduce some notation.

0937 **Situation 30.5.** Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a set of objects. We consider the following conditions

0938 (1) Every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$.
0939 (2) Every $U \in \mathcal{B}$ is quasi-compact (Sites, Section 17).
093A (3) For a covering $\{U_i \to U\}$ with $U_i, U \in \mathcal{B}$ the fibre products $U_i \times_U U_j$ are quasi-compact.

093B **Lemma 30.6.** *In Situation 30.5 assume (1) holds.*

(1) *Every sheaf of sets is the target of a surjective map whose source is a coproduct $\coprod h_{U_i}^\#$ with U_i in $\mathcal{B}$.*
(2) *If $\mathcal{O}$ is a sheaf of rings, then every $\mathcal{O}$-module is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$ with U_i in $\mathcal{B}$.*

Proof. Part (1) follows from Sites, Lemmas 12.5 and 12.4. Part (2) follows from Lemmas 28.8 and 30.1. □

093C **Lemma 30.7.** *In Situation 30.5 assume (1) and (2) hold.*

(1) *Every sheaf of sets is a filtered colimit of sheaves of the form*

09Y7 (30.7.1)
$$\textit{Coequalizer}\left(\coprod\nolimits_{j=1,\ldots,m} h_{V_j}^\# \rightrightarrows \coprod\nolimits_{i=1,\ldots,n} h_{U_i}^\# \right)$$

with U_i and V_j in $\mathcal{B}$.

(2) *If $\mathcal{O}$ is a sheaf of rings, then every $\mathcal{O}$-module is a filtered colimit of sheaves of the form*

093D (30.7.2)
$$\text{Coker}\left(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i} \right)$$

with U_i and V_j in $\mathcal{B}$.

Proof. Proof of (1). By Lemma 30.6 every sheaf of sets $\mathcal{F}$ is the target of a surjection whose source is a coprod $\mathcal{F}_0$ of sheaves the form $h_U^\#$ with $U \in \mathcal{B}$. Applying this to $\mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ we find that $\mathcal{F}$ is a coequalizer of a pair of maps

$$\coprod\nolimits_{j \in J} h_{V_j}^\# \rightrightarrows \coprod\nolimits_{i \in I} h_{U_i}^\#$$

for some index sets I, J and V_j and U_i in $\mathcal{B}$. For every finite subset $J' \subset J$ there is a finite subset $I' \subset I$ such that the coproduct over $j \in J'$ maps into the coprod over $i \in I'$ via both maps, see Sites, Lemma 17.7. (Details omitted; hint: an infinite coproduct is the filtered colimit of the finite sub-coproducts.) Thus our sheaf is the colimit of the cokernels of these maps between finite coproducts.

Proof of (2). By Lemma 30.6 every module is a quotient of a direct sum of modules of the form $j_{U!}\mathcal{O}_U$ with $U \in \mathcal{B}$. Thus every module is a cokernel

$$\text{Coker}\left(\bigoplus\nolimits_{j\in J} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i\in I} j_{U_i!}\mathcal{O}_{U_i}\right)$$

for some index sets I, J and V_j and U_i in $\mathcal{B}$. For every finite subset $J' \subset J$ there is a finite subset $I' \subset I$ such that the direct sum over $j \in J'$ maps into the direct sum over $i \in I'$, see Lemma 30.4. Thus our module is the colimit of the cokernels of these maps between finite direct sums. □

093E **Lemma 30.8.** *In Situation 30.5 assume (1) and (2) hold. Let $\mathcal{O}$ be a sheaf of rings. Then a cokernel of a map between modules as in (30.7.2) is another module as in (30.7.2).*

Proof. Let $\mathcal{F} = \text{Coker}(\bigoplus j_{V_j!}\mathcal{O}_{V_j} \to \bigoplus j_{U_i!}\mathcal{O}_{U_i})$ as in (30.7.2). It suffices to show that the cokernel of a map $\varphi : j_{W!}\mathcal{O}_W \to \mathcal{F}$ with $W \in \mathcal{B}$ is another module of the same type. The map φ corresponds to $s \in \mathcal{F}(W)$. Since $\bigoplus j_{U_i!}\mathcal{O}_{U_i} \to \mathcal{F}$ is surjective, by (1) we may choose a covering $\{W_k \to W\}_{k\in K}$ with $W_k \in \mathcal{B}$ such that $s|_{W_k}$ is the image of some section s_k of $\bigoplus j_{U_i!}\mathcal{O}_{U_i})$. By (2) the object W is quasi-compact. By Lemma 30.2 there is a finite subset $K' \subset K$ such that $\bigoplus_{k\in K'} j_{W_k!}\mathcal{O}_{W_k} \to j_{W!}\mathcal{O}_W$ is surjective. We conclude that $\text{Coker}(\varphi)$ is equal to

$$\text{Coker}\left(\bigoplus\nolimits_{k\in K'} j_{W_k!}\mathcal{O}_{W_k} \oplus \bigoplus j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus j_{U_i!}\mathcal{O}_{U_i}\right)$$

where the map $\bigoplus_{k\in K'} j_{W_k!}\mathcal{O}_{W_k} \to \bigoplus j_{U_i!}\mathcal{O}_{U_i}$ corresponds to $\sum_{k\in K'} s_k$. This finishes the proof. □

093F **Lemma 30.9.** *In Situation 30.5 assume (1), (2), and (3) hold. Let $\mathcal{O}$ be a sheaf of rings. Assume given a map*

$$\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}$$

with U_i and V_j in $\mathcal{B}$, and coverings $\{U_{ik} \to U_i\}_{k\in K_i}$ with $U_{ik} \in \mathcal{B}$. Then there exist finite subsets $K'_i \subset K_i$ and a finite set L of $W_l \in \mathcal{B}$ and a commutative diagram

$$\begin{tikzcd}
\bigoplus_{l\in L} j_{W_l!}\mathcal{O}_{W_l} \arrow[r] \arrow[d] & \bigoplus_{i=1,\ldots,n} \bigoplus_{k\in K'_i} j_{U_{ik}!}\mathcal{O}_{U_{ik}} \arrow[d] \\
\bigoplus_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \arrow[r] & \bigoplus_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}
\end{tikzcd}$$

inducing an isomorphism on cokernels of the horizontal maps.

Proof. Since U_i is quasi-compact, we may choose finite subsets $K'_i \subset K_i$ as in Lemma 30.2. Then since $\bigoplus_{i=1,\ldots,n} \bigoplus_{k\in K'_i} j_{U_{ik}!}\mathcal{O}_{U_{ik}} \to \bigoplus_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}$ is surjective, we can find coverings $\{V_{jm} \to V_j\}_{m\in M_j}$ with $V_{jm} \in \mathcal{B}$ such that we can find a commutative diagram

$$\begin{tikzcd}
\bigoplus_{j=1,\ldots,m} \bigoplus_{m\in M_j} j_{V_{jm}!}\mathcal{O}_{V_{jm}} \arrow[r] \arrow[d] & \bigoplus_{i=1,\ldots n} \bigoplus_{k\in K'_i} j_{U_{ik}!}\mathcal{O}_{U_{ik}} \arrow[d] \\
\bigoplus_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \arrow[r] & \bigoplus_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}
\end{tikzcd}$$

Since V_j is quasi-compact, we can choose finite subsets $M'_j \subset M_j$ as in Lemma 30.2. Set

$$L = \left(\coprod\nolimits_{i=1,\ldots,n} K'_i \times K'_i\right) \coprod \left(\coprod\nolimits_{j=1,\ldots,m} M'_j\right)$$

and for $l = (k, k') \in K'_i \times K'_i \subset L$ set $W_l = U_{ik} \times_{U_i} U_{ik'}$ and for $l = m \in M'_j \subset L$ set $W_l = V_{jm}$. Since we have the exact sequences of Lemma 30.2 for the families $\{U_{ik} \to U_i\}_{k \in K'_i}$ we conclude that we get a diagram as in the statement of the lemma (details omitted), except that it is not yet clear that $W_l \in \mathcal{B}$. However, since W_l is quasi-compact for all $l \in L$ we do another application of Lemma 30.2 and find finite families of maps $\{W_{lt} \to W_l\}_{t \in T_l}$ with $W_{lt} \in \mathcal{B}$ such that $\bigoplus j_{W_{lt}!}\mathcal{O}_{W_{lt}} \to j_{W_l!}\mathcal{O}_{W_l}$ is surjective. Then we replace L by $\coprod_{l \in L} T_l$ and everything is clear. $\square$

093G **Lemma 30.10.** *In Situation 30.5 assume (1), (2), and (3) hold. Let $\mathcal{O}$ be a sheaf of rings. Then an extension of modules as in (30.7.2) is another module as in (30.7.2).*

Proof. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}$-modules with $\mathcal{F}_1$ and $\mathcal{F}_3$ as in (30.7.2). Choose presentations

$$\bigoplus A_{V_j} \to \bigoplus A_{U_i} \to \mathcal{F}_1 \to 0 \quad \text{and} \quad \bigoplus A_{T_j} \to \bigoplus A_{W_i} \to \mathcal{F}_3 \to 0$$

In this proof the direct sums are always finite, and we write $A_U = j_{U!}\mathcal{O}_U$ for $U \in \mathcal{B}$. Since $\mathcal{F}_2 \to \mathcal{F}_3$ is surjective, we can choose coverings $\{W_{ik} \to W_i\}$ with $W_{ik} \in \mathcal{B}$ such that $A_{W_{ik}} \to \mathcal{F}_3$ lifts to a map $A_{W_{ik}} \to \mathcal{F}_2$. By Lemma 30.9 we may replace our collection $\{W_i\}$ by a finite subcollection of the collection $\{W_{ik}\}$ and assume the map $\bigoplus A_{W_i} \to \mathcal{F}_3$ lifts to a map into $\mathcal{F}_2$. Consider the kernel

$$\mathcal{K}_2 = \mathrm{Ker}(\bigoplus A_{U_i} \oplus \bigoplus A_{W_i} \longrightarrow \mathcal{F}_2)$$

By the snake lemma this kernel surjects onto $\mathcal{K}_3 = \mathrm{Ker}(\bigoplus A_{W_i} \to \mathcal{F}_3)$. Thus, arguing as above, after replacing each T_j by a finite family of elements of $\mathcal{B}$ (permissible by Lemma 30.2) we may assume there is a map $\bigoplus A_{T_j} \to \mathcal{K}_2$ lifting the given map $\bigoplus A_{T_j} \to \mathcal{K}_3$. Then $\bigoplus A_{V_j} \oplus \bigoplus A_{T_j} \to \mathcal{K}_2$ is surjective which finishes the proof. $\square$

093H **Lemma 30.11.** *In Situation 30.5 assume (1), (2), and (3) hold. Let $\mathcal{O}$ be a sheaf of rings. Let $\mathcal{A} \subset Mod(\mathcal{O})$ be the full subcategory of modules isomorphic to a cokernel as in (30.7.2). If the kernel of every map of $\mathcal{O}$-modules of the form*

$$\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}$$

with U_i and V_j in $\mathcal{B}$, is in $\mathcal{A}$, then $\mathcal{A}$ is weak Serre subcategory of $Mod(\mathcal{O})$.

Proof. We will use the criterion of Homology, Lemma 10.3. By the results of Lemmas 30.8 and 30.10 it suffices to see that the kernel of a map $\mathcal{F} \to \mathcal{G}$ between objects of $\mathcal{A}$ is in $\mathcal{A}$. To prove this choose presentations

$$\bigoplus A_{V_j} \to \bigoplus A_{U_i} \to \mathcal{F} \to 0 \quad \text{and} \quad \bigoplus A_{T_j} \to \bigoplus A_{W_i} \to \mathcal{G} \to 0$$

In this proof the direct sums are always finite, and we write $A_U = j_{U!}\mathcal{O}_U$ for $U \in \mathcal{B}$. Using Lemmas 30.1 and 30.9 and arguing as in the proof of Lemma 30.10 we may assume that the map $\mathcal{F} \to \mathcal{G}$ lifts to a map of presentations

$$\begin{array}{ccccccc} \bigoplus A_{V_j} & \longrightarrow & \bigoplus A_{U_i} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus A_{T_j} & \longrightarrow & \bigoplus A_{W_i} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

Then we see that

$$\mathrm{Ker}(\mathcal{F} \to \mathcal{G}) = \mathrm{Coker}\left(\bigoplus A_{V_j} \to \mathrm{Ker}\left(\bigoplus A_{T_j} \oplus \bigoplus A_{U_i} \to \bigoplus A_{W_i}\right)\right)$$

and the lemma follows from the assumption and Lemma 30.8. $\square$

31. Flat morphisms

04JA

04JB **Definition 31.1.** Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. We say $(f, f^\sharp)$ is *flat* if the ring map $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$ is flat. We say a morphism of ringed sites is *flat* if the associated morphism of ringed topoi is flat.

04JC **Lemma 31.2.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ be a morphism of ringed topoi. Then*

$$f^{-1} : Ab(\mathcal{C}') \longrightarrow Ab(\mathcal{C}), \quad \mathcal{F} \longmapsto f^{-1}\mathcal{F}$$

is exact. If $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ is a flat morphism of ringed topoi then

$$f^* : Mod(\mathcal{O}') \longrightarrow Mod(\mathcal{O}), \quad \mathcal{F} \longmapsto f^*\mathcal{F}$$

is exact.

Proof. Given an abelian sheaf $\mathcal{G}$ on $\mathcal{C}'$ the underlying sheaf of sets of $f^{-1}\mathcal{G}$ is the same as f^{-1} of the underlying sheaf of sets of $\mathcal{G}$, see Sites, Section 44. Hence the exactness of f^{-1} for sheaves of sets (required in the definition of a morphism of topoi, see Sites, Definition 15.1) implies the exactness of f^{-1} as a functor on abelian sheaves.

To see the statement on modules recall that $f^*\mathcal{F}$ is defined as the tensor product $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}', f^\sharp} \mathcal{O}$. Hence f^* is a composition of functors both of which are exact. $\square$

08M5 **Definition 31.3.** Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. We say that $\mathcal{F}$ is *flat over* $(Sh(\mathcal{D}), \mathcal{O}')$ if $\mathcal{F}$ is flat as an $f^{-1}\mathcal{O}'$-module.

This is compatible with the notion as defined for morphisms of ringed spaces, see Modules, Definition 20.3 and the discussion following.

0GN2 **Lemma 31.4.** *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_\mathcal{D}$-modules. If $\mathcal{F}$ is finitely presented and f is flat, then the canonical map*

$$f^* \mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G})$$

of Remark 27.3 is an isomorphism.

