

# DERIVED CATEGORIES OF SPACES

08EY

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## 1. Introduction

08EZ In this chapter we discuss derived categories of modules on algebraic spaces. There do not seem to be good introductory references addressing this topic; it is covered

in the literature by referring to papers dealing with derived categories of modules on algebraic stacks, for example see [Ols07].

## 2. Conventions

- 08F0 If  $\mathcal{A}$  is an abelian category and  $M$  is an object of  $\mathcal{A}$  then we also denote  $M$  the object of  $K(\mathcal{A})$  and/or  $D(\mathcal{A})$  corresponding to the complex which has  $M$  in degree 0 and is zero in all other degrees.

If we have a ring  $A$ , then  $K(A)$  denotes the homotopy category of complexes of  $A$ -modules and  $D(A)$  the associated derived category. Similarly, if we have a ringed space  $(X, \mathcal{O}_X)$  the symbol  $K(\mathcal{O}_X)$  denotes the homotopy category of complexes of  $\mathcal{O}_X$ -modules and  $D(\mathcal{O}_X)$  the associated derived category.

## 3. Generalities

- 08GD In this section we put some general results on cohomology of unbounded complexes of modules on algebraic spaces.

- 08GE **Lemma 3.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Given an étale morphism  $V \rightarrow Y$ , set  $U = V \times_Y X$  and denote  $g : U \rightarrow V$  the projection morphism. Then  $(Rf_*E)|_V = Rg_*(E|_U)$  for  $E$  in  $D(\mathcal{O}_X)$ .*

**Proof.** Represent  $E$  by a K-injective complex  $\mathcal{I}^\bullet$  of  $\mathcal{O}_X$ -modules. Then  $Rf_*(E) = f_*\mathcal{I}^\bullet$  and  $Rg_*(E|_U) = g_*(\mathcal{I}^\bullet|_U)$  by Cohomology on Sites, Lemma 20.1. Hence the result follows from Properties of Spaces, Lemma 26.2.  $\square$

- 08GF **Definition 3.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $E$  be an object of  $D(\mathcal{O}_X)$ . Let  $T \subset |X|$  be a closed subset. We say  $E$  is *supported on*  $T$  if the cohomology sheaves  $H^i(E)$  are supported on  $T$ .

## 4. Derived category of quasi-coherent modules on the small étale site

- 071P Let  $X$  be a scheme. In this section we show that  $D_{QCoh}(\mathcal{O}_X)$  can be defined in terms of the small étale site  $X_{\text{étale}}$  of  $X$ . Denote  $\mathcal{O}_{\text{étale}}$  the structure sheaf on  $X_{\text{étale}}$ . Consider the morphism of ringed sites

$$08H7 \quad (4.0.1) \quad \epsilon : (X_{\text{étale}}, \mathcal{O}_{\text{étale}}) \longrightarrow (X_{\text{Zar}}, \mathcal{O}_X).$$

denoted  $\text{id}_{\text{small}, \text{étale}, \text{Zar}}$  in Descent, Lemma 8.5.

- 08H8 **Lemma 4.1.** *The morphism  $\epsilon$  of (4.0.1) is a flat morphism of ringed sites. In particular the functor  $\epsilon^* : \text{Mod}(\mathcal{O}_X) \rightarrow \text{Mod}(\mathcal{O}_{\text{étale}})$  is exact. Moreover, if  $\epsilon^*\mathcal{F} = 0$ , then  $\mathcal{F} = 0$ .*

**Proof.** The flatness of the morphism  $\epsilon$  is Descent, Lemma 10.1. Here is another proof. We have to show that  $\mathcal{O}_{\text{étale}}$  is a flat  $\epsilon^{-1}\mathcal{O}_X$ -module. To do this it suffices to check  $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{\text{étale},\bar{x}}$  is flat for any geometric point  $\bar{x}$  of  $X$ , see Modules on Sites, Lemma 39.3, Sites, Lemma 34.2, and Étale Cohomology, Remarks 29.11. By Étale Cohomology, Lemma 33.1 we see that  $\mathcal{O}_{\text{étale},\bar{x}}$  is the strict henselization of  $\mathcal{O}_{X,x}$ . Thus  $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{\text{étale},\bar{x}}$  is faithfully flat by More on Algebra, Lemma 45.1.

The exactness of  $\epsilon^*$  follows from the flatness of  $\epsilon$  by Modules on Sites, Lemma 31.2.

Let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module. If  $\epsilon^*\mathcal{F} = 0$ , then with notation as above

$$0 = \epsilon^*\mathcal{F}_{\bar{x}} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{\text{étale},\bar{x}}$$

(Modules on Sites, Lemma 36.4) for all geometric points  $\bar{x}$ . By faithful flatness of  $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{\acute{e}tale,\bar{x}}$  we conclude  $\mathcal{F}_x = 0$  for all  $x \in X$ .  $\square$

Let  $X$  be a scheme. Notation as in (4.0.1). Recall that  $\epsilon^* : QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_{\acute{e}tale})$  is an equivalence by Descent, Proposition 8.9 and Remark 8.6. Moreover,  $QCoh(\mathcal{O}_{\acute{e}tale})$  forms a Serre subcategory of  $Mod(\mathcal{O}_{\acute{e}tale})$  by Descent, Lemma 10.2. Hence we can let  $D_{QCoh}(\mathcal{O}_{\acute{e}tale})$  be the triangulated subcategory of  $D(\mathcal{O}_{\acute{e}tale})$  whose objects are the complexes with quasi-coherent cohomology sheaves, see Derived Categories, Section 17. The functor  $\epsilon^*$  is exact (Lemma 4.1) hence induces  $\epsilon^* : D(\mathcal{O}_X) \rightarrow D(\mathcal{O}_{\acute{e}tale})$  and since pullbacks of quasi-coherent modules are quasi-coherent also  $\epsilon^* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_{\acute{e}tale})$ .

071Q **Lemma 4.2.** *Let  $X$  be a scheme. The functor  $\epsilon^* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_{\acute{e}tale})$  defined above is an equivalence.*

**Proof.** We will prove this by showing the functor  $R\epsilon_* : D(\mathcal{O}_{\acute{e}tale}) \rightarrow D(\mathcal{O}_X)$  induces a quasi-inverse. We will use freely that  $\epsilon_*$  is given by restriction to  $X_{Zar} \subset X_{\acute{e}tale}$  and the description of  $\epsilon^* = \text{id}_{small,\acute{e}tale,Zar}^*$  in Descent, Lemma 8.5.

For a quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  the adjunction map  $\mathcal{F} \rightarrow \epsilon_*\epsilon^*\mathcal{F}$  is an isomorphism by the fact that  $\mathcal{F}^a$  (Descent, Definition 8.2) is a sheaf as proved in Descent, Lemma 8.1. Conversely, every quasi-coherent  $\mathcal{O}_{\acute{e}tale}$ -module  $\mathcal{H}$  is of the form  $\epsilon^*\mathcal{F}$  for some quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$ , see Descent, Proposition 8.9. Then  $\mathcal{F} = \epsilon_*\mathcal{H}$  by what we just said and we conclude that the adjunction map  $\epsilon^*\epsilon_*\mathcal{H} \rightarrow \mathcal{H}$  is an isomorphism for all quasi-coherent  $\mathcal{O}_{\acute{e}tale}$ -modules  $\mathcal{H}$ .

Let  $E$  be an object of  $D_{QCoh}(\mathcal{O}_{\acute{e}tale})$  and denote  $\mathcal{H}^q = H^q(E)$  its  $q$ th cohomology sheaf. Let  $\mathcal{B}$  be the set of affine objects of  $X_{\acute{e}tale}$ . Then  $H^p(U, \mathcal{H}^q) = 0$  for all  $p > 0$ , all  $q \in \mathbf{Z}$ , and all  $U \in \mathcal{B}$ , see Descent, Proposition 9.3 and Cohomology of Schemes, Lemma 2.2. By Cohomology on Sites, Lemma 23.11 this means that

$$H^q(U, E) = H^0(U, \mathcal{H}^q)$$

for all  $U \in \mathcal{B}$ . In particular, we find that this holds for affine opens  $U \subset X$ . It follows that the  $q$ th cohomology of  $R\epsilon_*E$  over  $U$  is the value of the sheaf  $\epsilon_*\mathcal{H}^q$  over  $U$ . Applying sheafification we obtain

$$H^q(R\epsilon_*E) = \epsilon_*\mathcal{H}^q$$

which in particular shows that  $R\epsilon_*$  induces a functor  $D_{QCoh}(\mathcal{O}_{\acute{e}tale}) \rightarrow D_{QCoh}(\mathcal{O}_X)$ . Since  $\epsilon^*$  is exact we then obtain  $H^q(\epsilon^*R\epsilon_*E) = \epsilon^*\epsilon_*\mathcal{H}^q = \mathcal{H}^q$  (by discussion above). Thus the adjunction map  $\epsilon^*R\epsilon_*E \rightarrow E$  is an isomorphism.

Conversely, for  $F \in D_{QCoh}(\mathcal{O}_X)$  the adjunction map  $F \rightarrow R\epsilon_*\epsilon^*F$  is an isomorphism for the same reason, i.e., because the cohomology sheaves of  $R\epsilon_*\epsilon^*F$  are isomorphic to  $\epsilon_*H^m(\epsilon^*F) = \epsilon_*\epsilon^*H^m(F) = H^m(F)$ .  $\square$

## 5. Derived category of quasi-coherent modules

071W Let  $S$  be a scheme. Lemma 4.2 shows that the category  $D_{QCoh}(\mathcal{O}_S)$  can be defined in terms of complexes of  $\mathcal{O}_S$ -modules on the scheme  $S$  or by complexes of  $\mathcal{O}$ -modules on the small étale site of  $S$ . Hence the following definition is compatible with the definition in the case of schemes.