Proof. Say f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. We have to show that the restriction of the map to $\mathcal{C}/U$ for any $U \in \text{Ob}(\mathcal{C})$ is an isomorphism. We may replace U by the members of a covering of U. Hence by Sites, Lemma 14.10 we may assume there exists a morphism $U \to u(V)$ for some $V \in \text{Ob}(\mathcal{C})$. Of course, then we may replace U by $u(V)$. Then since u is continuous, we may replace V by a covering and assume there is a presentation $\mathcal{O}_V^{\oplus m} \to \mathcal{O}_V^{\oplus n} \to \mathcal{F}|_V \to 0$ over $\mathcal{D}/V$. Since formation of $\mathcal{H}om$ commutes with localization (Lemma 27.2) we may replace f by the morphism $(\mathcal{C}/u(V), \mathcal{O}_{u(V)}) \to (\mathcal{D}/V, \mathcal{O}_V)$ induced by f. Hence we reduce to the case where $\mathcal{F}$ has a global presentation $\mathcal{O}_\mathcal{D}^{\oplus m} \to \mathcal{O}_\mathcal{D}^{\oplus n} \to \mathcal{F} \to 0$. Since f is flat and $f^*\mathcal{O}_\mathcal{D} = \mathcal{O}_\mathcal{C}$ we obtain a corresponding presentation $\mathcal{O}_\mathcal{C}^{\oplus m} \to \mathcal{O}_\mathcal{C}^{\oplus n} \to f^*\mathcal{F} \to 0$, see Lemma 31.2. Using that $\mathcal{H}om$ commutes with finite direct sums in the first variable, using that both $\mathcal{H}om_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_\mathcal{C}, -)$ and $\mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{O}_\mathcal{D}, -)$ are the identity functor, and using the functoriality of the construction of Remark 27.3 we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & f^* \mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{F}, \mathcal{G}) \arrow[r] \arrow[d] & f^*\mathcal{G}^{\oplus n} \arrow[r] \arrow[d] & f^*\mathcal{G}^{\oplus n} \arrow[d] \\
0 \arrow[r] & \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G}) \arrow[r] & f^*\mathcal{G}^{\oplus n} \arrow[r] & f^*\mathcal{G}^{\oplus n}
\end{tikzcd}$$

where the right two vertical arrows are isomorphisms. By Lemma 27.5 the rows are exact. We conclude by the 5 lemma. $\square$

32. Invertible modules

0408 Here is the definition.

0409 **Definition 32.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site.

(1) A finite locally free $\mathcal{O}$-module $\mathcal{F}$ is said to have *rank r* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ of U such that $\mathcal{F}|_{U_i}$ is isomorphic to $\mathcal{O}_{U_i}^{\oplus r}$ as an $\mathcal{O}_{U_i}$-module.
(2) An $\mathcal{O}$-module $\mathcal{L}$ is *invertible* if the functor

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{F} \longmapsto \mathcal{F} \otimes_\mathcal{O} \mathcal{L}$$

is an equivalence.
(3) The sheaf $\mathcal{O}^*$ is the subsheaf of $\mathcal{O}$ defined by the rule

$$U \longmapsto \mathcal{O}^*(U) = \{f \in \mathcal{O}(U) \mid \exists g \in \mathcal{O}(U) \text{ such that } fg = 1\}$$

It is a sheaf of abelian groups with multiplication as the group law.

Lemma 40.7 below explains the relationship with locally free modules of rank 1.

0B8N **Lemma 32.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{L}$ be an $\mathcal{O}$-module. The following are equivalent:*

(1) *$\mathcal{L}$ is invertible, and*
(2) *there exists an $\mathcal{O}$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_\mathcal{O} \mathcal{N} \cong \mathcal{O}$.*

In this case we have

(a) *$\mathcal{L}$ is a flat $\mathcal{O}$-module of finite presentation,*
(b) *for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{L}|_{U_i}$ is a direct summand of a finite free module, and*
(c) *the module $\mathcal{N}$ in (2) is isomorphic to $\mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O})$.*

Proof. Assume (1). Then the functor $- \otimes_\mathcal{O} \mathcal{L}$ is essentially surjective, hence there exists an $\mathcal{O}$-module $\mathcal{N}$ as in (2). If (2) holds, then the functor $- \otimes_\mathcal{O} \mathcal{N}$ is a quasi-inverse to the functor $- \otimes_\mathcal{O} \mathcal{L}$ and we see that (1) holds.

Assume (1) and (2) hold. Since $- \otimes_\mathcal{O} \mathcal{L}$ is an equivalence, it is exact, and hence $\mathcal{L}$ is flat. Denote $\psi : \mathcal{L} \otimes_\mathcal{O} \mathcal{N} \to \mathcal{O}$ the given isomorphism. Let U be an object of $\mathcal{C}$. We will show that the restriction $\mathcal{L}$ to the members of a covering of U is a direct summand of a free module, which will certainly imply that $\mathcal{L}$ is of finite presentation. By construction of $\otimes$ we may assume (after replacing U by the members of a covering) that there exists an integer $n \geq 1$ and sections $x_i \in \mathcal{L}(U)$, $y_i \in \mathcal{N}(U)$ such that $\psi(\sum x_i \otimes y_i) = 1$. Consider the isomorphisms

$$\mathcal{L}|_U \to \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{N}|_U \to \mathcal{L}|_U$$

where the first arrow sends x to $\sum x_i \otimes x \otimes y_i$ and the second arrow sends $x \otimes x' \otimes y$ to $\psi(x' \otimes y)x$. We conclude that $x \mapsto \sum \psi(x \otimes y_i)x_i$ is an automorphism of $\mathcal{L}|_U$. This automorphism factors as

$$\mathcal{L}|_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{L}|_U$$

where the first arrow is given by $x \mapsto (\psi(x \otimes y_1), \ldots, \psi(x \otimes y_n))$ and the second arrow by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$. In this way we conclude that $\mathcal{L}|_U$ is a direct summand of a finite free $\mathcal{O}_U$-module.

Assume (1) and (2) hold. Consider the evaluation map

$$\mathcal{L} \otimes_\mathcal{O} \mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

To finish the proof of the lemma we will show this is an isomorphism. By Lemma 27.6 we have

$$\mathrm{Hom}_\mathcal{O}(\mathcal{O}, \mathcal{O}) = \mathrm{Hom}_\mathcal{O}(\mathcal{N} \otimes_\mathcal{O} \mathcal{L}, \mathcal{O}) \longrightarrow \mathrm{Hom}_\mathcal{O}(\mathcal{N}, \mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O}))$$

The image of 1 gives a morphism $\mathcal{N} \to \mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O})$. Tensoring with $\mathcal{L}$ we obtain

$$\mathcal{O} = \mathcal{L} \otimes_\mathcal{O} \mathcal{N} \longrightarrow \mathcal{L} \otimes_\mathcal{O} \mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O})$$

This map is the inverse to the evaluation map; computation omitted. $\square$

0B8P **Lemma 32.3.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. The pullback $f^*\mathcal{L}$ of an invertible $\mathcal{O}_\mathcal{D}$-module is invertible.*

Proof. By Lemma 32.2 there exists an $\mathcal{O}_\mathcal{D}$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{N} \cong \mathcal{O}_\mathcal{D}$. Pulling back we get $f^*\mathcal{L} \otimes_{\mathcal{O}_\mathcal{C}} f^*\mathcal{N} \cong \mathcal{O}_\mathcal{C}$ by Lemma 26.2. Thus $f^*\mathcal{L}$ is invertible by Lemma 32.2. $\square$

040A **Lemma 32.4.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed space.*

1. *If $\mathcal{L}$, $\mathcal{N}$ are invertible $\mathcal{O}$-modules, then so is $\mathcal{L} \otimes_\mathcal{O} \mathcal{N}$.*
2. *If $\mathcal{L}$ is an invertible $\mathcal{O}$-module, then so is $\mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O})$ and the evaluation map $\mathcal{L} \otimes_\mathcal{O} \mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O}) \to \mathcal{O}$ is an isomorphism.*

Proof. Part (1) is clear from the definition and part (2) follows from Lemma 32.2 and its proof. $\square$

040B **Lemma 32.5.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed space. There exists a set of invertible modules $\{\mathcal{L}_i\}_{i \in I}$ such that each invertible module on $(\mathcal{C}, \mathcal{O})$ is isomorphic to exactly one of the $\mathcal{L}_i$.*

Proof. Omitted, but see Sheaves of Modules, Lemma 25.8. $\square$

Lemma 32.5 says that the collection of isomorphism classes of invertible sheaves forms a set. Lemma 32.4 says that tensor product defines the structure of an abelian group on this set with inverse of $\mathcal{L}$ given by $\mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O})$.

In fact, given an invertible $\mathcal{O}$-module $\mathcal{L}$ and $n \in \mathbf{Z}$ we define the nth *tensor power* $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$ as the image of $\mathcal{O}$ under applying the equivalence $\mathcal{F} \mapsto \mathcal{F} \otimes_\mathcal{O} \mathcal{L}$ exactly n times. This makes sense also for negative n as we've defined an invertible $\mathcal{O}$-module as one for which tensoring is an equivalence. More explicitly, we have

$$\mathcal{L}^{\otimes n} = \left\{ \begin{matrix} \mathcal{O} & \text{if} & n = 0 \\ \mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O}) & \text{if} & n = -1 \\ \mathcal{L} \otimes_\mathcal{O} \ldots \otimes_\mathcal{O} \mathcal{L} & \text{if} & n > 0 \\ \mathcal{L}^{\otimes -1} \otimes_\mathcal{O} \ldots \otimes_\mathcal{O} \mathcal{L}^{\otimes -1} & \text{if} & n < -1 \end{matrix} \right.$$

see Lemma 32.4. With this definition we have canonical isomorphisms $\mathcal{L}^{\otimes n} \otimes_\mathcal{O} \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes n+m}$, and these isomorphisms satisfy a commutativity and an associativity constraint (formulation omitted).

040C **Definition 32.6.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The *Picard group* $\text{Pic}(\mathcal{O})$ of the ringed site is the abelian group whose elements are isomorphism classes of invertible $\mathcal{O}$-modules, with addition corresponding to tensor product.

33. Modules of differentials

04BJ In this section we briefly explain how to define the module of relative differentials for a morphism of ringed topoi. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 131).

04BK **Definition 33.1.** Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be an $\mathcal{O}_2$-module. A $\mathcal{O}_1$*-derivation* or more precisely a φ*-derivation* into $\mathcal{F}$ is a map $D : \mathcal{O}_2 \to \mathcal{F}$ which is additive, annihilates the image of $\mathcal{O}_1 \to \mathcal{O}_2$, and satisfies the *Leibniz rule*

$$D(ab) = aD(b) + D(a)b$$

for all a, b local sections of $\mathcal{O}_2$ (wherever they are both defined). We denote $\text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ the set of φ-derivations into $\mathcal{F}$.

This is the sheaf theoretic analogue of Algebra, Definition 33.1. Given a derivation $D : \mathcal{O}_2 \to \mathcal{F}$ as in the definition the map on global sections

$$D : \Gamma(\mathcal{O}_2) \longrightarrow \Gamma(\mathcal{F})$$

clearly is a $\Gamma(\mathcal{O}_1)$-derivation as in the algebra definition. Note that if $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_2$-modules, then there is an induced map

$$\text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) \longrightarrow \text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G})$$

given by the rule $D \mapsto \alpha \circ D$. In other words we obtain a functor.

04BL **Lemma 33.2.** *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The functor*

$$Mod(\mathcal{O}_2) \longrightarrow Ab, \quad \mathcal{F} \longmapsto Der_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$$

is representable.

Proof. This is proved in exactly the same way as the analogous statement in algebra. During this proof, for any sheaf of sets $\mathcal{F}$ on $\mathcal{C}$, let us denote $\mathcal{O}_2[\mathcal{F}]$ the sheafification of the presheaf $U \mapsto \mathcal{O}_2(U)[\mathcal{F}(U)]$ where this denotes the free $\mathcal{O}_2(U)$-module on the set $\mathcal{F}(U)$. For $s \in \mathcal{F}(U)$ we denote $[s]$ the corresponding section of $\mathcal{O}_2[\mathcal{F}]$ over U. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules, then there is a canonical map

$$c : \mathcal{O}_2[\mathcal{F}] \longrightarrow \mathcal{F}$$

which on the presheaf level is given by the rule $\sum f_s[s] \mapsto \sum f_s s$. We will employ the short hand $[s] \mapsto s$ to describe this map and similarly for other maps below. Consider the map of $\mathcal{O}_2$-modules

04BM (33.2.1)
$$\begin{matrix} \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_1] & \longrightarrow & \mathcal{O}_2[\mathcal{O}_2] \\ [(a,b)] \oplus [(f,g)] \oplus [h] & \longmapsto & [a+b]-[a]-[b]+ \\ & & [fg]-g[f]-f[g]+ \\ & & [\varphi(h)] \end{matrix}$$

with short hand notation as above. Set $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ equal to the cokernel of this map. Then it is clear that there exists a map of sheaves of sets

$$\mathrm{d} : \mathcal{O}_2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

mapping a local section f to the image of $[f]$ in $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$. By construction d is a $\mathcal{O}_1$-derivation. Next, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules and let $D : \mathcal{O}_2 \to \mathcal{F}$ be a $\mathcal{O}_1$-derivation. Then we can consider the $\mathcal{O}_2$-linear map $\mathcal{O}_2[\mathcal{O}_2] \to \mathcal{F}$ which sends $[g]$ to $D(g)$. It follows from the definition of a derivation that this map annihilates sections in the image of the map (33.2.1) and hence defines a map

$$\alpha_D : \Omega_{\mathcal{O}_2/\mathcal{O}_1} \longrightarrow \mathcal{F}$$

Since it is clear that $D = \alpha_D \circ \mathrm{d}$ the lemma is proved. □

04BN **Definition 33.3.** Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The *module of differentials* of the ring map φ is the object representing the functor $\mathcal{F} \mapsto \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ which exists by Lemma 33.2. It is denoted $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$, and the *universal φ-derivation* is denoted $\mathrm{d} : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$.

Since this module and the derivation form the universal object representing a functor, this notion is clearly intrinsic (i.e., does not depend on the choice of the site underlying the ringed topos, see Section 18). Note that $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (33.2.1) of $\mathcal{O}_2$-modules. Moreover the map d is described by the rule that $\mathrm{d}f$ is the image of the local section $[f]$.