071X **Definition 5.1.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The derived category of  $\mathcal{O}_X$ -modules with quasi-coherent cohomology sheaves is denoted  $D_{QCoh}(\mathcal{O}_X)$ .

This makes sense by Properties of Spaces, Lemma 29.7 and Derived Categories, Section 17. Thus we obtain a canonical functor

$$08F1 \quad (5.1.1) \quad D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

see Derived Categories, Equation (17.1.1).

Observe that a flat morphism  $f : Y \rightarrow X$  of algebraic spaces induces an exact functor  $f^* : Mod(\mathcal{O}_X) \rightarrow Mod(\mathcal{O}_Y)$ , see Morphisms of Spaces, Lemma 30.9 and Modules on Sites, Lemma 31.2. In particular  $Lf^* : D(\mathcal{O}_X) \rightarrow D(\mathcal{O}_Y)$  is computed on any representative complex (Derived Categories, Lemma 16.9). We will write  $Lf^* = f^*$  when  $f$  is flat and we have  $H^i(f^*E) = f^*H^i(E)$  for  $E$  in  $D(\mathcal{O}_X)$  in this case. We will use this often when  $f$  is étale. Of course in the étale case the pullback functor is just the restriction to  $Y_{\text{étale}}$ , see Properties of Spaces, Equation (26.1.1).

08F2 **Lemma 5.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $E$  be an object of  $D(\mathcal{O}_X)$ . The following are equivalent

- (1)  $E$  is in  $D_{QCoh}(\mathcal{O}_X)$ ,
- (2) for every étale morphism  $\varphi : U \rightarrow X$  where  $U$  is an affine scheme  $\varphi^*E$  is an object of  $D_{QCoh}(\mathcal{O}_U)$ ,
- (3) for every étale morphism  $\varphi : U \rightarrow X$  where  $U$  is a scheme  $\varphi^*E$  is an object of  $D_{QCoh}(\mathcal{O}_U)$ ,
- (4) there exists a surjective étale morphism  $\varphi : U \rightarrow X$  where  $U$  is a scheme such that  $\varphi^*E$  is an object of  $D_{QCoh}(\mathcal{O}_U)$ , and
- (5) there exists a surjective étale morphism of algebraic spaces  $f : Y \rightarrow X$  such that  $Lf^*E$  is an object of  $D_{QCoh}(\mathcal{O}_Y)$ .

**Proof.** This follows immediately from the discussion preceding the lemma and Properties of Spaces, Lemma 29.6.  $\square$

08F3 **Lemma 5.3.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Then  $D_{QCoh}(\mathcal{O}_X)$  has direct sums.

**Proof.** By Injectives, Lemma 13.4 the derived category  $D(\mathcal{O}_X)$  has direct sums and they are computed by taking termwise direct sums of any representatives. Thus it is clear that the cohomology sheaf of a direct sum is the direct sum of the cohomology sheaves as taking direct sums is an exact functor (in any Grothendieck abelian category). The lemma follows as the direct sum of quasi-coherent sheaves is quasi-coherent, see Properties of Spaces, Lemma 29.7.  $\square$

We will need some information on derived limits. We warn the reader that in the lemma below the derived limit will typically not be an object of  $D_{QCoh}$ .

0D3E **Lemma 5.4.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $(K_n)$  be an inverse system of  $D_{QCoh}(\mathcal{O}_X)$  with derived limit  $K = R\lim K_n$  in  $D(\mathcal{O}_X)$ . Assume  $H^q(K_{n+1}) \rightarrow H^q(K_n)$  is surjective for all  $q \in \mathbf{Z}$  and  $n \geq 1$ . Then

- (1)  $H^q(K) = \lim H^q(K_n)$ ,
- (2)  $R\lim H^q(K_n) = \lim H^q(K_n)$ , and
- (3) for every affine open  $U \subset X$  we have  $H^p(U, \lim H^q(K_n)) = 0$  for  $p > 0$ .

**Proof.** Let  $\mathcal{B} \subset \text{Ob}(X_{\text{étale}})$  be the set of affine objects. Since  $H^q(K_n)$  is quasi-coherent we have  $H^p(U, H^q(K_n)) = 0$  for  $U \in \mathcal{B}$  by the discussion in Cohomology of Spaces, Section 3 and Cohomology of Schemes, Lemma 2.2. Moreover, the maps  $H^0(U, H^q(K_{n+1})) \rightarrow H^0(U, H^q(K_n))$  are surjective for  $U \in \mathcal{B}$  by similar reasoning. Part (1) follows from Cohomology on Sites, Lemma 23.12 whose conditions we have just verified. Parts (2) and (3) follow from Cohomology on Sites, Lemma 23.5.  $\square$

08F4 **Lemma 5.5.** *Let  $S$  be a scheme. Let  $f : Y \rightarrow X$  be a morphism of algebraic spaces over  $S$ . The functor  $Lf^*$  sends  $D_{QCoh}(\mathcal{O}_X)$  into  $D_{QCoh}(\mathcal{O}_Y)$ .*

**Proof.** Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ a \downarrow & & \downarrow b \\ X & \xrightarrow{f} & Y \end{array}$$

where  $U$  and  $V$  are schemes, the vertical arrows are étale, and  $a$  is surjective. Since  $a^* \circ Lf^* = Lh^* \circ b^*$  the result follows from Lemma 5.2 and the case of schemes which is Derived Categories of Schemes, Lemma 3.8.  $\square$

08F5 **Lemma 5.6.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . For objects  $K, L$  of  $D_{QCoh}(\mathcal{O}_X)$  the derived tensor product  $K \otimes^{\mathbf{L}} L$  is in  $D_{QCoh}(\mathcal{O}_X)$ .*

**Proof.** Let  $\varphi : U \rightarrow X$  be a surjective étale morphism from a scheme  $U$ . Since  $\varphi^*(K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L) = \varphi^*K \otimes_{\mathcal{O}_U}^{\mathbf{L}} \varphi^*L$  we see from Lemma 5.2 that this follows from the case of schemes which is Derived Categories of Schemes, Lemma 3.9.  $\square$

The following lemma will help us to “compute” a right derived functor on an object of  $D_{QCoh}(\mathcal{O}_X)$ .

08F6 **Lemma 5.7.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $E$  be an object of  $D_{QCoh}(\mathcal{O}_X)$ . Then the map  $E \rightarrow R\lim \tau_{\geq -n} E$  of Derived Categories, Remark 34.4 is an isomorphism<sup>1</sup>.*

**Proof.** Denote  $\mathcal{H}^i = H^i(E)$  the  $i$ th cohomology sheaf of  $E$ . Let  $\mathcal{B}$  be the set of affine objects of  $X_{\text{étale}}$ . Then  $H^p(U, \mathcal{H}^i) = 0$  for all  $p > 0$ , all  $i \in \mathbf{Z}$ , and all  $U \in \mathcal{B}$  as  $U$  is an affine scheme. See discussion in Cohomology of Spaces, Section 3 and Cohomology of Schemes, Lemma 2.2. Thus the lemma follows from Cohomology on Sites, Lemma 23.10 with  $d = 0$ .  $\square$

08F7 **Lemma 5.8.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $F : \text{Mod}(\mathcal{O}_X) \rightarrow \text{Ab}$  be a functor and  $N \geq 0$  an integer. Assume that*

- (1)  $F$  is left exact,
- (2)  $F$  commutes with countable direct products,
- (3)  $R^p F(\mathcal{F}) = 0$  for all  $p \geq N$  and  $\mathcal{F}$  quasi-coherent.

Then for  $E \in D_{QCoh}(\mathcal{O}_X)$

- (1)  $H^i(RF(\tau_{\leq a} E)) \rightarrow H^i(RF(E))$  is an isomorphism for  $i \leq a$ ,
- (2)  $H^i(RF(E)) \rightarrow H^i(RF(\tau_{\geq b-N+1} E))$  is an isomorphism for  $i \geq b$ ,
- (3) if  $H^i(E) = 0$  for  $i \notin [a, b]$  for some  $-\infty \leq a \leq b \leq \infty$ , then  $H^i(RF(E)) = 0$  for  $i \notin [a, b + N - 1]$ .

<sup>1</sup>In particular,  $E$  has a K-injective representative, see Derived Categories, Lemma 34.5.

**Proof.** Statement (1) is Derived Categories, Lemma 16.1.

Proof of statement (2). Write  $E_n = \tau_{\geq -n} E$ . We have  $E = R \lim E_n$ , see Lemma 5.7. Thus  $RF(E) = R \lim RF(E_n)$  in  $D(Ab)$  by Injectives, Lemma 13.6. Thus for every  $i \in \mathbf{Z}$  we have a short exact sequence

$$0 \rightarrow R^1 \lim H^{i-1}(RF(E_n)) \rightarrow H^i(RF(E)) \rightarrow \lim H^i(RF(E_n)) \rightarrow 0$$

see More on Algebra, Remark 86.10. To prove (2) we will show that the term on the left is zero and that the term on the right equals  $H^i(RF(E_{-b+N-1}))$  for any  $b$  with  $i \geq b$ .

For every  $n$  we have a distinguished triangle

$$H^{-n}(E)[n] \rightarrow E_n \rightarrow E_{n-1} \rightarrow H^{-n}(E)[n+1]$$

(Derived Categories, Remark 12.4) in  $D(\mathcal{O}_X)$ . Since  $H^{-n}(E)$  is quasi-coherent we have

$$H^i(RF(H^{-n}(E)[n])) = R^{i+n}F(H^{-n}(E)) = 0$$

for  $i + n \geq N$  and

$$H^i(RF(H^{-n}(E)[n+1])) = R^{i+n+1}F(H^{-n}(E)) = 0$$

for  $i + n + 1 \geq N$ . We conclude that

$$H^i(RF(E_n)) \rightarrow H^i(RF(E_{n-1}))$$

is an isomorphism for  $n \geq N - i$ . Thus the systems  $H^i(RF(E_n))$  all satisfy the ML condition and the  $R^1 \lim$  term in our short exact sequence is zero (see discussion in More on Algebra, Section 86). Moreover, the system  $H^i(RF(E_n))$  is constant starting with  $n = N - i - 1$  as desired.

Proof of (3). Under the assumption on  $E$  we have  $\tau_{\leq a-1} E = 0$  and we get the vanishing of  $H^i(RF(E))$  for  $i \leq a - 1$  from (1). Similarly, we have  $\tau_{\geq b+1} E = 0$  and hence we get the vanishing of  $H^i(RF(E))$  for  $i \geq b + n$  from part (2).  $\square$

## 6. Total direct image

08F9 The following lemma is the analogue of Cohomology of Spaces, Lemma 8.1.

08FA **Lemma 6.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a quasi-separated and quasi-compact morphism of algebraic spaces over  $S$ .*

- (1) *The functor  $Rf_*$  sends  $D_{QCoh}(\mathcal{O}_X)$  into  $D_{QCoh}(\mathcal{O}_Y)$ .*
- (2) *If  $Y$  is quasi-compact, there exists an integer  $N = N(X, Y, f)$  such that for an object  $E$  of  $D_{QCoh}(\mathcal{O}_X)$  with  $H^m(E) = 0$  for  $m > 0$  we have  $H^m(Rf_* E) = 0$  for  $m \geq N$ .*
- (3) *In fact, if  $Y$  is quasi-compact we can find  $N = N(X, Y, f)$  such that for every morphism of algebraic spaces  $Y' \rightarrow Y$  the same conclusion holds for the functor  $R(f')_*$  where  $f' : X' \rightarrow Y'$  is the base change of  $f$ .*

**Proof.** Let  $E$  be an object of  $D_{QCoh}(\mathcal{O}_X)$ . To prove (1) we have to show that  $Rf_* E$  has quasi-coherent cohomology sheaves. This question is local on  $Y$ , hence we may assume  $Y$  is quasi-compact. Pick  $N = N(X, Y, f)$  as in Cohomology of Spaces, Lemma 8.1. Thus  $R^p f_* \mathcal{F} = 0$  for all quasi-coherent  $\mathcal{O}_X$ -modules  $\mathcal{F}$  and all  $p \geq N$ . Moreover  $R^p f_* \mathcal{F}$  is quasi-coherent for all  $p$  by Cohomology of Spaces, Lemma 3.1. These statements remain true after base change.

First, assume  $E$  is bounded below. We will show (1) and (2) and (3) hold for such  $E$  with our choice of  $N$ . In this case we can for example use the spectral sequence

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E$$

(Derived Categories, Lemma 21.3), the quasi-coherence of  $R^p f_* H^q(E)$ , and the vanishing of  $R^p f_* H^q(E)$  for  $p \geq N$  to see that (1), (2), and (3) hold in this case.

Next we prove (2) and (3). Say  $H^m(E) = 0$  for  $m > 0$ . Let  $V$  be an affine object of  $Y_{\text{étale}}$ . We have  $H^p(V \times_Y X, \mathcal{F}) = 0$  for  $p \geq N$ , see Cohomology of Spaces, Lemma 3.2. Hence we may apply Lemma 5.8 to the functor  $\Gamma(V \times_Y X, -)$  to see that

$$R\Gamma(V, Rf_* E) = R\Gamma(V \times_Y X, E)$$

has vanishing cohomology in degrees  $\geq N$ . Since this holds for all  $V$  affine in  $Y_{\text{étale}}$  we conclude that  $H^m(Rf_* E) = 0$  for  $m \geq N$ .

Next, we prove (1) in the general case. Recall that there is a distinguished triangle

$$\tau_{\leq -n-1} E \rightarrow E \rightarrow \tau_{\geq -n} E \rightarrow (\tau_{\leq -n-1} E)[1]$$

in  $D(\mathcal{O}_X)$ , see Derived Categories, Remark 12.4. By (2) we see that  $Rf_* \tau_{\leq -n-1} E$  has vanishing cohomology sheaves in degrees  $\geq -n+N$ . Thus, given an integer  $q$  we see that  $R^q f_* E$  is equal to  $R^q f_* \tau_{\geq -n} E$  for some  $n$  and the result above applies.  $\square$

08FB **Lemma 6.2.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a quasi-separated and quasi-compact morphism of algebraic spaces over  $S$ . Then  $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$  commutes with direct sums.*

**Proof.** Let  $E_i$  be a family of objects of  $D_{QCoh}(\mathcal{O}_X)$  and set  $E = \bigoplus E_i$ . We want to show that the map

$$\bigoplus Rf_* E_i \longrightarrow Rf_* E$$

is an isomorphism. We will show it induces an isomorphism on cohomology sheaves in degree 0 which will imply the lemma. Choose an integer  $N$  as in Lemma 6.1. Then  $R^0 f_* E = R^0 f_* \tau_{\geq -N} E$  and  $R^0 f_* E_i = R^0 f_* \tau_{\geq -N} E_i$  by the lemma cited. Observe that  $\tau_{\geq -N} E = \bigoplus \tau_{\geq -N} E_i$ . Thus we may assume all of the  $E_i$  have vanishing cohomology sheaves in degrees  $< -N$ . Next we use the spectral sequences

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E \quad \text{and} \quad R^p f_* H^q(E_i) \Rightarrow R^{p+q} f_* E_i$$

(Derived Categories, Lemma 21.3) to reduce to the case of a direct sum of quasi-coherent sheaves. This case is handled by Cohomology of Spaces, Lemma 5.2.  $\square$

08GH **Remark 6.3.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of representable algebraic spaces  $X$  and  $Y$  over  $S$ . Let  $f_0 : X_0 \rightarrow Y_0$  be a morphism of schemes representing  $f$  (awkward but temporary notation). Then the diagram

$$\begin{array}{ccc} D_{QCoh}(\mathcal{O}_{X_0}) & \xlongequal{\text{Lemma 4.2}} & D_{QCoh}(\mathcal{O}_X) \\ \uparrow Lf_0^* & & \uparrow Lf^* \\ D_{QCoh}(\mathcal{O}_{Y_0}) & \xlongequal{\text{Lemma 4.2}} & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

(Lemma 5.5 and Derived Categories of Schemes, Lemma 3.8) is commutative. This follows as the equivalences  $D_{QCoh}(\mathcal{O}_{X_0}) \rightarrow D_{QCoh}(\mathcal{O}_X)$  and  $D_{QCoh}(\mathcal{O}_{Y_0}) \rightarrow$

$D_{QCoh}(\mathcal{O}_Y)$  of Lemma 4.2 come from pulling back by the (flat) morphisms of ringed sites  $\epsilon : X_{\acute{e}tale} \rightarrow X_{0,Zar}$  and  $\epsilon : Y_{\acute{e}tale} \rightarrow Y_{0,Zar}$  and the diagram of ringed sites

$$\begin{array}{ccc} X_{0,Zar} & \xleftarrow{\epsilon} & X_{\acute{e}tale} \\ f_0 \downarrow & & \downarrow f \\ Y_{0,Zar} & \xleftarrow{\epsilon} & Y_{\acute{e}tale} \end{array}$$

is commutative (details omitted). If  $f$  is quasi-compact and quasi-separated, equivalently if  $f_0$  is quasi-compact and quasi-separated, then we claim

$$\begin{array}{ccc} D_{QCoh}(\mathcal{O}_{X_0}) & \xlongequal{\text{Lemma 4.2}} & D_{QCoh}(\mathcal{O}_X) \\ Rf_{0,*} \downarrow & & \downarrow Rf_* \\ D_{QCoh}(\mathcal{O}_{Y_0}) & \xlongequal{\text{Lemma 4.2}} & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

(Lemma 6.1 and Derived Categories of Schemes, Lemma 4.1) is commutative as well. This also follows from the commutative diagram of sites displayed above as the proof of Lemma 4.2 shows that the functor  $R\epsilon_*$  gives the equivalences  $D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_{X_0})$  and  $D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_{Y_0})$ .