08TP **Lemma 33.4.** *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of presheaves of rings. Then $\Omega_{\mathcal{O}_2^\#/\mathcal{O}_1^\#}$ is the sheaf associated to the presheaf $U \mapsto \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$.*

Proof. Consider the map (33.2.1). There is a similar map of presheaves whose value on $U \in \mathrm{Ob}(\mathcal{C})$ is

$$\mathcal{O}_2(U)[\mathcal{O}_2(U) \times \mathcal{O}_2(U)] \oplus \mathcal{O}_2(U)[\mathcal{O}_2(U) \times \mathcal{O}_2(U)] \oplus \mathcal{O}_2(U)[\mathcal{O}_1(U)] \longrightarrow \mathcal{O}_2(U)[\mathcal{O}_2(U)]$$

The cokernel of this map has value $\Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$ over U by the construction of the module of differentials in Algebra, Definition 131.2. On the other hand, the sheaves in (33.2.1) are the sheafifications of the presheaves above. Thus the result follows as sheafification is exact. □

08TQ **Lemma 33.5.** *Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then there is a canonical identification $f^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1} = \Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$ compatible with universal derivations.*

Proof. This holds because the sheaf $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (33.2.1) and a similar statement holds for $\Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$, because the functor f^{-1} is exact, and because $f^{-1}(\mathcal{O}_2[\mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2]$, $f^{-1}(\mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2 \times f^{-1}\mathcal{O}_2]$, and $f^{-1}(\mathcal{O}_2[\mathcal{O}_1]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_1]$. □

04BO **Lemma 33.6.** *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. For any object U of $\mathcal{C}$ there is a canonical isomorphism*

$$\Omega_{\mathcal{O}_2/\mathcal{O}_1}|_U = \Omega_{(\mathcal{O}_2|_U)/(\mathcal{O}_1|_U)}$$

compatible with universal derivations.

Proof. This is a special case of Lemma 33.5. □

08TR **Lemma 33.7.** *Let $\mathcal{C}$ be a site. Let*

$$\begin{tikzcd} \mathcal{O}_2 \arrow[r, "\varphi"'] & \mathcal{O}'_2 \\ \mathcal{O}_1 \arrow[u] \arrow[r] & \mathcal{O}'_1 \arrow[u] \end{tikzcd}$$

be a commutative diagram of sheaves of rings on $\mathcal{C}$. The map $\mathcal{O}_2 \to \mathcal{O}'_2$ composed with the map $d : \mathcal{O}'_2 \to \Omega_{\mathcal{O}'_2/\mathcal{O}'_1}$ is a $\mathcal{O}_1$-derivation. Hence we obtain a canonical map of $\mathcal{O}_2$-modules $\Omega_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega_{\mathcal{O}'_2/\mathcal{O}'_1}$. It is uniquely characterized by the property that $d(f)$ mapsto $d(\varphi(f))$ for any local section f of $\mathcal{O}_2$. In this way $\Omega_{-/-}$ becomes a functor on the category of arrows of sheaves of rings.

Proof. This lemma proves itself. □

08TS **Lemma 33.8.** *In Lemma 33.7 suppose that $\mathcal{O}_2 \to \mathcal{O}'_2$ is surjective with kernel $\mathcal{I} \subset \mathcal{O}_2$ and assume that $\mathcal{O}_1 = \mathcal{O}'_1$. Then there is a canonical exact sequence of $\mathcal{O}'_2$-modules*

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}'_2 \longrightarrow \Omega_{\mathcal{O}'_2/\mathcal{O}_1} \longrightarrow 0$$

The leftmost map is characterized by the rule that a local section f of $\mathcal{I}$ maps to $df \otimes 1$.

Proof. For a local section f of $\mathcal{I}$ denote $\overline{f}$ the image of f in $\mathcal{I}/\mathcal{I}^2$. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1f_2 \otimes 1 = 0$ if f_1, f_2 are local sections of $\mathcal{I}$. And this is clear from the Leibniz rule $\mathrm{d}f_1f_2 \otimes 1 = (f_1\mathrm{d}f_2 + f_2\mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_2 \otimes f_1 = 0$. A similar computation show this map is $\mathcal{O}'_2 = \mathcal{O}_2/\mathcal{I}$-linear. The map on the right is the one from Lemma 33.7.

To see that the sequence is exact, we argue as follows. Let $\mathcal{O}''_2 \subset \mathcal{O}'_2$ be the presheaf of $\mathcal{O}_1$-algebras whose value on U is the image of $\mathcal{O}_2(U) \to \mathcal{O}'_2(U)$. By Algebra, Lemma 131.9 the sequences

$$\mathcal{I}(U)/\mathcal{I}(U)^2 \longrightarrow \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)} \otimes_{\mathcal{O}_2(U)} \mathcal{O}''_2(U) \longrightarrow \Omega_{\mathcal{O}''_2(U)/\mathcal{O}_1(U)} \longrightarrow 0$$

are exact for all objects U of $\mathcal{C}$. Since sheafification is exact this gives an exact sequence of sheaves of $(\mathcal{O}''_2)^\#$-modules. By Lemma 33.4 and the fact that $(\mathcal{O}''_2)^\# = \mathcal{O}'_2$ we conclude. □

Here is a particular situation where derivations come up naturally.

04BP **Lemma 33.9.** *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Consider a short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{A} \to \mathcal{O}_2 \to 0$$

Here $\mathcal{A}$ is a sheaf of $\mathcal{O}_1$-algebras, $\pi : \mathcal{A} \to \mathcal{O}_2$ is a surjection of sheaves of $\mathcal{O}_1$-algebras, and $\mathcal{F} = \mathrm{Ker}(\pi)$ is its kernel. Assume $\mathcal{F}$ an ideal sheaf with square zero in $\mathcal{A}$. So $\mathcal{F}$ has a natural structure of an $\mathcal{O}_2$-module. A section $s : \mathcal{O}_2 \to \mathcal{A}$ of π is a $\mathcal{O}_1$-algebra map such that $\pi \circ s = id$. Given any section $s : \mathcal{O}_2 \to \mathcal{F}$ of π and any φ-derivation $D : \mathcal{O}_1 \to \mathcal{F}$ the map

$$s + D : \mathcal{O}_1 \to \mathcal{A}$$

is a section of π and every section s' is of the form $s + D$ for a unique φ-derivation D.

Proof. Recall that the $\mathcal{O}_2$-module structure on $\mathcal{F}$ is given by $h\tau = \tilde{h}\tau$ (multiplication in $\mathcal{A}$) where h is a local section of $\mathcal{O}_2$, and $\tilde{h}$ is a local lift of h to a local section of $\mathcal{A}$, and τ is a local section of $\mathcal{F}$. In particular, given s, we may use $\tilde{h} = s(h)$. To verify that $s + D$ is a homomorphism of sheaves of rings we compute

$$\begin{aligned} (s+D)(ab) &= s(ab) + D(ab) \\ &= s(a)s(b) + aD(b) + D(a)b \\ &= s(a)s(b) + s(a)D(b) + D(a)s(b) \\ &= (s(a) + D(a))(s(b) + D(b)) \end{aligned}$$

by the Leibniz rule. In the same manner one shows $s + D$ is a $\mathcal{O}_1$-algebra map because D is an $\mathcal{O}_1$-derivation. Conversely, given s' we set $D = s' - s$. Details omitted. ☐

04BQ **Definition 33.10.** Let $X = (Sh(\mathcal{C}), \mathcal{O})$ and $Y = (Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. Let $(f, f^\sharp) : X \to Y$ be a morphism of ringed topoi. In this situation

(1) for a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules a *Y-derivation* $D : \mathcal{O} \to \mathcal{F}$ is just a $f^\sharp$-derivation, and
(2) the *sheaf of differentials* $\Omega_{X/Y}$ *of X over Y* is the module of differentials of $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, see Definition 33.3.

Thus $\Omega_{X/Y}$ comes equipped with a *universal Y-derivation* $\mathrm{d}_{X/Y} : \mathcal{O} \longrightarrow \Omega_{X/Y}$. We sometimes write $\Omega_{X/Y} = \Omega_f$.

Recall that $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$ so that this definition makes sense.

04BR **Lemma 33.11.** *Let $X = (Sh(\mathcal{C}_X), \mathcal{O}_X)$, $Y = (Sh(\mathcal{C}_Y), \mathcal{O}_Y)$, $X' = (Sh(\mathcal{C}_{X'}), \mathcal{O}_{X'})$, and $Y' = (Sh(\mathcal{C}_{Y'}), \mathcal{O}_{Y'})$ be ringed topoi. Let*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram of morphisms of ringed topoi. The map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*\mathrm{d}_{X'/Y'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/Y'}$ is a Y-derivation.

Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/Y} \to f_*\Omega_{X'/Y'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}.$$

It is uniquely characterized by the property that $f^*d_{X/Y}(t)$ mapsto $d_{X'/Y'}(f^*t)$ for any local section t of $\mathcal{O}_X$.

Proof. This is clear except for the last assertion. Let us explain the meaning of this. Let $U \in \mathrm{Ob}(\mathcal{C}_X)$ and let $t \in \mathcal{O}_X(U)$. This is what it means for t to be a local section of $\mathcal{O}_X$. Now, we may think of t as a map of sheaves of sets $t : h_U^\# \to \mathcal{O}_X$. Then $f^{-1}t : f^{-1}h_U^\# \to f^{-1}\mathcal{O}_X$. By f^*t we mean the composition

$$\begin{tikzcd} f^{-1}h_U^\# \arrow[r, "f^{-1}t"] \arrow[rr, bend left, "f^*t"] & f^{-1}\mathcal{O}_X \arrow[r, "f^\sharp"] & \mathcal{O}_{X'} \end{tikzcd}$$

Note that $\mathrm{d}_{X/Y}(t) \in \Omega_{X/Y}(U)$. Hence we may think of $\mathrm{d}_{X/Y}(t)$ as a map $\mathrm{d}_{X/Y}(t) : h_U^\# \to \Omega_{X/Y}$. Then $f^{-1}\mathrm{d}_{X/Y}(t) : f^{-1}h_U^\# \to f^{-1}\Omega_{X/Y}$. By $f^*\mathrm{d}_{X/Y}(t)$ we mean the composition

$$\begin{tikzcd} f^{-1}h_U^\# \arrow[r, "f^{-1}\mathrm{d}_{X/Y}(t)"] \arrow[rr, bend left, "f^*\mathrm{d}_{X/Y}(t)"] & f^{-1}\Omega_{X/Y} \arrow[r, "1\otimes\mathrm{id}"] & f^*\Omega_{X/Y} \end{tikzcd}$$

OK, and now the statement of the lemma means that we have

$$c_f \circ f^*t = f^*\mathrm{d}_{X/Y}(t)$$

as maps from $f^{-1}h_U^\#$ to $\Omega_{X'/Y'}$. We omit the verification that this property holds for c_f as defined in the lemma. (Hint: The first map $c'_f : \Omega_{X/Y} \to f_*\Omega_{X'/Y'}$ satisfies $c'_f(\mathrm{d}_{X/Y}(t)) = f_*\mathrm{d}_{X'/Y'}(f^\sharp(t))$ as sections of $f_*\Omega_{X'/Y'}$ over U, and you have to turn this into the equality above by using adjunction.) The reason that this uniquely characterizes c_f is that the images of $f^*\mathrm{d}_{X/Y}(t)$ generate the $\mathcal{O}_{X'}$-module $f^*\Omega_{X/Y}$ simply because the local sections $\mathrm{d}_{X/Y}(t)$ generate the $\mathcal{O}_X$-module $\Omega_{X/Y}$. $\square$

34. Finite order differential operators

09CQ In this section we introduce differential operators of finite order. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 133).

09CR **Definition 34.1.** Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $k \geq 0$ be an integer. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. A *differential operator $D : \mathcal{F} \to \mathcal{G}$ of order k* is an is an $\mathcal{O}_1$-linear map such that for all local sections g of $\mathcal{O}_2$ the map $s \mapsto D(gs) - gD(s)$ is a differential operator of order $k-1$. For the base case $k = 0$ we define a differential operator of order 0 to be an $\mathcal{O}_2$-linear map.

If $D : \mathcal{F} \to \mathcal{G}$ is a differential operator of order k, then for all local sections g of $\mathcal{O}_2$ the map gD is a differential operator of order k. The sum of two differential operators of order k is another. Hence the set of all these

$$\mathrm{Diff}^k(\mathcal{F}, \mathcal{G}) = \mathrm{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G})$$

is a $\Gamma(\mathcal{C}, \mathcal{O}_2)$-module. We have

$$\text{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \text{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \text{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \ldots$$

The rule which maps $U \in \text{Ob}(\mathcal{C})$ to the module of differential operators $D : \mathcal{F}|_U \to \mathcal{G}|_U$ of order k is a sheaf of $\mathcal{O}_2$-modules on the site $\mathcal{C}$. Thus we obtain a sheaf of differential operators (if we ever need this we will add a definition here).

09CS **Lemma 34.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings. Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. If $D : \mathcal{E} \to \mathcal{F}$ and $D' : \mathcal{F} \to \mathcal{G}$ are differential operators of order k and k', then $D' \circ D$ is a differential operator of order $k + k'$.*

Proof. Let g be a local section of $\mathcal{O}_2$. Then the map which sends a local section x of $\mathcal{E}$ to

$$D'(D(gx)) - gD'(D(x)) = D'(D(gx)) - D'(gD(x)) + D'(gD(x)) - gD'(D(x))$$

is a sum of two compositions of differential operators of lower order. Hence the lemma follows by induction on $k + k'$. $\square$

09CT **Lemma 34.3.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. Let $k \geq 0$. There exists a sheaf of $\mathcal{O}_2$-modules $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ and a canonical isomorphism*

$$\textit{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathcal{O}_2}(\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}), \mathcal{G})$$

functorial in the $\mathcal{O}_2$-module $\mathcal{G}$.

Proof. The existence follows from general category theoretic arguments (insert future reference here), but we will also give a direct construction as this construction will be useful in the future proofs. We will freely use the notation introduced in the proof of Lemma 33.2. Given any differential operator $D : \mathcal{F} \to \mathcal{G}$ we obtain an $\mathcal{O}_2$-linear map $L_D : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ sending $[m]$ to $D(m)$. If D has order 0 then L_D annihilates the local sections

$$[m + m'] - [m] - [m'], \quad g_0[m] - [g_0 m]$$

where g_0 is a local section of $\mathcal{O}_2$ and m, m' are local sections of $\mathcal{F}$. If D has order 1, then L_D annihilates the local sections

$$[m + m'] - [m] - [m'], \quad f[m] - [fm], \quad g_0 g_1[m] - g_0[g_1 m] - g_1[g_0 m] + [g_1 g_0 m]$$

where f is a local section of $\mathcal{O}_1$, g_0, g_1 are local sections of $\mathcal{O}_2$, and m, m' are local sections of $\mathcal{F}$. If D has order k, then L_D annihilates the local sections $[m + m'] - [m] - [m']$, $f[m] - [fm]$, and the local sections

$$g_0 g_1 \ldots g_k[m] - \sum g_0 \ldots \hat{g}_i \ldots g_k[g_i m] + \ldots + (-1)^{k+1}[g_0 \ldots g_k m]$$

Conversely, if $L : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ is an $\mathcal{O}_2$-linear map annihilating all the local sections listed in the previous sentence, then $m \mapsto L([m])$ is a differential operator of order k. Thus we see that $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ is the quotient of $\mathcal{O}_2[\mathcal{F}]$ by the $\mathcal{O}_2$-submodule generated by these local sections. $\square$

09CU **Definition 34.4.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. The module $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ constructed in Lemma 34.3 is called the *module of principal parts of order k* of $\mathcal{F}$.