08II **Lemma 6.4.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be an affine morphism of algebraic spaces over  $S$ . Then  $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$  reflects isomorphisms.*

**Proof.** The statement means that a morphism  $\alpha : E \rightarrow F$  of  $D_{QCoh}(\mathcal{O}_X)$  is an isomorphism if  $Rf_*\alpha$  is an isomorphism. We may check this on cohomology sheaves. In particular, the question is étale local on  $Y$ . Hence we may assume  $Y$  and therefore  $X$  is affine. In this case the problem reduces to the case of schemes (Derived Categories of Schemes, Lemma 5.2) via Lemma 4.2 and Remark 6.3.  $\square$

08IJ **Lemma 6.5.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be an affine morphism of algebraic spaces over  $S$ . For  $E$  in  $D_{QCoh}(\mathcal{O}_Y)$  we have  $Rf_*Lf^*E = E \otimes_{\mathcal{O}_Y}^L f_*\mathcal{O}_X$ .*

**Proof.** Since  $f$  is affine the map  $f_*\mathcal{O}_X \rightarrow Rf_*\mathcal{O}_X$  is an isomorphism (Cohomology of Spaces, Lemma 8.2). There is a canonical map  $E \otimes^L f_*\mathcal{O}_X = E \otimes^L Rf_*\mathcal{O}_X \rightarrow Rf_*Lf^*E$  adjoint to the map

$$Lf^*(E \otimes^L Rf_*\mathcal{O}_X) = Lf^*E \otimes^L Lf^*Rf_*\mathcal{O}_X \longrightarrow Lf^*E \otimes^L \mathcal{O}_X = Lf^*E$$

coming from  $1 : Lf^*E \rightarrow Lf^*E$  and the canonical map  $Lf^*Rf_*\mathcal{O}_X \rightarrow \mathcal{O}_X$ . To check the map so constructed is an isomorphism we may work locally on  $Y$ . Hence we may assume  $Y$  and therefore  $X$  is affine. In this case the problem reduces to the case of schemes (Derived Categories of Schemes, Lemma 5.3) via Lemma 4.2 and Remark 6.3.  $\square$

## 7. Being proper over a base

0CZB This section is the analogue of Cohomology of Schemes, Section 26. As usual with material having to do with topology on the sets of points, we have to be careful translating the material to algebraic spaces.

0CZC **Lemma 7.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $T \subset |X|$  be a closed subset. The following are equivalent*

- (1) *the morphism  $Z \rightarrow Y$  is proper if  $Z$  is the reduced induced algebraic space structure on  $T$  (Properties of Spaces, Definition 12.5),*
- (2) *for some closed subspace  $Z \subset X$  with  $|Z| = T$  the morphism  $Z \rightarrow Y$  is proper, and*
- (3) *for any closed subspace  $Z \subset X$  with  $|Z| = T$  the morphism  $Z \rightarrow Y$  is proper.*

**Proof.** The implications (3)  $\Rightarrow$  (1) and (1)  $\Rightarrow$  (2) are immediate. Thus it suffices to prove that (2) implies (3). We urge the reader to find their own proof of this fact. Let  $Z'$  and  $Z''$  be closed subspaces with  $T = |Z'| = |Z''|$  such that  $Z' \rightarrow Y$  is a proper morphism of algebraic spaces. We have to show that  $Z'' \rightarrow Y$  is proper too. Let  $Z''' = Z' \cup Z''$  be the scheme theoretic union, see Morphisms of Spaces, Definition 14.4. Then  $Z'''$  is another closed subspace with  $|Z'''| = T$ . This follows for example from the description of scheme theoretic unions in Morphisms of Spaces, Lemma 14.6. Since  $Z'' \rightarrow Z'''$  is a closed immersion it suffices to prove that  $Z''' \rightarrow Y$  is proper (see Morphisms of Spaces, Lemmas 40.5 and 40.4). The morphism  $Z' \rightarrow Z'''$  is a bijective closed immersion and in particular surjective and universally closed. Then the fact that  $Z' \rightarrow Y$  is separated implies that  $Z''' \rightarrow Y$  is separated, see Morphisms of Spaces, Lemma 9.8. Moreover  $Z''' \rightarrow Y$  is locally of finite type as  $X \rightarrow Y$  is locally of finite type (Morphisms of Spaces, Lemmas 23.7 and 23.2). Since  $Z' \rightarrow Y$  is quasi-compact and  $Z' \rightarrow Z'''$  is a universal homeomorphism we see that  $Z''' \rightarrow Y$  is quasi-compact. Finally, since  $Z' \rightarrow Y$  is universally closed, we see that the same thing is true for  $Z''' \rightarrow Y$  by Morphisms of Spaces, Lemma 40.7. This finishes the proof.  $\square$

0CZD **Definition 7.2.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $T \subset |X|$  be a closed subset. We say  $T$  is *proper over  $Y$*  if the equivalent conditions of Lemma 7.1 are satisfied.

The lemma used in the definition above is false if the morphism  $f : X \rightarrow Y$  is not locally of finite type. Therefore we urge the reader not to use this terminology if  $f$  is not locally of finite type.

0CZE **Lemma 7.3.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $T' \subset T \subset |X|$  be closed subsets. If  $T$  is proper over  $Y$ , then the same is true for  $T'$ .

**Proof.** Omitted.  $\square$

0CZF **Lemma 7.4.** Let  $S$  be a scheme. Consider a cartesian diagram of algebraic spaces over  $S$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

with  $f$  locally of finite type. If  $T$  is a closed subset of  $|X|$  proper over  $Y$ , then  $|g'|^{-1}(T)$  is a closed subset of  $|X'|$  proper over  $Y'$ .

**Proof.** Observe that the statement makes sense as  $f'$  is locally of finite type by Morphisms of Spaces, Lemma 23.3. Let  $Z \subset X$  be the reduced induced closed subspace structure on  $T$ . Denote  $Z' = (g')^{-1}(Z)$  the scheme theoretic inverse image. Then  $Z' = X' \times_X Z = (Y' \times_Y X) \times_X Z = Y' \times_Y Z$  is proper over  $Y'$  as a

base change of  $Z$  over  $Y$  (Morphisms of Spaces, Lemma 40.3). On the other hand, we have  $T' = |Z'|$ . Hence the lemma holds.  $\square$

0CZG **Lemma 7.5.** *Let  $S$  be a scheme. Let  $B$  be an algebraic space over  $S$ . Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces which are locally of finite type over  $B$ .*

- (1) *If  $Y$  is separated over  $B$  and  $T \subset |X|$  is a closed subset proper over  $B$ , then  $|f|(T)$  is a closed subset of  $|Y|$  proper over  $B$ .*
- (2) *If  $f$  is universally closed and  $T \subset |X|$  is a closed subset proper over  $B$ , then  $|f|(T)$  is a closed subset of  $Y$  proper over  $B$ .*
- (3) *If  $f$  is proper and  $T \subset |Y|$  is a closed subset proper over  $B$ , then  $|f|^{-1}(T)$  is a closed subset of  $|X|$  proper over  $B$ .*

**Proof.** Proof of (1). Assume  $Y$  is separated over  $B$  and  $T \subset |X|$  is a closed subset proper over  $B$ . Let  $Z$  be the reduced induced closed subspace structure on  $T$  and apply Morphisms of Spaces, Lemma 40.8 to  $Z \rightarrow Y$  over  $B$  to conclude.

Proof of (2). Assume  $f$  is universally closed and  $T \subset |X|$  is a closed subset proper over  $B$ . Let  $Z$  be the reduced induced closed subspace structure on  $T$  and let  $Z'$  be the reduced induced closed subspace structure on  $|f|(T)$ . We obtain an induced morphism  $Z \rightarrow Z'$ . Denote  $Z'' = f^{-1}(Z')$  the scheme theoretic inverse image. Then  $Z'' \rightarrow Z'$  is universally closed as a base change of  $f$  (Morphisms of Spaces, Lemma 40.3). Hence  $Z \rightarrow Z'$  is universally closed as a composition of the closed immersion  $Z \rightarrow Z''$  and  $Z'' \rightarrow Z'$  (Morphisms of Spaces, Lemmas 40.5 and 40.4). We conclude that  $Z' \rightarrow B$  is separated by Morphisms of Spaces, Lemma 9.8. Since  $Z \rightarrow B$  is quasi-compact and  $Z \rightarrow Z'$  is surjective we see that  $Z' \rightarrow B$  is quasi-compact. Since  $Z' \rightarrow B$  is the composition of  $Z' \rightarrow Y$  and  $Y \rightarrow B$  we see that  $Z' \rightarrow B$  is locally of finite type (Morphisms of Spaces, Lemmas 23.7 and 23.2). Finally, since  $Z \rightarrow B$  is universally closed, we see that the same thing is true for  $Z' \rightarrow B$  by Morphisms of Spaces, Lemma 40.7. This finishes the proof.

Proof of (3). Assume  $f$  is proper and  $T \subset |Y|$  is a closed subset proper over  $B$ . Let  $Z$  be the reduced induced closed subspace structure on  $T$ . Denote  $Z' = f^{-1}(Z)$  the scheme theoretic inverse image. Then  $Z' \rightarrow Z$  is proper as a base change of  $f$  (Morphisms of Spaces, Lemma 40.3). Whence  $Z' \rightarrow B$  is proper as the composition of  $Z' \rightarrow Z$  and  $Z \rightarrow B$  (Morphisms of Spaces, Lemma 40.4). This finishes the proof.  $\square$

0CZH **Lemma 7.6.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $T_i \subset |X|$ ,  $i = 1, \dots, n$  be closed subsets. If  $T_i$ ,  $i = 1, \dots, n$  are proper over  $Y$ , then the same is true for  $T_1 \cup \dots \cup T_n$ .*

**Proof.** Let  $Z_i$  be the reduced induced closed subscheme structure on  $T_i$ . The morphism

$$Z_1 \amalg \dots \amalg Z_n \longrightarrow X$$

is finite by Morphisms of Spaces, Lemmas 45.10 and 45.11. As finite morphisms are universally closed (Morphisms of Spaces, Lemma 45.9) and since  $Z_1 \amalg \dots \amalg Z_n$  is proper over  $S$  we conclude by Lemma 7.5 part (2) that the image  $Z_1 \cup \dots \cup Z_n$  is proper over  $S$ .  $\square$

Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $\mathcal{F}$  be a finite type, quasi-coherent  $\mathcal{O}_X$ -module.

Then the support  $\text{Supp}(\mathcal{F})$  of  $\mathcal{F}$  is a closed subset of  $|X|$ , see Morphisms of Spaces, Lemma 15.2. Hence it makes sense to say “the support of  $\mathcal{F}$  is proper over  $Y$ ”.

0CZI **Lemma 7.7.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $\mathcal{F}$  be a finite type, quasi-coherent  $\mathcal{O}_X$ -module. The following are equivalent*

- (1) *the support of  $\mathcal{F}$  is proper over  $Y$ ,*
- (2) *the scheme theoretic support of  $\mathcal{F}$  (Morphisms of Spaces, Definition 15.4) is proper over  $Y$ , and*
- (3) *there exists a closed subspace  $Z \subset X$  and a finite type, quasi-coherent  $\mathcal{O}_Z$ -module  $\mathcal{G}$  such that (a)  $Z \rightarrow Y$  is proper, and (b)  $(Z \rightarrow X)_*\mathcal{G} = \mathcal{F}$ .*

**Proof.** The support  $\text{Supp}(\mathcal{F})$  of  $\mathcal{F}$  is a closed subset of  $|X|$ , see Morphisms of Spaces, Lemma 15.2. Hence we can apply Definition 7.2. Since the scheme theoretic support of  $\mathcal{F}$  is a closed subspace whose underlying closed subset is  $\text{Supp}(\mathcal{F})$  we see that (1) and (2) are equivalent by Definition 7.2. It is clear that (2) implies (3). Conversely, if (3) is true, then  $\text{Supp}(\mathcal{F}) \subset |Z|$  and hence  $\text{Supp}(\mathcal{F})$  is proper over  $Y$  for example by Lemma 7.3.  $\square$

0CZJ **Lemma 7.8.** *Let  $S$  be a scheme. Consider a cartesian diagram of algebraic spaces over  $S$*

$$\begin{array}{ccc} X' & \xrightarrow{\quad g' \quad} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{\quad g \quad} & Y \end{array}$$

*with  $f$  locally of finite type. Let  $\mathcal{F}$  be a finite type, quasi-coherent  $\mathcal{O}_X$ -module. If the support of  $\mathcal{F}$  is proper over  $Y$ , then the support of  $(g')^*\mathcal{F}$  is proper over  $Y'$ .*

**Proof.** Observe that the statement makes sense because  $(g')^*\mathcal{F}$  is of finite type by Modules on Sites, Lemma 23.4. We have  $\text{Supp}((g')^*\mathcal{F}) = |g'|^{-1}(\text{Supp}(\mathcal{F}))$  by Morphisms of Spaces, Lemma 15.2. Thus the lemma follows from Lemma 7.4.  $\square$

0CZK **Lemma 7.9.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $\mathcal{F}, \mathcal{G}$  be finite type, quasi-coherent  $\mathcal{O}_X$ -module.*

- (1) *If the supports of  $\mathcal{F}, \mathcal{G}$  are proper over  $Y$ , then the same is true for  $\mathcal{F} \oplus \mathcal{G}$ , for any extension of  $\mathcal{G}$  by  $\mathcal{F}$ , for  $\text{Im}(u)$  and  $\text{Coker}(u)$  given any  $\mathcal{O}_X$ -module map  $u : \mathcal{F} \rightarrow \mathcal{G}$ , and for any quasi-coherent quotient of  $\mathcal{F}$  or  $\mathcal{G}$ .*
- (2) *If  $Y$  is locally Noetherian, then the category of coherent  $\mathcal{O}_X$ -modules with support proper over  $Y$  is a Serre subcategory (Homology, Definition 10.1) of the abelian category of coherent  $\mathcal{O}_X$ -modules.*

**Proof.** Proof of (1). Let  $T, T'$  be the support of  $\mathcal{F}$  and  $\mathcal{G}$ . Then all the sheaves mentioned in (1) have support contained in  $T \cup T'$ . Thus the assertion itself is clear from Lemmas 7.3 and 7.6 provided we check that these sheaves are finite type and quasi-coherent. For quasi-coherence we refer the reader to Properties of Spaces, Section 29. For “finite type” we refer the reader to Properties of Spaces, Section 30.

Proof of (2). The proof is the same as the proof of (1). Note that the assertions make sense as  $X$  is locally Noetherian by Morphisms of Spaces, Lemma 23.5 and

by the description of the category of coherent modules in Cohomology of Spaces, Section 12.  $\square$

08GC **Lemma 7.10.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Assume  $f$  is locally of finite type and  $Y$  locally Noetherian. Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module with support proper over  $Y$ . Then  $R^p f_* \mathcal{F}$  is a coherent  $\mathcal{O}_Y$ -module for all  $p \geq 0$ .*

**Proof.** By Lemma 7.7 there exists a closed immersion  $i : Z \rightarrow X$  with  $g = f \circ i : Z \rightarrow Y$  proper and  $\mathcal{F} = i_* \mathcal{G}$  for some coherent module  $\mathcal{G}$  on  $Z$ . We see that  $R^p g_* \mathcal{G}$  is coherent on  $S$  by Cohomology of Spaces, Lemma 20.2. On the other hand,  $R^q i_* \mathcal{G} = 0$  for  $q > 0$  (Cohomology of Spaces, Lemma 12.9). By Cohomology on Sites, Lemma 14.7 we get  $R^p f_* \mathcal{F} = R^p g_* \mathcal{G}$  and the lemma follows.  $\square$

## 8. Derived category of coherent modules

08GI Let  $S$  be a scheme. Let  $X$  be a locally Noetherian algebraic space over  $S$ . In this case the category  $\text{Coh}(\mathcal{O}_X) \subset \text{Mod}(\mathcal{O}_X)$  of coherent  $\mathcal{O}_X$ -modules is a weak Serre subcategory, see Homology, Section 10 and Cohomology of Spaces, Lemma 12.3. Denote

$$D_{\text{Coh}}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are coherent, see Derived Categories, Section 17. Thus we obtain a canonical functor

08GJ (8.0.1) 
$$D(\text{Coh}(\mathcal{O}_X)) \longrightarrow D_{\text{Coh}}(\mathcal{O}_X)$$

see Derived Categories, Equation (17.1.1).

08GK **Lemma 8.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Assume  $f$  is locally of finite type and  $Y$  is Noetherian. Let  $E$  be an object of  $D_{\text{Coh}}^b(\mathcal{O}_X)$  such that the support of  $H^i(E)$  is proper over  $Y$  for all  $i$ . Then  $Rf_* E$  is an object of  $D_{\text{Coh}}^b(\mathcal{O}_Y)$ .*

**Proof.** Consider the spectral sequence

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E$$

see Derived Categories, Lemma 21.3. By assumption and Lemma 7.10 the sheaves  $R^p f_* H^q(E)$  are coherent. Hence  $R^{p+q} f_* E$  is coherent, i.e.,  $E \in D_{\text{Coh}}(\mathcal{O}_Y)$ . Boundedness from below is trivial. Boundedness from above follows from Cohomology of Spaces, Lemma 8.1 or from Lemma 6.1.  $\square$

0D0R **Lemma 8.2.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Assume  $f$  is locally of finite type and  $Y$  is Noetherian. Let  $E$  be an object of  $D_{\text{Coh}}^+(\mathcal{O}_X)$  such that the support of  $H^i(E)$  is proper over  $S$  for all  $i$ . Then  $Rf_* E$  is an object of  $D_{\text{Coh}}^+(\mathcal{O}_Y)$ .*

**Proof.** The proof is the same as the proof of Lemma 8.1. You can also deduce it from Lemma 8.1 by considering what the exact functor  $Rf_*$  does to the distinguished triangles  $\tau_{\leq a} E \rightarrow E \rightarrow \tau_{\geq a+1} E \rightarrow \tau_{\leq a} E[1]$ .  $\square$

0D0S **Lemma 8.3.** *Let  $S$  be a scheme. Let  $X$  be a locally Noetherian algebraic space over  $S$ . If  $L$  is in  $D_{\text{Coh}}^+(\mathcal{O}_X)$  and  $K$  in  $D_{\text{Coh}}^-(\mathcal{O}_X)$ , then  $R\mathcal{H}om(K, L)$  is in  $D_{\text{Coh}}^+(\mathcal{O}_X)$ .*

**Proof.** We can check whether an object of  $D(\mathcal{O}_X)$  is in  $D_{\text{Coh}}(\mathcal{O}_X)$  étale locally on  $X$ , see Cohomology of Spaces, Lemma 12.2. Hence this lemma follows from the case of schemes, see Derived Categories of Schemes, Lemma 11.5.  $\square$

0D0T **Lemma 8.4.** *Let  $A$  be a Noetherian ring. Let  $X$  be a proper algebraic space over  $A$ . For  $L$  in  $D_{\text{Coh}}^+(\mathcal{O}_X)$  and  $K$  in  $D_{\text{Coh}}^-(\mathcal{O}_X)$ , the  $A$ -modules  $\text{Ext}_{\mathcal{O}_X}^n(K, L)$  are finite.*

**Proof.** Recall that

$$\text{Ext}_{\mathcal{O}_X}^n(K, L) = H^n(X, R\mathcal{H}om_{\mathcal{O}_X}(K, L)) = H^n(\text{Spec}(A), Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K, L))$$

see Cohomology on Sites, Lemma 35.1 and Cohomology on Sites, Section 14. Thus the result follows from Lemmas 8.3 and 8.2.  $\square$

## 9. Induction principle

08GL In this section we discuss an induction principle for algebraic spaces analogous to what is Cohomology of Schemes, Lemma 4.1 for schemes. To formulate it we introduce the notion of an *elementary distinguished square*; this terminology is borrowed from [MV99]. The principle as formulated here is implicit in the paper [GR71] by Raynaud and Gruson. A related principle for algebraic stacks is [Ryd10, Theorem D] by David Rydh.