Note that the inclusions

$$\mathrm{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \ldots$$

correspond via Yoneda's lemma (Categories, Lemma 3.5) to surjections

$$\ldots \to \mathcal{P}^2_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^0_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) = \mathcal{F}$$

09CV **Lemma 34.5.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of presheaves of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_2$-modules. Then $\mathcal{P}^k_{\mathcal{O}_2^\#/\mathcal{O}_1^\#}(\mathcal{F}^\#)$ is the sheaf associated to the presheaf $U \mapsto P^k_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}(\mathcal{F}(U))$.*

Proof. This can be proved in exactly the same way as is done for the sheaf of differentials in Lemma 33.4. Perhaps a more pleasing approach is to use the universal property of Lemma 34.3 directly to see the equality. We omit the details. $\square$

09CW **Lemma 34.6.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. There is a canonical short exact sequence*

$$0 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{F} \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{F} \to 0$$

functorial in $\mathcal{F}$ called the sequence of principal parts.

Proof. Follows from the commutative algebra version (Algebra, Lemma 133.6) and Lemmas 33.4 and 34.5. $\square$

09CX **Remark 34.7.** Let $\mathcal{C}$ be a site. Suppose given a commutative diagram of sheaves of rings

$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}' \arrow[u] \end{tikzcd}$$

a $\mathcal{B}$-module $\mathcal{F}$, a $\mathcal{B}'$-module $\mathcal{F}'$, and a $\mathcal{B}$-linear map $\mathcal{F} \to \mathcal{F}'$. Then we get a compatible system of module maps

$$\begin{tikzcd} \ldots \arrow[r] & \mathcal{P}^2_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \arrow[r] & \mathcal{P}^1_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \arrow[r] & \mathcal{P}^0_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \\ \ldots \arrow[r] & \mathcal{P}^2_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \arrow[u] \arrow[r] & \mathcal{P}^1_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \arrow[u] \arrow[r] & \mathcal{P}^0_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \arrow[u] \end{tikzcd}$$

These maps are compatible with further composition of maps of this type. The easiest way to see this is to use the description of the modules $\mathcal{P}^k_{\mathcal{B}/\mathcal{A}}(\mathcal{M})$ in terms of (local) generators and relations in the proof of Lemma 34.3 but it can also be seen directly from the universal property of these modules. Moreover, these maps are compatible with the short exact sequences of Lemma 34.6.

35. The naive cotangent complex

08TT This section is the analogue of Algebra, Section 134 and Modules, Section 31. We advise the reader to read those sections first.

Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. In this section, for any sheaf of sets $\mathcal{E}$ on $\mathcal{C}$ we denote $\mathcal{A}[\mathcal{E}]$ the sheafification of the presheaf $U \mapsto \mathcal{A}(U)[\mathcal{E}(U)]$. Here $\mathcal{A}(U)[\mathcal{E}(U)]$ denotes the polynomial algebra over $\mathcal{A}(U)$ whose variables correspond to the elements of $\mathcal{E}(U)$. We denote $[e] \in \mathcal{A}(U)[\mathcal{E}(U)]$

the variable corresponding to $e \in \mathcal{E}(U)$. There is a canonical surjection of $\mathcal{A}$-algebras

08TU (35.0.1)
$$\mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{B}, \quad [b] \longmapsto b$$

whose kernel we denote $\mathcal{I} \subset \mathcal{A}[\mathcal{B}]$. It is a simple observation that $\mathcal{I}$ is generated by the local sections $[b][b'] - [bb']$ and $[a] - a$. According to Lemma 33.8 there is a canonical map

08TV (35.0.2)
$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$$

whose cokernel is canonically isomorphic to $\Omega_{\mathcal{B}/\mathcal{A}}$.

08TW **Definition 35.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. The *naive cotangent complex* $NL_{\mathcal{B}/\mathcal{A}}$ is the chain complex (35.0.2)

$$NL_{\mathcal{B}/\mathcal{A}} = \left(\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}\right)$$

with $\mathcal{I}/\mathcal{I}^2$ placed in degree -1 and $\Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$ placed in degree 0.

This construction satisfies a functoriality similar to that discussed in Lemma 33.7 for modules of differentials. Namely, given a commutative diagram

08TX (35.1.1)
$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}' \arrow[u] \end{tikzcd}$$

of sheaves of rings on $\mathcal{C}$ there is a canonical $\mathcal{B}$-linear map of complexes

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'}$$

Namely, the maps in the commutative diagram give rise to a canonical map $\mathcal{A}[\mathcal{B}] \to \mathcal{A}'[\mathcal{B}']$ which maps $\mathcal{I}$ into $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}'[\mathcal{B}'] \to \mathcal{B}')$. Thus a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{I}'/(\mathcal{I}')^2$ and a map between modules of differentials, which together give the desired map between the naive cotangent complexes.

We can choose a different presentation of $\mathcal{B}$ as a quotient of a polynomial algebra over $\mathcal{A}$ and still obtain the same object of $D(\mathcal{B})$. To explain this, suppose that $\mathcal{E}$ is a sheaves of sets on $\mathcal{C}$ and $\alpha : \mathcal{E} \to \mathcal{B}$ a map of sheaves of sets. Then we obtain an $\mathcal{A}$-algebra homomorphism $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$. Assume this map is surjective, and let $\mathcal{J} \subset \mathcal{A}[\mathcal{E}]$ be the kernel. Set

$$NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}\right)$$

Here is the result.

08TY **Lemma 35.2.** *In the situation above there is a canonical isomorphism $NL(\alpha) = NL_{\mathcal{B}/\mathcal{A}}$ in $D(\mathcal{B})$.*

Proof. Observe that $NL_{\mathcal{B}/\mathcal{A}} = NL(\mathrm{id}_{\mathcal{B}})$. Thus it suffices to show that given two maps $\alpha_i : \mathcal{E}_i \to \mathcal{B}$ as above, there is a canonical quasi-isomorphism $NL(\alpha_1) = NL(\alpha_2)$ in $D(\mathcal{B})$. To see this set $\mathcal{E} = \mathcal{E}_1 \amalg \mathcal{E}_2$ and $\alpha = \alpha_1 \amalg \alpha_2 : \mathcal{E} \to \mathcal{B}$. Set $\mathcal{J}_i = \mathrm{Ker}(\mathcal{A}[\mathcal{E}_i] \to \mathcal{B})$ and $\mathcal{J} = \mathrm{Ker}(\mathcal{A}[\mathcal{E}] \to \mathcal{B})$. We obtain maps $\mathcal{A}[\mathcal{E}_i] \to \mathcal{A}[\mathcal{E}]$ which send $\mathcal{J}_i$ into $\mathcal{J}$. Thus we obtain canonical maps of complexes

$$NL(\alpha_i) \longrightarrow NL(\alpha)$$

and it suffices to show these maps are quasi-isomorphism. To see this we argue as follows. First, observe that $H^0(NL(\alpha_i)) = \Omega_{\mathcal{B}/\mathcal{A}}$ and $H^0(NL(\alpha)) = \Omega_{\mathcal{B}/\mathcal{A}}$ by

Lemma 33.8 hence the map is an isomorphism on cohomology sheaves in degree 0. Similarly, we claim that $H^{-1}(NL(\alpha_i))$ and $H^{-1}(NL(\alpha))$ are the sheaves associated to the presheaf $U \mapsto H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$ where $H_1(L_{-/-})$ is as in Algebra, Definition 134.1. If the claim holds, then the proof is finished.

Proof of the claim. Let $\alpha : \mathcal{E} \to \mathcal{B}$ be as above. Let $\mathcal{B}' \subset \mathcal{B}$ be the subpresheaf of $\mathcal{A}$-algebras whose value on U is the image of $\mathcal{A}(U)[\mathcal{E}(U)] \to \mathcal{B}(U)$. Let $\mathcal{I}'$ be the presheaf whose value on U is the kernel of $\mathcal{A}(U)[\mathcal{E}(U)] \to \mathcal{B}(U)$. Then $\mathcal{I}$ is the sheafification of $\mathcal{I}'$ and $\mathcal{B}$ is the sheafification of $\mathcal{B}'$. Similarly, $H^{-1}(NL(\alpha))$ is the sheafification of the presheaf

$$U \longmapsto \mathrm{Ker}(\mathcal{I}'(U)/\mathcal{I}'(U)^2 \to \Omega_{\mathcal{A}(U)[\mathcal{E}(U)]/\mathcal{A}(U)} \otimes_{\mathcal{A}(U)[\mathcal{E}(U)]} \mathcal{B}'(U))$$

by Lemma 33.4. By Algebra, Lemma 134.2 we conclude $H^{-1}(NL(\alpha))$ is the sheaf associated to the presheaf $U \mapsto H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)})$. Thus we have to show that the maps $H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)}) \to H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$ induce an isomorphism $\mathcal{H}'_1 \to \mathcal{H}_1$ of sheafifications.

Injectivity of $\mathcal{H}'_1 \to \mathcal{H}_1$. Let $f \in H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)})$ map to zero in $\mathcal{H}_1(U)$. To show: f maps to zero in $\mathcal{H}'_1(U)$. The assumption means there is a covering $\{U_i \to U\}$ such that f maps to zero in $H_1(L_{\mathcal{B}(U_i)/\mathcal{A}(U_i)})$ for all i. Replace U by U_i to get to the point where f maps to zero in $H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$. By Algebra, Lemma 134.9 we can find a finitely generated subalgebra $\mathcal{B}'(U) \subset B \subset \mathcal{B}(U)$ such that f maps to zero in $H_1(L_{B/\mathcal{A}(U)})$. Since $\mathcal{B} = (\mathcal{B}')^\#$ we can find a covering $\{U_i \to U\}$ such that $B \to \mathcal{B}(U_i)$ factors through $\mathcal{B}'(U_i)$. Hence f maps to zero in $H_1(L_{\mathcal{B}'(U_i)/\mathcal{A}(U_i)})$ as desired.

The surjectivity of $\mathcal{H}'_1 \to \mathcal{H}_1$ is proved in exactly the same way. □

08TZ **Lemma 35.3.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be morphism of topoi. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{D}$. Then $f^{-1} NL_{\mathcal{B}/\mathcal{A}} = NL_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.*

Proof. Omitted. Hint: Use Lemma 33.5. □

The cotangent complex of a morphism of ringed topoi is defined in terms of the cotangent complex we defined above.

08U0 **Definition 35.4.** Let $X = (Sh(\mathcal{C}), \mathcal{O})$ and $Y = (Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. Let $(f, f^\sharp) : X \to Y$ be a morphism of ringed topoi. The *naive cotangent complex* $NL_f = NL_{X/Y}$ of the given morphism of ringed topoi is $NL_{\mathcal{O}/f^{-1}\mathcal{O}'}$. We sometimes write $NL_{X/Y} = NL_{\mathcal{O}/\mathcal{O}'}$.

36. Stalks of modules

04EM We have to be a bit careful when taking stalks at points, since the colimit defining a stalk (see Sites, Equation 32.1.1) may not be filtered[3]. On the other hand, by definition of a point of a site the stalk functor is exact and commutes with arbitrary colimits. In other words, it behaves exactly as if the colimit were filtered.

04EN **Lemma 36.1.** *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$.*

(1) *We have $(\mathcal{F}^\#)_p = \mathcal{F}_p$ for any presheaf of sets on $\mathcal{C}$.*

[3]Of course in almost any naturally occurring case the colimit is filtered and some of the discussion in this section may be simplified.

(2) *The stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact (see Categories, Definition 23.1) and commutes with arbitrary colimits.*
(3) *The stalk functor $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact (see Categories, Definition 23.1) and commutes with arbitrary colimits.*

Proof. By Sites, Lemma 32.5 we have (1). By Sites, Lemmas 32.4 we see that $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is a left adjoint, and by Sites, Lemma 32.5 we see the same thing for $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$. Hence the stalk functor commutes with arbitrary colimits (see Categories, Lemma 24.5). It follows from the definition of a point of a site, see Sites, Definition 32.2 that $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact. Since sheafification is exact (Sites, Lemma 10.14) it follows that $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact. □

In particular, since the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ on presheaves commutes with all finite limits and colimits we may apply the reasoning of the proof of Sites, Proposition 44.3. The result of such an argument is that if $\mathcal{F}$ is a (pre)sheaf of algebraic structures listed in Sites, Proposition 44.3 then the stalk $\mathcal{F}_p$ is naturally an algebraic structure of the same kind. Let us explain this in detail when $\mathcal{F}$ is an abelian presheaf. In this case the addition map $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ induces a map

$$+ : \mathcal{F}_p \times \mathcal{F}_p = (\mathcal{F} \times \mathcal{F})_p \longrightarrow \mathcal{F}_p$$

where the equal sign uses that stalk functor on presheaves of sets commutes with finite limits. This defines a group structure on the stalk $\mathcal{F}_p$. In this way we obtain our stalk functor

$$PAb(\mathcal{C}) \longrightarrow Ab, \quad \mathcal{F} \longmapsto \mathcal{F}_p$$

By construction the underlying set of $\mathcal{F}_p$ is the stalk of the underlying presheaf of sets. This also defines our stalk functor for sheaves of abelian groups by precomposing with the inclusion $Ab(\mathcal{C}) \subset PAb(\mathcal{C})$.

04EP **Lemma 36.2.** *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$.*
(1) *The functor $Ab(\mathcal{C}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
(2) *The stalk functor $PAb(\mathcal{C}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
(3) *For $\mathcal{F} \in \mathrm{Ob}(PAb(\mathcal{C}))$ we have $\mathcal{F}_p = \mathcal{F}_p^\#$.*

Proof. This is formal from the results of Lemma 36.1 and the construction of the stalk functor above. □

Next, we turn to the case of sheaves of modules. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. (It suffices for the discussion that $\mathcal{O}$ be a presheaf of rings.) Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let p be a point of $\mathcal{C}$. In this case we get a map

$$\cdot : \mathcal{O}_p \times \mathcal{O}_p = (\mathcal{O} \times \mathcal{O})_p \longrightarrow \mathcal{O}_p$$

which is the stalk of the multiplication map and

$$\cdot : \mathcal{O}_p \times \mathcal{F}_p = (\mathcal{O} \times \mathcal{F})_p \longrightarrow \mathcal{F}_p$$

which is the stalk of the multiplication map. We omit the verification that this defines a ring structure on $\mathcal{O}_p$ and an $\mathcal{O}_p$-module structure on $\mathcal{F}_p$. In this way we obtain a functor

$$PMod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}_p), \quad \mathcal{F} \longmapsto \mathcal{F}_p$$

By construction the underlying set of $\mathcal{F}_p$ is the stalk of the underlying presheaf of sets. This also defines our stalk functor for sheaves of $\mathcal{O}$-modules by precomposing with the inclusion $Mod(\mathcal{O}) \subset PMod(\mathcal{O})$.