08GM **Definition 9.1.** Let  $S$  be a scheme. A commutative diagram

$$\begin{array}{ccc} U \times_W V & \longrightarrow & V \\ \downarrow & & \downarrow f \\ U & \xrightarrow{j} & W \end{array}$$

of algebraic spaces over  $S$  is called an *elementary distinguished square* if

- (1)  $U$  is an open subspace of  $W$  and  $j$  is the inclusion morphism,
- (2)  $f$  is étale, and
- (3) setting  $T = W \setminus U$  (with reduced induced subspace structure) the morphism  $f^{-1}(T) \rightarrow T$  is an isomorphism.

We will indicate this by saying: “Let  $(U \subset W, f : V \rightarrow W)$  be an elementary distinguished square.”

Note that if  $(U \subset W, f : V \rightarrow W)$  is an elementary distinguished square, then we have  $W = U \cup f(V)$ . Thus  $\{U \rightarrow W, V \rightarrow W\}$  is an étale covering of  $W$ . It turns out that these étale coverings have nice properties and that in some sense there are “enough” of them.

08GN **Lemma 9.2.** *Let  $S$  be a scheme. Let  $(U \subset W, f : V \rightarrow W)$  be an elementary distinguished square of algebraic spaces over  $S$ .*

- (1) *If  $V' \subset V$  and  $U \subset U' \subset W$  are open subspaces and  $W' = U' \cup f(V')$  then  $(U' \subset W', f|_{V'} : V' \rightarrow W')$  is an elementary distinguished square.*
- (2) *If  $p : W' \rightarrow W$  is a morphism of algebraic spaces, then  $(p^{-1}(U) \subset W', V \times_W W' \rightarrow W')$  is an elementary distinguished square.*
- (3) *If  $S' \rightarrow S$  is a morphism of schemes, then  $(S' \times_S U \subset S' \times_S W, S' \times_S V \rightarrow S' \times_S W)$  is an elementary distinguished square.*

**Proof.** Omitted.  $\square$

08GP **Lemma 9.3.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $P$  be a property of the quasi-compact and quasi-separated objects of  $X_{\text{spaces}, \text{étale}}$ . Assume that*

- (1)  *$P$  holds for every affine object of  $X_{\text{spaces}, \text{étale}}$ ,*
- (2) *for every elementary distinguished square  $(U \subset W, f : V \rightarrow W)$  such that*
  - (a)  *$W$  is a quasi-compact and quasi-separated object of  $X_{\text{spaces}, \text{étale}}$ ,*
  - (b)  *$U$  is quasi-compact,*
  - (c)  *$V$  is affine, and*
  - (d)  *$P$  holds for  $U$ ,  $V$ , and  $U \times_W V$ ,**then  $P$  holds for  $W$ .*

*Then  $P$  holds for every quasi-compact and quasi-separated object of  $X_{\text{spaces}, \text{étale}}$  and in particular for  $X$ .*

**Proof.** We first claim that  $P$  holds for every representable quasi-compact and quasi-separated object of  $X_{\text{spaces}, \text{étale}}$ . Namely, suppose that  $U \rightarrow X$  is étale and  $U$  is a quasi-compact and quasi-separated scheme. By assumption (1) property  $P$  holds for every affine open of  $U$ . Moreover, if  $W, V \subset U$  are quasi-compact open with  $V$  affine and  $P$  holds for  $W$ ,  $V$ , and  $W \cap V$ , then  $P$  holds for  $W \cup V$  by (2) (as the pair  $(W \subset W \cup V, V \rightarrow W \cup V)$  is an elementary distinguished square). Thus  $P$  holds for  $U$  by the induction principle for schemes, see Cohomology of Schemes, Lemma 4.1.

To finish the proof it suffices to prove  $P$  holds for  $X$  (because we can simply replace  $X$  by any quasi-compact and quasi-separated object of  $X_{\text{spaces}, \text{étale}}$  we want to prove the result for). We will use the filtration

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \dots \subset U_1 = X$$

and the morphisms  $f_p : V_p \rightarrow U_p$  of Decent Spaces, Lemma 8.6. We will prove that  $P$  holds for  $U_p$  by descending induction on  $p$ . Note that  $P$  holds for  $U_{n+1}$  by (1) as an empty algebraic space is affine. Assume  $P$  holds for  $U_{p+1}$ . Note that  $(U_{p+1} \subset U_p, f_p : V_p \rightarrow U_p)$  is an elementary distinguished square, but (2) may not apply as  $V_p$  may not be affine. However, as  $V_p$  is a quasi-compact scheme we may choose a finite affine open covering  $V_p = V_{p,1} \cup \dots \cup V_{p,m}$ . Set  $W_{p,0} = U_{p+1}$  and

$$W_{p,i} = U_{p+1} \cup f_p(V_{p,1} \cup \dots \cup V_{p,i})$$

for  $i = 1, \dots, m$ . These are quasi-compact open subspaces of  $X$ . Then we have

$$U_{p+1} = W_{p,0} \subset W_{p,1} \subset \dots \subset W_{p,m} = U_p$$

and the pairs

$$(W_{p,0} \subset W_{p,1}, f_p|_{V_{p,1}}), (W_{p,1} \subset W_{p,2}, f_p|_{V_{p,2}}), \dots, (W_{p,m-1} \subset W_{p,m}, f_p|_{V_{p,m}})$$

are elementary distinguished squares by Lemma 9.2. Note that  $P$  holds for each  $V_{p,i}$  (as affine schemes) and for  $W_{p,i} \times_{W_{p,i+1}} V_{p,i+1}$  as this is a quasi-compact open of  $V_{p,i+1}$  and hence  $P$  holds for it by the first paragraph of this proof. Thus (2) applies to each of these and we inductively conclude  $P$  holds for  $W_{p,1}, \dots, W_{p,m} = U_p$ .  $\square$

08GQ **Lemma 9.4.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $\mathcal{B} \subset \text{Ob}(X_{\text{spaces}, \text{étale}})$ . Let  $P$  be a property of the elements of  $\mathcal{B}$ . Assume that*

- (1) *every  $W \in \mathcal{B}$  is quasi-compact and quasi-separated,*
- (2) *if  $W \in \mathcal{B}$  and  $U \subset W$  is quasi-compact open, then  $U \in \mathcal{B}$ ,*

- (3) if  $V \in \text{Ob}(X_{\text{spaces}, \text{étale}})$  is affine, then (a)  $V \in \mathcal{B}$  and (b)  $P$  holds for  $V$ ,
- (4) for every elementary distinguished square  $(U \subset W, f : V \rightarrow W)$  such that
  - (a)  $W \in \mathcal{B}$ ,
  - (b)  $U$  is quasi-compact,
  - (c)  $V$  is affine, and
  - (d)  $P$  holds for  $U$ ,  $V$ , and  $U \times_W V$ ,
 then  $P$  holds for  $W$ .

Then  $P$  holds for every  $W \in \mathcal{B}$ .

**Proof.** This is proved in exactly the same manner as the proof of Lemma 9.3. (We remark that (4)(d) makes sense as  $U \times_W V$  is a quasi-compact open of  $V$  hence an element of  $\mathcal{B}$  by conditions (2) and (3).)  $\square$

08GR **Remark 9.5.** How to choose the collection  $\mathcal{B}$  in Lemma 9.4? Here are some examples:

- (1) If  $X$  is quasi-compact and separated, then we can choose  $\mathcal{B}$  to be the set of quasi-compact and separated objects of  $X_{\text{spaces}, \text{étale}}$ . Then  $X \in \mathcal{B}$  and  $\mathcal{B}$  satisfies (1), (2), and (3)(a). With this choice of  $\mathcal{B}$  Lemma 9.4 reproduces Lemma 9.3.
- (2) If  $X$  is quasi-compact with affine diagonal over  $\mathbf{Z}$  (as in Properties of Spaces, Definition 3.1), then we can choose  $\mathcal{B}$  to be the set of objects of  $X_{\text{spaces}, \text{étale}}$  which are quasi-compact and have affine diagonal over  $\mathbf{Z}$ . Again  $X \in \mathcal{B}$  and  $\mathcal{B}$  satisfies (1), (2), and (3)(a).
- (3) If  $X$  is quasi-compact and quasi-separated, then the smallest subset  $\mathcal{B}$  which contains  $X$  and satisfies (1), (2), and (3)(a) is given by the rule  $W \in \mathcal{B}$  if and only if either  $W$  is a quasi-compact open subspace of  $X$ , or  $W$  is a quasi-compact open of an affine object of  $X_{\text{spaces}, \text{étale}}$ .

Here is a variant where we extend the truth from an open to larger opens.

09IT **Lemma 9.6.** Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $W \subset X$  be a quasi-compact open subspace. Let  $P$  be a property of quasi-compact open subspaces of  $X$ . Assume that

- (1)  $P$  holds for  $W$ , and
- (2) for every elementary distinguished square  $(W_1 \subset W_2, f : V \rightarrow W_2)$  where such that
  - (a)  $W_1, W_2$  are quasi-compact open subspaces of  $X$ ,
  - (b)  $W \subset W_1$ ,
  - (c)  $V$  is affine, and
  - (d)  $P$  holds for  $W_1$ ,
 then  $P$  holds for  $W_2$ .