04EQ **Lemma 36.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$.*

(1) *The functor $Mod(\mathcal{O}) \to Mod(\mathcal{O}_p)$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
(2) *The stalk functor $PMod(\mathcal{O}) \to Mod(\mathcal{O}_p)$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
(3) *For $\mathcal{F} \in \mathrm{Ob}(PMod(\mathcal{O}))$ we have $\mathcal{F}_p = \mathcal{F}^\#_p$.*

Proof. This is formal from the results of Lemma 36.2, the construction of the stalk functor above, and Lemma 14.1. $\square$

05V5 **Lemma 36.4.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Let p be a point of $\mathcal{C}$ or $Sh(\mathcal{C})$ and set $q = f \circ p$. Then*

$$(f^*\mathcal{F})_p = \mathcal{F}_q \otimes_{\mathcal{O}_{\mathcal{D},q}} \mathcal{O}_{\mathcal{C},p}$$

for any $\mathcal{O}_\mathcal{D}$-module $\mathcal{F}$.

Proof. We have

$$f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}_\mathcal{C}$$

by definition. Since taking stalks at p (i.e., applying p^{-1}) commutes with $\otimes$ by Lemma 26.2 we win by the relation between the stalk of pullbacks at p and stalks at q explained in Sites, Lemma 34.2 or Sites, Lemma 34.3. $\square$

37. Skyscraper sheaves

05V6 Let p be a point of a site $\mathcal{C}$ or a topos $Sh(\mathcal{C})$. In this section we study the exactness properties of the functor which associates to an abelian group A the skyscraper sheaf p_*A. First, recall that $p_* : Sets \to Sh(\mathcal{C})$ has a lot of exactness properties, see Sites, Lemmas 32.9 and 32.10.

05V7 **Lemma 37.1.** *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ or of its associated topos.*

(1) *The functor $p_* : Ab \to Ab(\mathcal{C})$, $A \mapsto p_*A$ is exact.*
(2) *There is a functorial direct sum decomposition*

$$p^{-1}p_*A = A \oplus I(A)$$

for $A \in \mathrm{Ob}(Ab)$.

Proof. By Sites, Lemma 32.9 there are functorial maps $A \to p^{-1}p_*A \to A$ whose composition equals id_A. Hence a functorial direct sum decomposition as in (2) with $I(A)$ the kernel of the adjunction map $p^{-1}p_*A \to A$. The functor p_* is left exact by Lemma 14.3. The functor p_* transforms surjections into surjections by Sites, Lemma 32.10. Hence (1) holds. $\square$

To do the same thing for sheaves of modules, suppose given a point p of a ringed topos $(Sh(\mathcal{C}), \mathcal{O})$. Recall that p^{-1} is just the stalk functor. Hence we can think of p as a morphism of ringed topoi

$$(p, \mathrm{id}_{\mathcal{O}_p}) : (Sh(pt), \mathcal{O}_p) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}).$$

Thus we get a pullback functor $p^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_p)$ which equals the stalk functor, and which we discussed in Lemma 36.3. In this section we consider the functor $p_* : Mod(\mathcal{O}_p) \to Mod(\mathcal{O})$.

05V8 **Lemma 37.2.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let p be a point of the topos $Sh(\mathcal{C})$.*

(1) *The functor $p_* : Mod(\mathcal{O}_p) \to Mod(\mathcal{O})$, $M \mapsto p_*M$ is exact.*
(2) *The canonical surjection $p^{-1}p_*M \to M$ is $\mathcal{O}_p$-linear.*

(3) *The functorial direct sum decomposition* $p^{-1}p_*M = M \oplus I(M)$ *of Lemma 37.1 is* **not** $\mathcal{O}_p$*-linear in general.*

Proof. Part (1) and surjectivity in (2) follow immediately from the corresponding result for abelian sheaves in Lemma 37.1. Since $p^{-1}\mathcal{O} = \mathcal{O}_p$ we have $p^{-1} = p^*$ and hence $p^{-1}p_*M \to M$ is the same as the counit $p^*p_*M \to M$ of the adjunction for modules, whence linear.

Proof of (3). Suppose that G is a group. Consider the topos $G\textit{-Sets} = Sh(\mathcal{T}_G)$ and the point $p : \textit{Sets} \to G\textit{-Sets}$. See Sites, Section 9 and Example 33.7. Here p^{-1} is the functor forgetting about the G-action. And p_* is the right adjoint of the forgetful functor, sending M to $\text{Map}(G, M)$. The maps in the direct sum decomposition are the maps

$$M \to \text{Map}(G, M) \to M$$

where the first sends $m \in M$ to the constant map with value m and where the second map is evaluation at the identity element 1 of G. Next, suppose that R is a ring endowed with an action of G. This determines a sheaf of rings $\mathcal{O}$ on $\mathcal{T}_G$. The category of $\mathcal{O}$-modules is the category of R-modules M endowed with an action of G compatible with the action on R. The R-module structure on $\text{Map}(G, M)$ is given by

$$(rf)(\sigma) = \sigma(r)f(\sigma)$$

for $r \in R$ and $f \in \text{Map}(G, M)$. This is true because it is the unique G-invariant R-module structure compatible with evaluation at 1. The reader observes that in general the image of $M \to \text{Map}(G, M)$ is not an R-submodule (for example take $M = R$ and assume the G-action is nontrivial), which concludes the proof. $\square$

05V9 **Example 37.3.** Let G be a group. Consider the site $\mathcal{T}_G$ and its point p, see Sites, Example 33.7. Let R be a ring with a G-action which corresponds to a sheaf of rings $\mathcal{O}$ on $\mathcal{T}_G$. Then $\mathcal{O}_p = R$ where we forget the G-action. In this case $p^{-1}p_*M = \text{Map}(G, M)$ and $I(M) = \{f : G \to M \mid f(1_G) = 0\}$ and $M \to \text{Map}(G, M)$ assigns to $m \in M$ the constant function with value m.

38. Localization and points

070Z

0710 **Lemma 38.1.** *Let* $(\mathcal{C}, \mathcal{O})$ *be a ringed site. Let* p *be a point of* $\mathcal{C}$*. Let* U *be an object of* $\mathcal{C}$*. For* $\mathcal{G}$ *in* $\textit{Mod}(\mathcal{O}_U)$ *we have*

$$(j_{U!}\mathcal{G})_p = \bigoplus\nolimits_q \mathcal{G}_q$$

where the coproduct is over the points q *of* $\mathcal{C}/U$ *lying over* p*, see Sites, Lemma 35.2.*

Proof. We use the description of $j_{U!}\mathcal{G}$ as the sheaf associated to the presheaf $V \mapsto \bigoplus_{\varphi \in \text{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V/_\varphi U)$ of Lemma 19.2. The stalk of $j_{U!}\mathcal{G}$ at p is equal to the stalk of this presheaf, see Lemma 36.3. Let $u : \mathcal{C} \to \textit{Sets}$ be the functor corresponding to p (see Sites, Section 32). Hence we see that

$$(j_{U!}\mathcal{G})_p = \text{colim}_{(V,y)} \bigoplus\nolimits_{\varphi : V \to U} \mathcal{G}(V/_\varphi U)$$

where the colimit is taken in the category of abelian groups. To a quadruple (V, y, φ, s) occurring in this colimit, we can assign $x = u(\varphi)(y) \in u(U)$. Hence we obtain

$$(j_{U!}\mathcal{G})_p = \bigoplus\nolimits_{x \in u(U)} \mathrm{colim}_{(\varphi : V \to U, y),\ u(\varphi)(y) = x}\, \mathcal{G}(V/_{\varphi}U).$$

This is equal to the expression of the lemma by the description of the points q lying over x in Sites, Lemma 35.2. $\square$

0711 **Remark 38.2.** Warning: The result of Lemma 38.1 has no analogue for $j_{U,*}$.

39. Pullbacks of flat modules

05VA The pullback of a flat module along a morphism of ringed topoi is flat. This is a bit tricky to prove.

05VD **Lemma 39.1.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Then $f^*\mathcal{F}$ is a flat $\mathcal{O}_\mathcal{C}$-module whenever $\mathcal{F}$ is a flat $\mathcal{O}_\mathcal{D}$-module.* [AGV71, Exposé V, Corollary 1.7.1]

Proof. Choose a diagram as in Lemma 7.2. Recall that being a flat module is intrinsic (see Section 18 and Definition 28.1). Hence it suffices to prove the lemma for the morphism $(h, h^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \to (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'})$. In other words, we may assume that our sites $\mathcal{C}$ and $\mathcal{D}$ have all finite limits and that f is a morphism of sites induced by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ which commutes with finite limits.

Recall that $f^*\mathcal{F} = \mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{F}$ (Definition 13.1). By Lemma 28.13 it suffices to prove that $f^{-1}\mathcal{F}$ is a flat $f^{-1}\mathcal{O}_\mathcal{D}$-module. Combined with the previous paragraph this reduces us to the situation of the next paragraph.

Assume $\mathcal{C}$ and $\mathcal{D}$ are sites which have all finite limits and that $u : \mathcal{D} \to \mathcal{C}$ is a continuous functor which commutes with finite limits. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$ and let $\mathcal{F}$ be a flat $\mathcal{O}$-module. Then u defines a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ (Sites, Proposition 14.7). To show: $f^{-1}\mathcal{F}$ is a flat $f^{-1}\mathcal{O}$-module. Let U be an object of $\mathcal{C}$ and let

$$f^{-1}\mathcal{O}|_U \xrightarrow{(f_1, \ldots, f_n)} f^{-1}\mathcal{O}|_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} f^{-1}\mathcal{F}|_U$$

be a complex of $f^{-1}\mathcal{O}|_U$-modules. Our goal is to construct a factorization of $(s_1, \ldots, s_n)$ on the members of a covering of U as in Lemma 28.14 part (2). Consider the elements $s_a \in f^{-1}\mathcal{F}(U)$ and $f_a \in f^{-1}\mathcal{O}(U)$. Since $f^{-1}\mathcal{F}$, resp. $f^{-1}\mathcal{O}$ is the sheafification of $u_p\mathcal{F}$ we may, after replacing U by the members of a covering, assume that s_a is the image of an element $s'_a \in u_p\mathcal{F}(U)$ and f_a is the image of an element $f'_a \in u_p\mathcal{O}(U)$. Then after another replacement of U by the members of a covering we may assume that $\sum f'_a s'_a$ is zero in $u_p\mathcal{F}(U)$. Recall that the category $(\mathcal{I}_U^u)^{opp}$ is directed (Sites, Lemma 5.2) and that $u_p\mathcal{F}(U) = \mathrm{colim}_{(\mathcal{I}_U^u)^{opp}} \mathcal{F}(V)$ and $u_p\mathcal{O}(U) = \mathrm{colim}_{(\mathcal{I}_U^u)^{opp}} \mathcal{O}(V)$. Hence we may assume there is a pair $(V, \phi) \in \mathrm{Ob}(\mathcal{I}_U^u)$ where V is an object of $\mathcal{D}$ and ϕ is a morphism $\phi : U \to u(V)$ of $\mathcal{D}$ and elements $s''_a \in \mathcal{F}(V)$ and $f''_a \in \mathcal{O}(V)$ whose images in $u_p\mathcal{F}(U)$ and $u_p\mathcal{O}(U)$ are equal to s'_a and f'_a and such that $\sum f''_a s''_a = 0$ in $\mathcal{F}(V)$. Then we obtain a complex

$$\mathcal{O}|_V \xrightarrow{(f''_1, \ldots, f''_n)} \mathcal{O}|_V^{\oplus n} \xrightarrow{(s''_1, \ldots, s''_n)} \mathcal{F}|_V$$

and we can apply the other direction of Lemma 28.14 to see there exists a covering $\{V_i \to V\}$ of $\mathcal{D}$ and for each i a factorization

$$\mathcal{O}|_{V_i}^{\oplus n} \xrightarrow{B''_i} \mathcal{O}|_{V_i}^{\oplus l_i} \xrightarrow{(t''_{i1}, \ldots, t''_{il_i})} \mathcal{F}|_{V_i}$$

of $(s''_1, \ldots, s''_n)|_{V_i}$ such that $B_i \circ (f''_1, \ldots, f''_n)|_{V_i} = 0$. Set $U_i = U \times_{\phi, u(V)} u(V_i)$, denote $B_i \in \text{Mat}(l_i \times n, f^{-1}\mathcal{O}(U_i))$ the image of B''_i, and denote $t_{ij} \in f^{-1}\mathcal{F}(U_i)$ the image of t''_{ij}. Then we get a factorization

$$f^{-1}\mathcal{O}|_{U_i}^{\oplus n} \xrightarrow{B_i} f^{-1}\mathcal{O}|_{U_i}^{\oplus l_i} \xrightarrow{(t_{i1}, \ldots, t_{il_i})} \mathcal{F}|_{U_i}$$

of $(s_1, \ldots, s_n)|_{U_i}$ such that $B_i \circ (f_1, \ldots, f_n)|_{U_i} = 0$. This finishes the proof. □

05VB **Lemma 39.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}_p$ is a flat $\mathcal{O}_p$-module.*

Proof. In Section 37 we have seen that we can think of p as a morphism of ringed topoi

$$(p, \text{id}_{\mathcal{O}_p}) : (Sh(pt), \mathcal{O}_p) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}).$$

such that the pullback functor $p^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_p)$ equals the stalk functor. Thus the lemma follows from Lemma 39.1. □

05VC **Lemma 39.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\{p_i\}_{i \in I}$ be a conservative family of points of $\mathcal{C}$. Then $\mathcal{F}$ is flat if and only if $\mathcal{F}_{p_i}$ is a flat $\mathcal{O}_{p_i}$-module for all $i \in I$.*

Proof. By Lemma 39.2 we see one of the implications. For the converse, use that $(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_p = \mathcal{F}_p \otimes_{\mathcal{O}_p} \mathcal{G}_p$ by Lemma 26.2 (as taking stalks at p is given by p^{-1}) and Lemma 14.4. □

0G6R **Lemma 39.4.** *Let $f : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}'), \mathcal{O})$ be a morphism of ringed topoi. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of $\mathcal{O}$-modules with $\mathcal{H}$ a flat $\mathcal{O}$-module. Then the sequence $0 \to f^*\mathcal{F} \to f^*\mathcal{G} \to f^*\mathcal{H} \to 0$ is exact as well.*

Proof. Since f^{-1} is exact we have the short exact sequence $0 \to f^{-1}\mathcal{F} \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H} \to 0$ of $f^{-1}\mathcal{O}$-modules. By Lemma 39.1 the $f^{-1}\mathcal{O}$-module $f^{-1}\mathcal{H}$ is flat. By Lemma 28.9 this implies that tensoring the sequence over $f^{-1}\mathcal{O}$ with $\mathcal{O}'$ the sequence remains exact. Since $f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}} \mathcal{O}'$ and similarly for $\mathcal{G}$ and $\mathcal{H}$ we conclude. □

40. Locally ringed topoi

04ER A reference for this section is [AGV71, Exposé IV, Exercice 13.9].