Then  $P$  holds for  $X$ .

**Proof.** We can deduce this from Lemma 9.4, but instead we will give a direct argument by explicitly redoing the proof of Lemma 9.3. We will use the filtration

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \dots \subset U_1 = X$$

and the morphisms  $f_p : V_p \rightarrow U_p$  of Decent Spaces, Lemma 8.6. We will prove that  $P$  holds for  $W_p = W \cup U_p$  by descending induction on  $p$ . This will finish the proof as  $W_1 = X$ . Note that  $P$  holds for  $W_{n+1} = W \cap U_{n+1} = W$  by (1). Assume  $P$  holds for  $W_{p+1}$ . Observe that  $W_p \setminus W_{p+1}$  (with reduced induced subspace structure) is

a closed subspace of  $U_p \setminus U_{p+1}$ . Since  $(U_{p+1} \subset U_p, f_p : V_p \rightarrow U_p)$  is an elementary distinguished square, the same is true for  $(W_{p+1} \subset W_p, f_p : V_p \rightarrow W_p)$ . However (2) may not apply as  $V_p$  may not be affine. However, as  $V_p$  is a quasi-compact scheme we may choose a finite affine open covering  $V_p = V_{p,1} \cup \dots \cup V_{p,m}$ . Set  $W_{p,0} = W_{p+1}$  and

$$W_{p,i} = W_{p+1} \cup f_p(V_{p,1} \cup \dots \cup V_{p,i})$$

for  $i = 1, \dots, m$ . These are quasi-compact open subspaces of  $X$  containing  $W$ . Then we have

$$W_{p+1} = W_{p,0} \subset W_{p,1} \subset \dots \subset W_{p,m} = W_p$$

and the pairs

$$(W_{p,0} \subset W_{p,1}, f_p|_{V_{p,1}}), (W_{p,1} \subset W_{p,2}, f_p|_{V_{p,2}}), \dots, (W_{p,m-1} \subset W_{p,m}, f_p|_{V_{p,m}})$$

are elementary distinguished squares by Lemma 9.2. Now (2) applies to each of these and we inductively conclude  $P$  holds for  $W_{p,1}, \dots, W_{p,m} = W_p$ .  $\square$

## 10. Mayer-Vietoris

08GS In this section we prove that an elementary distinguished triangle gives rise to various Mayer-Vietoris sequences.

Let  $S$  be a scheme. Let  $U \rightarrow X$  be an étale morphism of algebraic spaces over  $S$ . In Properties of Spaces, Section 27 it was shown that  $U_{spaces, \acute{e}tale} = X_{spaces, \acute{e}tale}/U$  compatible with structure sheaves. Hence in this situation we often think of the morphism  $j_U : U \rightarrow X$  as a localization morphism (see Modules on Sites, Definition 19.1). In particular we think of pullback  $j_U^*$  as restriction to  $U$  and we often denote it by  $|_U$ ; this is compatible with Properties of Spaces, Equation (26.1.1). In particular we see that

08GT (10.0.1)  $(\mathcal{F}|_U)_{\bar{u}} = \mathcal{F}_{\bar{x}}$

if  $\bar{u}$  is a geometric point of  $U$  and  $\bar{x}$  the image of  $\bar{u}$  in  $X$ . Moreover, restriction has an exact left adjoint  $j_{U!}$ , see Modules on Sites, Lemmas 19.2 and 19.3. Finally, recall that if  $\mathcal{G}$  is an  $\mathcal{O}_X$ -module, then

08GU (10.0.2)  $(j_{U!}\mathcal{G})_{\bar{x}} = \bigoplus_{\bar{u}} \mathcal{G}_{\bar{u}}$

for any geometric point  $\bar{x} : \text{Spec}(k) \rightarrow X$  where the direct sum is over those morphisms  $\bar{u} : \text{Spec}(k) \rightarrow U$  such that  $j_U \circ \bar{u} = \bar{x}$ , see Modules on Sites, Lemma 38.1 and Properties of Spaces, Lemma 19.13.

08GV **Lemma 10.1.** *Let  $S$  be a scheme. Let  $(U \subset X, V \rightarrow X)$  be an elementary distinguished square of algebraic spaces over  $S$ .*

(1) *For a sheaf of  $\mathcal{O}_X$ -modules  $\mathcal{F}$  we have a short exact sequence*

$$0 \rightarrow j_{U \times_X V!} \mathcal{F}|_{U \times_X V} \rightarrow j_{U!} \mathcal{F}|_U \oplus j_{V!} \mathcal{F}|_V \rightarrow \mathcal{F} \rightarrow 0$$

(2) *For an object  $E$  of  $D(\mathcal{O}_X)$  we have a distinguished triangle*

$$j_{U \times_X V!} E|_{U \times_X V} \rightarrow j_{U!} E|_U \oplus j_{V!} E|_V \rightarrow E \rightarrow j_{U \times_X V!} E|_{U \times_X V}[1]$$

*in  $D(\mathcal{O}_X)$ .*

**Proof.** To show the sequence of (1) is exact we may check on stalks at geometric points by Properties of Spaces, Theorem 19.12. Let  $\bar{x}$  be a geometric point of  $X$ . By Equations (10.0.1) and (10.0.2) taking stalks at  $\bar{x}$  we obtain the sequence

$$0 \rightarrow \bigoplus_{(\bar{u}, \bar{v})} \mathcal{F}_{\bar{x}} \rightarrow \bigoplus_{\bar{u}} \mathcal{F}_{\bar{x}} \oplus \bigoplus_{\bar{v}} \mathcal{F}_{\bar{x}} \rightarrow \mathcal{F}_{\bar{x}} \rightarrow 0$$

This sequence is exact because for every  $\bar{x}$  there either is exactly one  $\bar{u}$  mapping to  $\bar{x}$ , or there is no  $\bar{u}$  and exactly one  $\bar{v}$  mapping to  $\bar{x}$ .

Proof of (2). We have seen in Cohomology on Sites, Section 20 that the restriction functors and the extension by zero functors on derived categories are computed by just applying the functor to any complex. Let  $\mathcal{E}^\bullet$  be a complex of  $\mathcal{O}_X$ -modules representing  $E$ . The distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes of  $\mathcal{O}_X$ -modules

$$0 \rightarrow j_{U \times_X V!} \mathcal{E}^\bullet|_{U \times_X V} \rightarrow j_{U!} \mathcal{E}^\bullet|_U \oplus j_{V!} \mathcal{E}^\bullet|_V \rightarrow \mathcal{E}^\bullet \rightarrow 0$$

which is short exact by (1).  $\square$

08GW **Lemma 10.2.** *Let  $S$  be a scheme. Let  $(U \subset X, V \rightarrow X)$  be an elementary distinguished square of algebraic spaces over  $S$ .*

(1) *For every sheaf of  $\mathcal{O}_X$ -modules  $\mathcal{F}$  we have a short exact sequence*

$$0 \rightarrow \mathcal{F} \rightarrow j_{U,*} \mathcal{F}|_U \oplus j_{V,*} \mathcal{F}|_V \rightarrow j_{U \times_X V,*} \mathcal{F}|_{U \times_X V} \rightarrow 0$$

(2) *For any object  $E$  of  $D(\mathcal{O}_X)$  we have a distinguished triangle*

$$E \rightarrow Rj_{U,*} E|_U \oplus Rj_{V,*} E|_V \rightarrow Rj_{U \times_X V,*} E|_{U \times_X V} \rightarrow E[1]$$

*in  $D(\mathcal{O}_X)$ .*

**Proof.** Let  $W$  be an object of  $X_{\text{étale}}$ . We claim the sequence

$$0 \rightarrow \mathcal{F}(W) \rightarrow \mathcal{F}(W \times_X U) \oplus \mathcal{F}(W \times_X V) \rightarrow \mathcal{F}(W \times_X U \times_X V)$$

is exact and that an element of the last group can locally on  $W$  be lifted to the middle one. By Lemma 9.2 the pair  $(W \times_X U \subset W, V \times_X W \rightarrow W)$  is an elementary distinguished square. Thus we may assume  $W = X$  and it suffices to prove the same thing for

$$0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{F}(U) \oplus \mathcal{F}(V) \rightarrow \mathcal{F}(U \times_X V)$$

We have seen that

$$0 \rightarrow j_{U \times_X V!} \mathcal{O}_{U \times_X V} \rightarrow j_{U!} \mathcal{O}_U \oplus j_{V!} \mathcal{O}_V \rightarrow \mathcal{O}_X \rightarrow 0$$

is a exact sequence of  $\mathcal{O}_X$ -modules in Lemma 10.1 and applying the right exact functor  $\text{Hom}_{\mathcal{O}_X}(-, \mathcal{F})$  gives the sequence above. This also means that the obstruction to lifting  $s \in \mathcal{F}(U \times_X V)$  to an element of  $\mathcal{F}(U) \oplus \mathcal{F}(V)$  lies in  $\text{Ext}_{\mathcal{O}_X}^1(\mathcal{O}_X, \mathcal{F}) = H^1(X, \mathcal{F})$ . By locality of cohomology (Cohomology on Sites, Lemma 7.3) this obstruction vanishes étale locally on  $X$  and the proof of (1) is complete.