04ES **Lemma 40.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The following are equivalent*

(1) *For every object U of $\mathcal{C}$ and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j either $f|_{U_j}$ is invertible or $(1-f)|_{U_j}$ is invertible.*
(2) *For $U \in \text{Ob}(\mathcal{C})$, $n \geq 1$, and $f_1, \ldots, f_n \in \mathcal{O}(U)$ which generate the unit ideal in $\mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j there exists an i such that $f_i|_{U_j}$ is invertible.*
(3) *The map of sheaves of sets*

$$(\mathcal{O} \times \mathcal{O}) \amalg (\mathcal{O} \times \mathcal{O}) \longrightarrow \mathcal{O} \times \mathcal{O}$$

which maps (f, a) in the first component to (f, af) and (f, b) in the second component to $(f, b(1-f))$ is surjective.

Proof. It is clear that (2) implies (1). To show that (1) implies (2) we argue by induction on n. The first case is $n = 2$ (since $n = 1$ is trivial). In this case we have $a_1 f_1 + a_2 f_2 = 1$ for some $a_1, a_2 \in \mathcal{O}(U)$. By assumption we can find a covering $\{U_j \to U\}$ such that for each j either $a_1 f_1|_{U_j}$ is invertible or $a_2 f_2|_{U_j}$ is invertible. Hence either $f_1|_{U_j}$ is invertible or $f_2|_{U_j}$ is invertible as desired. For $n > 2$ we have $a_1 f_1 + \ldots + a_n f_n = 1$ for some $a_1, \ldots, a_n \in \mathcal{O}(U)$. By the case $n = 2$ we see that we have some covering $\{U_j \to U\}_{j \in J}$ such that for each j either $f_n|_{U_j}$ is invertible or $a_1 f_1 + \ldots + a_{n-1} f_{n-1}|_{U_j}$ is invertible. Say the first case happens for $j \in J_n$. Set $J' = J \setminus J_n$. By induction hypothesis, for each $j \in J'$ we can find a covering $\{U_{jk} \to U_j\}_{k \in K_j}$ such that for each $k \in K_j$ there exists an $i \in \{1, \ldots, n-1\}$ such that $f_i|_{U_{jk}}$ is invertible. By the axioms of a site the family of morphisms $\{U_j \to U\}_{j \in J_n} \cup \{U_{jk} \to U\}_{j \in J', k \in K_j}$ is a covering which has the desired property.

Assume (1). To see that the map in (3) is surjective, let (f, c) be a section of $\mathcal{O} \times \mathcal{O}$ over U. By assumption there exists a covering $\{U_j \to U\}$ such that for each j either f or $1 - f$ restricts to an invertible section. In the first case we can take $a = c|_{U_j}(f|_{U_j})^{-1}$, and in the second case we can take $b = c|_{U_j}(1 - f|_{U_j})^{-1}$. Hence (f, c) is in the image of the map on each of the members. Conversely, assume (3) holds. For any U and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ of U such that the section $(f, 1)|_{U_j}$ is in the image of the map in (3) on sections over U_j. This means precisely that either f or $1 - f$ restricts to an invertible section over U_j, and we see that (1) holds. $\square$

04ET **Lemma 40.2.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the following conditions*

1. *For every object U of $\mathcal{C}$ and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j either $f|_{U_j}$ is invertible or $(1 - f)|_{U_j}$ is invertible.*
2. *For every point p of $\mathcal{C}$ the stalk $\mathcal{O}_p$ is either the zero ring or a local ring.*

We always have (1) $\Rightarrow$ (2). If $\mathcal{C}$ has enough points then (1) and (2) are equivalent.

Proof. Assume (1). Let p be a point of $\mathcal{C}$ given by a functor $u : \mathcal{C} \to Sets$. Let $f_p \in \mathcal{O}_p$. Since $\mathcal{O}_p$ is computed by Sites, Equation (32.1.1) we may represent f_p by a triple (U, x, f) where $x \in U(U)$ and $f \in \mathcal{O}(U)$. By assumption there exists a covering $\{U_i \to U\}$ such that for each i either f or $1 - f$ is invertible on U_i. Because u defines a point of the site we see that for some i there exists an $x_i \in u(U_i)$ which maps to $x \in u(U)$. By the discussion surrounding Sites, Equation (32.1.1) we see that (U, x, f) and $(U_i, x_i, f|_{U_i})$ define the same element of $\mathcal{O}_p$. Hence we conclude that either f_p or $1 - f_p$ is invertible. Thus $\mathcal{O}_p$ is a ring such that for every element a either a or $1 - a$ is invertible. This means that $\mathcal{O}_p$ is either zero or a local ring, see Algebra, Lemma 18.3.

Assume (2) and assume that $\mathcal{C}$ has enough points. Consider the map of sheaves of sets

$$\mathcal{O} \times \mathcal{O} \amalg \mathcal{O} \times \mathcal{O} \longrightarrow \mathcal{O} \times \mathcal{O}$$

of Lemma 40.1 part (3). For any local ring R the corresponding map $(R \times R) \amalg (R \times R) \to R \times R$ is surjective, see for example Algebra, Lemma 18.3. Since each $\mathcal{O}_p$ is a local ring or zero the map is surjective on stalks. Hence, by our assumption that $\mathcal{C}$ has enough points it is surjective and we win. $\square$

In Modules, Section 2 we pointed out how in a ringed space $(X, \mathcal{O}_X)$ there can be an open subspace over which the structure sheaf is zero. To prevent this we can require the sections 1 and 0 to have different values in every stalk of the space X. In the setting of ringed topoi and ringed sites the condition is that

05D7 (40.2.1)
$$\emptyset^\# \longrightarrow \text{Equalizer}(0, 1 : * \longrightarrow \mathcal{O})$$

is an isomorphism of sheaves. Here $*$ is the singleton sheaf, resp. $\emptyset^\#$ is the "empty sheaf", i.e., the final, resp. initial object in the category of sheaves, see Sites, Example 10.2, resp. Section 42. In other words, the condition is that whenever $U \in \text{Ob}(\mathcal{C})$ is not sheaf theoretically empty, then $1, 0 \in \mathcal{O}(U)$ are not equal. Let us state the obligatory lemma.

05D8 **Lemma 40.3.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the statements*

1. *(40.2.1) is an isomorphism, and*
2. *for every point p of $\mathcal{C}$ the stalk $\mathcal{O}_p$ is not the zero ring.*

We always have (1) $\Rightarrow$ (2) and if $\mathcal{C}$ has enough points then (1) $\Leftrightarrow$ (2).

Proof. Omitted. □

Lemmas 40.1, 40.2, and 40.3 motivate the following definition.

04EU **Definition 40.4.** A ringed site $(\mathcal{C}, \mathcal{O})$ is said to be *locally ringed site* if (40.2.1) is an isomorphism, and the equivalent properties of Lemma 40.1 are satisfied.

In [AGV71, Exposé IV, Exercice 13.9] the condition that (40.2.1) be an isomorphism is missing leading to a slightly different notion of a locally ringed site and locally ringed topos. As we are motivated by the notion of a locally ringed space we decided to add this condition (see explanation above).

04H7 **Lemma 40.5.** *Being a locally ringed site is an intrinsic property. More precisely,*

1. *if $f : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ is a morphism of topoi and $(\mathcal{C}, \mathcal{O})$ is a locally ringed site, then $(\mathcal{C}', f^{-1}\mathcal{O})$ is a locally ringed site, and*
2. *if $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ is an equivalence of ringed topoi, then $(\mathcal{C}, \mathcal{O})$ is locally ringed if and only if $(\mathcal{C}', \mathcal{O}')$ is locally ringed.*

Proof. It is clear that (2) follows from (1). To prove (1) note that as f^{-1} is exact we have $f^{-1}* = *$, $f^{-1}\emptyset^\# = \emptyset^\#$, and f^{-1} commutes with products, equalizers and transforms isomorphisms and surjections into isomorphisms and surjections. Thus f^{-1} transforms the isomorphism (40.2.1) into its analogue for $f^{-1}\mathcal{O}$ and transforms the surjection of Lemma 40.1 part (3) into the corresponding surjection for $f^{-1}\mathcal{O}$. □

In fact Lemma 40.5 part (2) is the analogue of Schemes, Lemma 2.2. It assures us that the following definition makes sense.

04H8 **Definition 40.6.** A ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ is said to be *locally ringed* if the underlying ringed site $(\mathcal{C}, \mathcal{O})$ is locally ringed.

Here is an example of a consequence of being locally ringed.

0B8Q **Lemma 40.7.** *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Any locally free $\mathcal{O}$-module of rank 1 is invertible. If $(\mathcal{C}, \mathcal{O})$ is locally ringed, then the converse holds as well (but in general this is not the case).*

Proof. Assume $\mathcal{L}$ is locally free of rank 1 and consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O}) \longrightarrow \mathcal{O}$$

Given any object U of $\mathcal{C}$ and restricting to the members of a covering trivializing $\mathcal{L}$, we see that this map is an isomorphism (details omitted). Hence $\mathcal{L}$ is invertible by Lemma 32.2.

Assume $(Sh(\mathcal{C}), \mathcal{O})$ is locally ringed. Let U be an object of $\mathcal{C}$. In the proof of Lemma 32.2 we have seen that there exists a covering $\{U_i \to U\}$ such that $\mathcal{L}|_{\mathcal{C}/U_i}$ is a direct summand of a finite free $\mathcal{O}_{U_i}$-module. After replacing U by U_i, let $p : \mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus r}$ be a projector whose image is isomorphic to $\mathcal{L}|_{\mathcal{C}/U}$. Then p corresponds to a matrix

$$P = (p_{ij}) \in \mathrm{Mat}(r \times r, \mathcal{O}(U))$$

which is a projector: $P^2 = P$. Set $A = \mathcal{O}(U)$ so that $P \in \mathrm{Mat}(r \times r, A)$. By Algebra, Lemma 78.2 the image of P is a finite locally free module M over A. Hence there are $f_1, \ldots, f_t \in A$ generating the unit ideal, such that M_{f_i} is finite free. By Lemma 40.1 after replacing U by the members of an open covering, we may assume that M is free. This means that $\mathcal{L}|_U$ is free (details omitted). Of course, since $\mathcal{L}$ is invertible, this is only possible if the rank of $\mathcal{L}|_U$ is 1 and the proof is complete. $\square$

Next, we want to work out what it means to have a morphism of locally ringed spaces. In order to do this we have the following lemma.

04H9 **Lemma 40.8.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Consider the following conditions*

(1) *The diagram of sheaves*

$$\begin{tikzcd} f^{-1}(\mathcal{O}^*_\mathcal{D}) \arrow[r, "f^\sharp"'] \arrow[d] & \mathcal{O}^*_\mathcal{C} \arrow[d] \\ f^{-1}(\mathcal{O}_\mathcal{D}) \arrow[r, "f^\sharp"] & \mathcal{O}_\mathcal{C} \end{tikzcd}$$

is cartesian.

(2) *For any point p of $\mathcal{C}$, setting $q = f \circ p$, the diagram*

$$\begin{tikzcd} \mathcal{O}^*_{\mathcal{D},q} \arrow[r] \arrow[d] & \mathcal{O}^*_{\mathcal{C},p} \arrow[d] \\ \mathcal{O}_{\mathcal{D},q} \arrow[r] & \mathcal{O}_{\mathcal{C},p} \end{tikzcd}$$

of sets is cartesian.

We always have (1) $\Rightarrow$ (2). If $\mathcal{C}$ has enough points then (1) and (2) are equivalent. If $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ and $(Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ are locally ringed topoi then (2) is equivalent to

(3) *For any point p of $\mathcal{C}$, setting $q = f \circ p$, the ring map $\mathcal{O}_{\mathcal{D},q} \to \mathcal{O}_{\mathcal{C},p}$ is a local ring map.*

In fact, properties (2), or (3) for a conservative family of points implies (1).

Proof. This lemma proves itself, in other words, it follows by unwinding the definitions. $\square$

04HA **Definition 40.9.** Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Assume $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ and $(Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ are locally ringed topoi. We say that $(f, f^\sharp)$ is a *morphism of locally ringed topoi* if and only if the diagram of sheaves

$$\begin{tikzcd} f^{-1}(\mathcal{O}^*_\mathcal{D}) \arrow[r, "f^\sharp"'] \arrow[d] & \mathcal{O}^*_\mathcal{C} \arrow[d] \\ f^{-1}(\mathcal{O}_\mathcal{D}) \arrow[r, "f^\sharp"] & \mathcal{O}_\mathcal{C} \end{tikzcd}$$

(see Lemma 40.8) is cartesian. If $(f, f^\sharp)$ is a morphism of ringed sites, then we say that it is a *morphism of locally ringed sites* if the associated morphism of ringed topoi is a morphism of locally ringed topoi.

It is clear that an isomorphism of ringed topoi between locally ringed topoi is automatically an isomorphism of locally ringed topoi.

04IG **Lemma 40.10.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of locally ringed topoi. Then the composition $(g, g^\sharp) \circ (f, f^\sharp)$ (see Definition 7.1) is also a morphism of locally ringed topoi.*

Proof. Omitted. $\square$

04KR **Lemma 40.11.** *Let $f : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ be a morphism of topoi. If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$, then*

$$f^{-1}(\mathcal{O}^*) = (f^{-1}\mathcal{O})^*.$$

In particular, if $\mathcal{O}$ turns $\mathcal{C}$ into a locally ringed site, then setting $f^\sharp = id$ the morphism of ringed topoi

$$(f, f^\sharp) : (Sh(\mathcal{C}'), f^{-1}\mathcal{O}) \to (Sh(\mathcal{C}, \mathcal{O})$$

is a morphism of locally ringed topoi.

Proof. Note that the diagram

$$\begin{tikzcd} \mathcal{O}^* \arrow[r] \arrow[d, "u \mapsto (u, u^{-1})"'] & * \arrow[d, "1"] \\ \mathcal{O} \times \mathcal{O} \arrow[r, "(a,b) \mapsto ab"] & \mathcal{O} \end{tikzcd}$$

is cartesian. Since f^{-1} is exact we conclude that

$$\begin{tikzcd} f^{-1}(\mathcal{O}^*) \arrow[r] \arrow[d, "u \mapsto (u, u^{-1})"'] & * \arrow[d, "1"] \\ f^{-1}\mathcal{O} \times f^{-1}\mathcal{O} \arrow[r, "(a,b) \mapsto ab"] & f^{-1}\mathcal{O} \end{tikzcd}$$

is cartesian which implies the first assertion. For the second, note that $(\mathcal{C}', f^{-1}\mathcal{O})$ is a locally ringed site by Lemma 40.5 so that the assertion makes sense. Now the first part implies that the morphism is a morphism of locally ringed topoi. $\square$

04IH **Lemma 40.12.** *Localization of locally ringed sites and topoi.*

(1) *Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. Let U be an object of $\mathcal{C}$. Then the localization $(\mathcal{C}/U, \mathcal{O}_U)$ is a locally ringed site, and the localization morphism*

$$(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$$

is a morphism of locally ringed topoi.

(2) *Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then the morphism*

$$(j, j^\sharp) : (Sh(\mathcal{C}/V), \mathcal{O}_V) \to (Sh(\mathcal{C}/U), \mathcal{O}_U)$$

of Lemma 19.5 is a morphism of locally ringed topoi.

(3) *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of locally ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$ and let $U = u(V)$. Then the morphism*

$$(f', (f')^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$

of Lemma 20.1 is a morphism of locally ringed sites.

(4) *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of locally ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$, $U \in \mathrm{Ob}(\mathcal{C})$, and $c : U \to u(V)$. Then the morphism*

$$(f_c, (f_c)^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$

of Lemma 20.2 is a morphism of locally ringed topoi.

(5) *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a locally ringed topos. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then the localization $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a locally ringed topos and the localization morphism*

$$(j_\mathcal{F}, j_\mathcal{F}^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C}), \mathcal{O})$$

is a morphism of locally ringed topoi.

(6) *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a locally ringed topos. Let $s : \mathcal{G} \to \mathcal{F}$ be a map of sheaves on $\mathcal{C}$. Then the morphism*

$$(j, j^\sharp) : (Sh(\mathcal{C})/\mathcal{G}, \mathcal{O}_\mathcal{G}) \longrightarrow (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$$

of Lemma 21.4 is a morphism of locally ringed topoi.

(7) *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of locally ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then the morphism*

$$(f', (f')^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$

of Lemma 22.1 is a morphism of locally ringed topoi.

(8) *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of locally ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be a morphism of sheaves. Then the morphism*

$$(f_s, (f_s)^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$

of Lemma 22.3 is a morphism of locally ringed topoi.

Proof. Part (1) is clear since $\mathcal{O}_U$ is just the restriction of $\mathcal{O}$, so Lemmas 40.5 and 40.11 apply. Part (2) is clear as the morphism $(j, j^\sharp)$ is actually a localization of a locally ringed site so (1) applies. Part (3) is clear also since $(f')^\sharp$ is just the restriction of $f^\sharp$ to the topos $Sh(\mathcal{C})/\mathcal{F}$, see proof of Lemma 22.1 (hence the diagram of Definition 40.9 for the morphism f' is just the restriction of the corresponding diagram for f, and restriction is an exact functor). Part (4) follows formally on combining (2) and (3). Parts (5), (6), (7), and (8) follow from their counterparts (1), (2), (3), and (4) by enlarging the sites as in Lemma 7.2 and translating everything in terms of sites and morphisms of sites using the comparisons of Lemmas 21.3, 21.5, 22.2, and 22.4. (Alternatively one could use the same arguments as in the proofs of (1), (2), (3), and (4) to prove (5), (6), (7), and (8) directly.) □

41. Lower shriek for modules

0796 In this section we extend the construction of $g_!$ discussed in Section 16 to the case of sheaves of modules.

0797 **Lemma 41.1.** *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor between sites. Denote $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ the associated morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings on $\mathcal{D}$. Set $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$. Hence g becomes a morphism of ringed topoi with $g^* = g^{-1}$. In this case there exists a functor*

$$g_! : Mod(\mathcal{O}_\mathcal{C}) \longrightarrow Mod(\mathcal{O}_\mathcal{D})$$

which is left adjoint to g^*.

Proof. Let U be an object of $\mathcal{C}$. For any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{G}) &= g^{-1}\mathcal{G}(U) \\ &= \mathcal{G}(u(U)) \\ &= \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(j_{u(U)!}\mathcal{O}_{u(U)}, \mathcal{G}) \end{aligned}$$

because g^{-1} is described by restriction, see Sites, Lemma 21.5. Of course a similar formula holds a direct sum of modules of the form $j_{U!}\mathcal{O}_U$. By Homology, Lemma 29.6 and Lemma 28.8 we see that $g_!$ exists. □

0798 **Remark 41.2.** Warning! Let $u : \mathcal{C} \to \mathcal{D}$, g, $\mathcal{O}_\mathcal{D}$, and $\mathcal{O}_\mathcal{C}$ be as in Lemma 41.1. In general it is **not** the case that the diagram

$$\begin{tikzcd} Mod(\mathcal{O}_\mathcal{C}) \arrow[r, "g_!"'] \arrow[d, "forget"'] & Mod(\mathcal{O}_\mathcal{D}) \arrow[d, "forget"] \\ Ab(\mathcal{C}) \arrow[r, "g_!^{Ab}"] & Ab(\mathcal{D}) \end{tikzcd}$$

commutes (here $g_!^{Ab}$ is the one from Lemma 16.2). There is a transformation of functors

$$g_!^{Ab} \circ forget \longrightarrow forget \circ g_!$$

From the proof of Lemma 41.1 we see that this is an isomorphism if and only if $g_!^{Ab} j_{U!}\mathcal{O}_U \to g_! j_{U!}\mathcal{O}_U$ is an isomorphism for all objects U of $\mathcal{C}$. Since we have $g_! j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}$ this holds if and only if

$$g_!^{Ab} j_{U!}\mathcal{O}_U \longrightarrow j_{u(U)!}\mathcal{O}_{u(U)}$$

is an isomorphism for all objects U of $\mathcal{C}$. Note that for such a U we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{C}/U \arrow[r, "u'"'] \arrow[d, "j_U"'] & \mathcal{D}/u(U) \arrow[d, "j_{u(U)}"] \\ \mathcal{C} \arrow[r, "u"] & \mathcal{D} \end{tikzcd}$$

of cocontinuous functors of sites, see Sites, Lemma 28.4 and therefore $g_!^{Ab} j_{U!} = j_{u(U)!}(g')_!^{Ab}$ where $g' : Sh(\mathcal{C}/U) \to Sh(\mathcal{D}/u(U))$ is the morphism of topoi induced by the cocontinuous functor u'. Hence we see that $g_! = g_!^{Ab}$ if the canonical map

0799 (41.2.1)
$$(g')_!^{Ab}\mathcal{O}_U \longrightarrow \mathcal{O}_{u(U)}$$

is an isomorphism for all objects U of $\mathcal{C}$.

The following two results are of a slightly different nature.

0FN3 **Lemma 41.3.** *Assume given a commutative diagram*

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "{(g', (g')^\sharp)}"'] \arrow[d, "{(f', (f')^\sharp)}"'] & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \arrow[d, "{(f, f^\sharp)}"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "{(g, g^\sharp)}"] & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{tikzcd}$$

of ringed topoi. Assume

1. *f, f', g, and g' correspond to cocontinuous functors u, u', v, and v' as in Sites, Lemma 21.1,*
2. *$v \circ u' = u \circ v'$,*
3. *v and v' are continuous as well as cocontinuous,*
4. *for any object V' of $\mathcal{D}'$ the functor ${}^{u'}_{V'}\mathcal{I} \to {}^{u}_{v(V')}\mathcal{I}$ given by v is cofinal, and*
5. *$g^{-1}\mathcal{O}_\mathcal{D} = \mathcal{O}_{\mathcal{D}'}$ and $(g')^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}_{\mathcal{C}'}$.*

Then we have $f'_* \circ (g')^* = g^* \circ f_*$ and $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$ on modules.

Proof. We have $(g')^*\mathcal{F} = (g')^{-1}\mathcal{F}$ and $g^*\mathcal{G} = g^{-1}\mathcal{G}$ because of condition (5). Thus the first equality follows immediately from the corresponding equality in Sites, Lemma 28.6. Since the left adjoint functors $g_!$ and $g'_!$ to g^* and $(g')^*$ exist by Lemma 41.1 we see that the second equality follows by uniqueness of adjoint functors. $\square$

0FN4 **Lemma 41.4.** *Consider a commutative diagram*

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "{(g', (g')^\sharp)}"'] \arrow[d, "{(f', (f')^\sharp)}"'] & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \arrow[d, "{(f, f^\sharp)}"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "{(g, g^\sharp)}"] & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{tikzcd}$$

of ringed topoi and suppose we have functors

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "v'"'] & \mathcal{C} \\ \mathcal{D}' \arrow[u, "u'"] \arrow[r, "v"] & \mathcal{D} \arrow[u, "u"'] \end{tikzcd}$$

such that (with notation as in Sites, Sections 14 and 21) we have

1. *u and u' are continuous and give rise to the morphisms f and f',*
2. *v and v' are cocontinuous giving rise to the morphisms g and g',*
3. *$u \circ v = v' \circ u'$,*
4. *v and v' are continuous as well as cocontinuous, and*
5. *$g^{-1}\mathcal{O}_\mathcal{D} = \mathcal{O}_{\mathcal{D}'}$ and $(g')^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}_{\mathcal{C}'}$.*

Then $f'_* \circ (g')^* = g^* \circ f_*$ and $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$ on modules.

Proof. We have $(g')^*\mathcal{F} = (g')^{-1}\mathcal{F}$ and $g^*\mathcal{G} = g^{-1}\mathcal{G}$ because of condition (5). Thus the first equality follows immediately from the corresponding equality in Sites, Lemma 28.7. Since the left adjoint functors $g_!$ and $g'_!$ to g^* and $(g')^*$ exist by Lemma 41.1 we see that the second equality follows by uniqueness of adjoint functors. $\square$

42. Constant sheaves

093I Let E be a set and let $\mathcal{C}$ be a site. We will denote $\underline{E}$ the constant sheaf with value E on $\mathcal{C}$. If E is an abelian group, ring, module, etc, then $\underline{E}$ is a sheaf of abelian groups, rings, modules, etc.

093J **Lemma 42.1.** *Let $\mathcal{C}$ be a site. If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, then $0 \to \underline{A} \to \underline{B} \to \underline{C} \to 0$ is an exact sequence of abelian sheaves and in fact it is even exact as a sequence of abelian presheaves.*

Proof. Since sheafification is exact it is clear that $0 \to \underline{A} \to \underline{B} \to \underline{C} \to 0$ is an exact sequence of abelian sheaves. Thus $0 \to \underline{A} \to \underline{B} \to \underline{C}$ is an exact sequence of abelian presheaves. To see that $\underline{B} \to \underline{C}$ is surjective, pick a set theoretical section $s : C \to B$. This induces a section $\underline{s} : \underline{C} \to \underline{B}$ of sheaves of sets left inverse to the surjection $\underline{B} \to \underline{C}$. $\square$

093K **Lemma 42.2.** *Let $\mathcal{C}$ be a site. Let Λ be a ring and let M and Q be Λ-modules. If Q is a finitely presented Λ-module, then we have $\underline{M \otimes_\Lambda Q}(U) = \underline{M}(U) \otimes_\Lambda Q$ for all $U \in \mathrm{Ob}(\mathcal{C})$.*

Proof. Choose a presentation $\Lambda^{\oplus m} \to \Lambda^{\oplus n} \to Q \to 0$. This gives an exact sequence $M^{\oplus m} \to M^{\oplus n} \to M \otimes Q \to 0$. By Lemma 42.1 we obtain an exact sequence

$$\underline{M}(U)^{\oplus m} \to \underline{M}(U)^{\oplus n} \to \underline{M \otimes Q}(U) \to 0$$

which proves the lemma. (Note that taking sections over U always commutes with finite direct sums, but not arbitrary direct sums.) $\square$

093L **Lemma 42.3.** *Let $\mathcal{C}$ be a site. Let Λ be a coherent ring. Let M be a flat Λ-module. For $U \in \mathrm{Ob}(\mathcal{C})$ the module $\underline{M}(U)$ is a flat Λ-module.*

Proof. Let $I \subset \Lambda$ be a finitely generated ideal. By Algebra, Lemma 39.5 it suffices to show that $\underline{M}(U) \otimes_\Lambda I \to \underline{M}(U)$ is injective. As Λ is coherent I is finitely presented as a Λ-module. By Lemma 42.2 we see that $\underline{M}(U) \otimes I = \underline{M \otimes I}$. Since M is flat the map $M \otimes I \to M$ is injective, whence $\underline{M \otimes I} \to \underline{M}$ is injective. $\square$

093M **Lemma 42.4.** *Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. The sheaf $\underline{\Lambda}^\wedge = \lim \underline{\Lambda/I^n}$ is a flat $\underline{\Lambda}$-algebra. Moreover we have canonical identifications*

$$\underline{\Lambda}/I\underline{\Lambda} = \underline{\Lambda}/\underline{I} = \underline{\Lambda}^\wedge/I\underline{\Lambda}^\wedge = \underline{\Lambda}^\wedge/\underline{I} \cdot \underline{\Lambda}^\wedge = \underline{\Lambda}^\wedge/\underline{I}^\wedge = \underline{\Lambda/I}$$

where $\underline{I}^\wedge = \lim \underline{I/I^n}$.

Proof. To prove $\underline{\Lambda}^\wedge$ is flat, it suffices to show that $\underline{\Lambda}^\wedge(U)$ is flat as a Λ-module for each $U \in \mathrm{Ob}(\mathcal{C})$, see Lemmas 28.2 and 28.3. By Lemma 42.3 we see that

$$\underline{\Lambda}^\wedge(U) = \lim \underline{\Lambda/I^n}(U)$$

is a limit of a system of flat Λ/I^n-modules. By Lemma 42.1 we see that the transition maps are surjective. We conclude by More on Algebra, Lemma 27.4.

To see the equalities, note that $\underline{\Lambda}(U)/I\underline{\Lambda}(U) = \underline{\Lambda/I}(U)$ by Lemma 42.2. It follows that $\underline{\Lambda}/I\underline{\Lambda} = \underline{\Lambda}/\underline{I} = \underline{\Lambda/I}$. The system of short exact sequences

$$0 \to \underline{I/I^n}(U) \to \underline{\Lambda/I^n}(U) \to \underline{\Lambda/I}(U) \to 0$$

has surjective transition maps, hence gives a short exact sequence

$$0 \to \lim \underline{I/I^n}(U) \to \lim \underline{\Lambda/I^n}(U) \to \lim \underline{\Lambda/I}(U) \to 0$$

see Homology, Lemma 31.3. Thus we see that $\underline{\Lambda}^\wedge/\underline{I}^\wedge = \underline{\Lambda/I}$. Since

$$I\underline{\Lambda}^\wedge \subset \underline{I} \cdot \underline{\Lambda}^\wedge \subset \underline{I}^\wedge$$

it suffices to show that $I\underline{\Lambda}^\wedge(U) = \underline{I}^\wedge(U)$ for all U. Choose generators $I = (f_1, \ldots, f_r)$. For every n we obtain a short exact sequence

$$0 \to K_n/(I^n)^{\oplus r} \to (\Lambda/I^n)^{\oplus r} \xrightarrow{(f_1,\ldots,f_r)} I/I^{n+1} \to 0$$

where $K_n = \{(x_1, \ldots, x_r) \in \Lambda^{\oplus r} \mid \sum x_i f_i \in I^{n+1}\}$. We obtain short exact sequences

$$0 \to \underline{K_n/(I^n)^{\oplus r}}(U) \to \underline{(\Lambda/I^n)^{\oplus r}}(U) \to \underline{I/I^{n+1}}(U) \to 0$$

A calculation shows $K_n = K + (I^n)^{\oplus r}$, hence the transition maps $K_{n+1}/(I^{n+1})^{\oplus r} \to K_n/(I^n)^{\oplus r}$ are surjective. Hence the system of modules on the left hand side has surjective transition maps and a fortiori has ML. Thus we see that $(f_1, \ldots, f_r) : (\underline{\Lambda}^\wedge)^{\oplus r}(U) \to \underline{I}^\wedge(U)$ is surjective by Homology, Lemma 31.3 which is what we wanted to show. $\square$

093N **Lemma 42.5.** *Let $\mathcal{C}$ be a site. Let Λ be a ring and let M be a Λ-module. Assume $Sh(\mathcal{C})$ is not the empty topos. Then*

1. *$\underline{M}$ is a finite type sheaf of $\underline{\Lambda}$-modules if and only if M is a finite Λ-module, and*
2. *$\underline{M}$ is a finitely presented sheaf of $\underline{\Lambda}$-modules if and only if M is a finitely presented Λ-module.*

Proof. Proof of (1). If M is generated by $x_1, \ldots, x_r$ then $x_1, \ldots, x_r$ define global sections of $\underline{M}$ which generate it, hence $\underline{M}$ is of finite type. Conversely, assume $\underline{M}$ is of finite type. Let $U \in \mathcal{C}$ be an object which is not sheaf theoretically empty (Sites, Definition 42.1). Such an object exists as we assumed $Sh(\mathcal{C})$ is not the empty topos. Then there exists a covering $\{U_i \to U\}$ and finitely many sections $s_{ij} \in \underline{M}(U_i)$ generating $\underline{M}|_{U_i}$. After refining the covering we may assume that s_{ij} come from elements x_{ij} of M. Then x_{ij} define global sections of $\underline{M}$ whose restriction to U generate $\underline{M}$.

Assume there exist elements $x_1, \ldots, x_r$ of M which define global sections of $\underline{M}$ generating $\underline{M}$ as a sheaf of $\underline{\Lambda}$-modules. We will show that $x_1, \ldots, x_r$ generate M as a Λ-module. Let $x \in M$. We can find a covering $\{U_i \to U\}_{i \in I}$ and $f_{i,j} \in \underline{\Lambda}(U_i)$ such that $x|_{U_i} = \sum f_{i,j} x_j|_{U_i}$. After refining the covering we may assume $f_{i,j} \in \Lambda$. Since U is not sheaf theoretically empty, there is at least one $i \in I$ such that U_i is not sheaf theoretically empty. Then the map $M \to \underline{M}(U_i)$ is injective (details omitted). We conclude that $x = \sum f_{i,j} x_j$ in M as desired.

Proof of (2). Assume $\underline{M}$ is a $\underline{\Lambda}$-module of finite presentation. By (1) we see that M is of finite type. Choose generators $x_1, \ldots, x_r$ of M as a Λ-module. This determines a short exact sequence $0 \to K \to \Lambda^{\oplus r} \to M \to 0$ which turns into a short exact sequence

$$0 \to \underline{K} \to \underline{\Lambda}^{\oplus r} \to \underline{M} \to 0$$

by Lemma 42.1. By Lemma 24.1 we see that $\underline{K}$ is of finite type. Hence K is a finite Λ-module by (1). Thus M is a Λ-module of finite presentation. $\square$

43. Locally constant sheaves

093P Here is the general definition.

093Q **Definition 43.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc.

(1) We say $\mathcal{F}$ is a *constant sheaf* of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc if it is isomorphic as a sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc to a constant sheaf $\underline{E}$ as in Section 42.
(2) We say $\mathcal{F}$ is *locally constant* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
(3) If $\mathcal{F}$ is a sheaf of sets or groups, then we say $\mathcal{F}$ is *finite locally constant* if the constant values are finite sets or finite groups.

093R **Lemma 43.2.** *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. If $\mathcal{G}$ is a locally constant sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc on $\mathcal{D}$, the same is true for $f^{-1}\mathcal{G}$ on $\mathcal{C}$.*

Proof. Omitted. $\square$

093S **Lemma 43.3.** *Let $\mathcal{C}$ be a site with a final object X.*

(1) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of sets on $\mathcal{C}$. If $\mathcal{F}$ is finite locally constant, there exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves associated to a map of sets.*
(2) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of abelian groups on $\mathcal{C}$. If $\mathcal{F}$ is finite locally constant, there exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant abelian sheaves associated to a map of abelian groups.*
(3) *Let Λ be a ring. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of Λ-modules on $\mathcal{C}$. If $\mathcal{F}$ is of finite type, then there exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves of Λ-modules associated to a map of Λ-modules.*

Proof. Proof omitted. $\square$

093T **Lemma 43.4.** *Let $\mathcal{C}$ be a site. Let Λ be a ring. Let M, N be Λ-modules. Let $\mathcal{F}, \mathcal{G}$ be a locally constant sheaves of Λ-modules.*

(1) *If M is of finite presentation, then*
$$\underline{\mathrm{Hom}_\Lambda(M, N)} = \mathcal{H}om_{\underline{\Lambda}}(\underline{M}, \underline{N})$$
(2) *If M and N are both of finite presentation, then*
$$\underline{Isom_\Lambda(M, N)} = Isom_{\underline{\Lambda}}(\underline{M}, \underline{N})$$
(3) *If $\mathcal{F}$ is of finite presentation, then $\mathcal{H}om_{\underline{\Lambda}}(\mathcal{F}, \mathcal{G})$ is a locally constant sheaf of Λ-modules.*

(4) *If $\mathcal{F}$ and $\mathcal{G}$ are both of finite presentation, then $Isom_{\underline{\Lambda}}(\mathcal{F}, \mathcal{G})$ is a locally constant sheaf of sets.*

Proof. Proof of (1). Set $E = \mathrm{Hom}_\Lambda(M, N)$. We want to show the canonical map

$$\underline{E} \longrightarrow \mathcal{H}om_{\underline{\Lambda}}(\underline{M}, \underline{N})$$

is an isomorphism. The module M has a presentation $\Lambda^{\oplus s} \to \Lambda^{\oplus t} \to M \to 0$. Then E sits in an exact sequence

$$0 \to E \to \mathrm{Hom}_\Lambda(\Lambda^{\oplus t}, N) \to \mathrm{Hom}_\Lambda(\Lambda^{\oplus s}, N)$$

and we have similarly

$$0 \to \mathcal{H}om_{\underline{\Lambda}}(\underline{M}, \underline{N}) \to \mathcal{H}om_{\underline{\Lambda}}(\underline{\Lambda^{\oplus t}}, \underline{N}) \to \mathcal{H}om_{\underline{\Lambda}}(\underline{\Lambda^{\oplus s}}, \underline{N})$$

This reduces the question to the case where M is a finite free module where the result is clear.

Proof of (3). The question is local on $\mathcal{C}$, hence we may assume $\mathcal{F} = \underline{M}$ and $\mathcal{G} = \underline{N}$ for some Λ-modules M and N. By Lemma 42.5 the module M is of finite presentation. Thus the result follows from (1).

Parts (2) and (4) follow from parts (1) and (3) and the fact that *Isom* can be viewed as the subsheaf of sections of $\mathcal{H}om_{\underline{\Lambda}}(\mathcal{F}, \mathcal{G})$ which have an inverse in $\mathcal{H}om_{\underline{\Lambda}}(\mathcal{G}, \mathcal{F})$. □

093U **Lemma 43.5.** *Let $\mathcal{C}$ be a site.*

(1) *The category of finite locally constant sheaves of sets is closed under finite limits and colimits inside $Sh(\mathcal{C})$.*
(2) *The category of finite locally constant abelian sheaves is a weak Serre subcategory of $Ab(\mathcal{C})$.*
(3) *Let Λ be a Noetherian ring. The category of finite type, locally constant sheaves of Λ-modules on $\mathcal{C}$ is a weak Serre subcategory of $Mod(\mathcal{C}, \Lambda)$.*

Proof. Proof of (1). We may work locally on $\mathcal{C}$. Hence by Lemma 43.3 we may assume we are given a finite diagram of finite sets such that our diagram of sheaves is the associated diagram of constant sheaves. Then we just take the limit or colimit in the category of sets and take the associated constant sheaf. Some details omitted.

To prove (2) and (3) we use the criterion of Homology, Lemma 10.3. Existence of kernels and cokernels is argued in the same way as above. Of course, the reason for using a Noetherian ring in (3) is to assure us that the kernel of a map of finite Λ-modules is a finite Λ-module. To see that the category is closed under extensions (in the case of sheaves Λ-modules), assume given an extension of sheaves of Λ-modules

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

on $\mathcal{C}$ with $\mathcal{F}$, $\mathcal{G}$ finite type and locally constant. Localizing on $\mathcal{C}$ we may assume $\mathcal{F}$ and $\mathcal{G}$ are constant, i.e., we get

$$0 \to \underline{M} \to \mathcal{E} \to \underline{N} \to 0$$

for some Λ-modules M, N. Choose generators $y_1, \ldots, y_m$ of N, so that we get a short exact sequence $0 \to K \to \Lambda^{\oplus m} \to N \to 0$ of Λ-modules. Localizing further we may assume y_j lifts to a section s_j of $\mathcal{E}$. Thus we see that $\mathcal{E}$ is a pushout as in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \underline{K} \arrow[r] \arrow[d] & \underline{\Lambda^{\oplus m}} \arrow[r] \arrow[d] & \underline{N} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{M} \arrow[r] & \mathcal{E} \arrow[r] & \underline{N} \arrow[r] & 0 \end{tikzcd}$$

By Lemma 43.3 again (and the fact that K is a finite Λ-module as Λ is Noetherian) we see that the map $\underline{K} \to \underline{M}$ is locally constant, hence we conclude. $\square$

093V **Lemma 43.6.** *Let $\mathcal{C}$ be a site. Let Λ be a ring. The tensor product of two locally constant sheaves of Λ-modules on $\mathcal{C}$ is a locally constant sheaf of Λ-modules.*

Proof. Omitted. $\square$

44. Localizing sheaves of rings

0EMB Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{S} \subset \mathcal{O}$ be a sub-presheaf of sets such that for all $U \in \text{Ob}(\mathcal{C})$ the set $\mathcal{S}(U) \subset \mathcal{O}(U)$ is a multiplicative subset, see Algebra, Definition 9.1. In this case we can consider the presheaf of rings

$$\mathcal{S}^{-1}\mathcal{O} : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}(U).$$

The restriction mapping sends the section f/s, $f \in \mathcal{O}(U)$, $s \in \mathcal{S}(U)$ to $(f|_V)/(s|_V)$ for $V \to U$ in $\mathcal{C}$.

0EMC **Lemma 44.1.** *In the situation above the map to the sheafification*

$$\mathcal{O} \longrightarrow (\mathcal{S}^{-1}\mathcal{O})^\#$$

is a homomorphism of sheaves of rings with the following universal property: for any homomorphism of sheaves of rings $\mathcal{O} \to \mathcal{A}$ such that each local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{A}$ there exists a unique factorization $(\mathcal{S}^{-1}\mathcal{O})^\# \to \mathcal{A}$.

Proof. Omitted. $\square$

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{S} \subset \mathcal{O}$ be a sub-presheaf of sets such that for all $U \in \mathcal{C}$ the set $\mathcal{S}(U) \subset \mathcal{O}(U)$ is a multiplicative subset. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. In this case we can consider the presheaf of $\mathcal{S}^{-1}\mathcal{O}$-modules

$$\mathcal{S}^{-1}\mathcal{F} : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{F}(U).$$

The restriction mapping sends the section t/s, $t \in \mathcal{F}(U)$, $s \in \mathcal{S}(U)$ to $(t|_V)/(s|_V)$ if $V \to U$ is a morphism of $\mathcal{C}$. Then $\mathcal{S}^{-1}\mathcal{F}$ is a presheaf of $\mathcal{S}^{-1}\mathcal{O}$-modules.

0EMD **Lemma 44.2.** *In the situation above the map to the sheafification*

$$\mathcal{F} \longrightarrow (\mathcal{S}^{-1}\mathcal{F})^\#$$

has the following universal property: for any homomorphism of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ such that each local section of $\mathcal{S}$ acts invertibly on $\mathcal{G}$ there exists a unique factorization $(\mathcal{S}^{-1}\mathcal{F})^\# \to \mathcal{G}$. Moreover we have

$$(\mathcal{S}^{-1}\mathcal{F})^\# = (\mathcal{S}^{-1}\mathcal{O})^\# \otimes_\mathcal{O} \mathcal{F}$$

as sheaves of $(\mathcal{S}^{-1}\mathcal{O})^\#$-modules.

Proof. Omitted. $\square$

45. Sheaves of pointed sets

0F4H In this section we collect some facts about sheaves of pointed sets which we've previously mentioned only for abelian sheaves.

A pointed set is a pair $(S, 0)$ where S is a set and $0 \in S$ is an element of S. A morphism $(S, 0) \to (S', 0')$ of pointed sets is simply a map of sets $S \to S'$ sending 0 to $0'$. We'll abuse notation and say "let S be a pointed set" to mean S is endowed with a marked element $0 \in S$. A sheaf of pointed sets is the same thing as a sheaf of sets $\mathcal{F}$ endowed with a "marking" $0 : * \to \mathcal{F}$ where $*$ is the final sheaf (Sites, Example 10.2).

Given a morphism of sites or of topoi, there are pushforward and pullback functors on the categories of sheaves of pointed sets, see Sites, Section 44. These are constructed by taking the pushforward, resp. pullback of the underlying sheaf of sets and suitably marking it (using that the pullback of the final sheaf is the final sheaf).

Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor between sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated with u, see Sites, Lemma 21.1. Then g^{-1} on sheaves of pointed sets has an left adjoint $g_!$ as well. The construction of this functor is entirely analogous to the construction of $g_!$ on abelian sheaves in Section 16.

Similarly, if $j : \mathcal{C}/U \to \mathcal{C}$ is as in Section 19 then there is a left adjoint $j_!$ to the functor j^{-1} on sheaves of pointed sets

If we ever need these facts and constructions we will precisely state and prove here the corresponding lemmas.

46. Other chapters

References

[AGV71] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier, *Theorie de topos et cohomologie etale des schemas I, II, III*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer, 1971.

[DG67] Jean Dieudonné and Alexander Grothendieck, *Éléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32** (1961–1967).

[Ser55] Jean-Pierre Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), 197–278.