Proof of (2). Choose a K-injective complex  $\mathcal{I}^\bullet$  representing  $E$  whose terms  $\mathcal{I}^n$  are injective objects of  $\text{Mod}(\mathcal{O}_X)$ , see Injectives, Theorem 12.6. Then  $\mathcal{I}^\bullet|_U$  is a K-injective complex (Cohomology on Sites, Lemma 20.1). Hence  $Rj_{U,*} E|_U$  is represented by  $j_{U,*} \mathcal{I}^\bullet|_U$ . Similarly for  $V$  and  $U \times_X V$ . Hence the distinguished triangle

of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes

$$0 \rightarrow \mathcal{I}^\bullet \rightarrow j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \rightarrow j_{U \times_X V,*}\mathcal{I}^\bullet|_{U \times_X V} \rightarrow 0.$$

This sequence is exact by (1).  $\square$

08JK **Lemma 10.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $(U \subset X, V \rightarrow X)$  be an elementary distinguished square. Denote  $a = f|_U : U \rightarrow Y$ ,  $b = f|_V : V \rightarrow Y$ , and  $c = f|_{U \times_X V} : U \times_X V \rightarrow Y$  the restrictions. For every object  $E$  of  $D(\mathcal{O}_X)$  there exists a distinguished triangle*

$$Rf_*E \rightarrow Ra_*(E|_U) \oplus Rb_*(E|_V) \rightarrow Rc_*(E|_{U \times_X V}) \rightarrow Rf_*E[1]$$

*in  $D(\mathcal{O}_Y)$ . This triangle is functorial in  $E$ .*

**Proof.** Choose a K-injective complex  $\mathcal{I}^\bullet$  representing  $E$ . We may assume  $\mathcal{I}^n$  is an injective object of  $\text{Mod}(\mathcal{O}_X)$  for all  $n$ , see Injectives, Theorem 12.6. Then  $Rf_*E$  is computed by  $f_*\mathcal{I}^\bullet$ . Similarly for  $U$ ,  $V$ , and  $U \cap V$  by Cohomology on Sites, Lemma 20.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes

$$0 \rightarrow f_*\mathcal{I}^\bullet \rightarrow a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \rightarrow c_*\mathcal{I}^\bullet|_{U \times_X V} \rightarrow 0.$$

To see this is a short exact sequence of complexes we argue as follows. Pick an injective object  $\mathcal{I}$  of  $\text{Mod}(\mathcal{O}_X)$ . Apply  $f_*$  to the short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow j_{U,*}\mathcal{I}|_U \oplus j_{V,*}\mathcal{I}|_V \rightarrow j_{U \times_X V,*}\mathcal{I}|_{U \times_X V} \rightarrow 0$$

of Lemma 10.2 and use that  $R^1f_*\mathcal{I} = 0$  to get a short exact sequence

$$0 \rightarrow f_*\mathcal{I} \rightarrow f_*j_{U,*}\mathcal{I}|_U \oplus f_*j_{V,*}\mathcal{I}|_V \rightarrow f_*j_{U \times_X V,*}\mathcal{I}|_{U \times_X V} \rightarrow 0$$

The proof is finished by observing that  $a_* = f_*j_{U,*}$  and similarly for  $b_*$  and  $c_*$ .  $\square$

08H9 **Lemma 10.4.** *Let  $S$  be a scheme. Let  $(U \subset X, V \rightarrow X)$  be an elementary distinguished square of algebraic spaces over  $S$ . For objects  $E, F$  of  $D(\mathcal{O}_X)$  we have a Mayer-Vietoris sequence*

$$\begin{array}{ccc} \dots & \longrightarrow & \text{Ext}^{-1}(E_{U \times_X V}, F_{U \times_X V}) \\ & \nearrow & \\ \text{Hom}(E, F) & \xleftarrow{\quad} & \text{Hom}(E_U, F_U) \oplus \text{Hom}(E_V, F_V) \longrightarrow \text{Hom}(E_{U \times_X V}, F_{U \times_X V}) \end{array}$$

*where the subscripts denote restrictions to the relevant opens and the Hom's are taken in the relevant derived categories.*

**Proof.** Use the distinguished triangle of Lemma 10.1 to obtain a long exact sequence of Hom's (from Derived Categories, Lemma 4.2) and use that  $\text{Hom}(j_{U!}E|_U, F) = \text{Hom}(E|_U, F|_U)$  by Cohomology on Sites, Lemma 20.8.  $\square$

0CRS **Lemma 10.5.** *Let  $S$  be a scheme. Let  $(U \subset X, V \rightarrow X)$  be an elementary distinguished square of algebraic spaces over  $S$ . For an object  $E$  of  $D(\mathcal{O}_X)$  we have a distinguished triangle*

$$R\Gamma(X, E) \rightarrow R\Gamma(U, E) \oplus R\Gamma(V, E) \rightarrow R\Gamma(U \times_X V, E) \rightarrow R\Gamma(X, E)[1]$$

and in particular a long exact cohomology sequence

$$\dots \rightarrow H^n(X, E) \rightarrow H^n(U, E) \oplus H^n(V, E) \rightarrow H^n(U \times_X V, E) \rightarrow H^{n+1}(X, E) \rightarrow \dots$$

The construction of the distinguished triangle and the long exact sequence is functorial in  $E$ .

**Proof.** Choose a K-injective complex  $\mathcal{I}^\bullet$  representing  $E$  whose terms  $\mathcal{I}^n$  are injective objects of  $\text{Mod}(\mathcal{O}_X)$ , see Injectives, Theorem 12.6. In the proof of Lemma 10.2 we found a short exact sequence of complexes

$$0 \rightarrow \mathcal{I}^\bullet \rightarrow j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \rightarrow j_{U \times_X V,*}\mathcal{I}^\bullet|_{U \times_X V} \rightarrow 0$$

Since  $H^1(X, \mathcal{I}^n) = 0$ , we see that taking global sections gives an exact sequence of complexes

$$0 \rightarrow \Gamma(X, \mathcal{I}^\bullet) \rightarrow \Gamma(U, \mathcal{I}^\bullet) \oplus \Gamma(V, \mathcal{I}^\bullet) \rightarrow \Gamma(U \times_X V, \mathcal{I}^\bullet) \rightarrow 0$$

Since these complexes represent  $R\Gamma(X, E)$ ,  $R\Gamma(U, E)$ ,  $R\Gamma(V, E)$ , and  $R\Gamma(U \times_X V, E)$  we get a distinguished triangle by Derived Categories, Section 12 and especially Lemma 12.1.  $\square$

08HA **Lemma 10.6.** *Let  $S$  be a scheme. Let  $j : U \rightarrow X$  be a étale morphism of algebraic spaces over  $S$ . Given an étale morphism  $V \rightarrow Y$ , set  $W = V \times_X U$  and denote  $j_W : W \rightarrow V$  the projection morphism. Then  $(j_!E)|_V = j_{W!}(E|_W)$  for  $E$  in  $D(\mathcal{O}_U)$ .*

**Proof.** This is true because  $(j_!\mathcal{F})|_V = j_{W!}(\mathcal{F}|_W)$  for an  $\mathcal{O}_X$ -module  $\mathcal{F}$  as follows immediately from the construction of the functors  $j_!$  and  $j_{W!}$ , see Modules on Sites, Lemma 19.2.  $\square$

08GG **Lemma 10.7.** *Let  $S$  be a scheme. Let  $(U \subset X, j : V \rightarrow X)$  be an elementary distinguished square of algebraic spaces over  $S$ . Set  $T = |X| \setminus |U|$ .*

- (1) *If  $E$  is an object of  $D(\mathcal{O}_X)$  supported on  $T$ , then (a)  $E \rightarrow Rj_*(E|_V)$  and (b)  $j_!(E|_V) \rightarrow E$  are isomorphisms.*
- (2) *If  $F$  is an object of  $D(\mathcal{O}_V)$  supported on  $j^{-1}T$ , then (a)  $F \rightarrow (j_!F)|_V$ , (b)  $(Rj_*F)|_V \rightarrow F$ , and (c)  $j_!F \rightarrow Rj_*F$  are isomorphisms.*

**Proof.** Let  $E$  be an object of  $D(\mathcal{O}_X)$  whose cohomology sheaves are supported on  $T$ . Then we see that  $E|_U = 0$  and  $E|_{U \times_X V} = 0$  as  $T$  doesn't meet  $U$  and  $j^{-1}T$  doesn't meet  $U \times_X V$ . Thus (1)(a) follows from Lemma 10.2. In exactly the same way (1)(b) follows from Lemma 10.1.

Let  $F$  be an object of  $D(\mathcal{O}_V)$  whose cohomology sheaves are supported on  $j^{-1}T$ . By Lemma 3.1 we have  $(Rj_*F)|_U = Rj_{W,*}(F|_W) = 0$  because  $F|_W = 0$  by our assumption. Similarly  $(j_!F)|_U = j_{W!}(F|_W) = 0$  by Lemma 10.6. Thus  $j_!F$  and  $Rj_*F$  are supported on  $T$  and  $(j_!F)|_V$  and  $(Rj_*F)|_V$  are supported on  $j^{-1}(T)$ . To check that the maps (2)(a), (b), (c) are isomorphisms in the derived category, it suffices to check that these map induce isomorphisms on stalks of cohomology sheaves at geometric points of  $T$  and  $j^{-1}(T)$  by Properties of Spaces, Theorem 19.12. This we may do after replacing  $X$  by  $V$ ,  $U$  by  $U \times_X V$ ,  $V$  by  $V \times_X V$  and  $F$  by  $F|_{V \times_X V}$  (restriction via first projection), see Lemmas 3.1, 10.6, and 9.2. Since  $V \times_X V \rightarrow V$  has a section this reduces (2) to the case that  $j : V \rightarrow X$  has a section.

Assume  $j$  has a section  $\sigma : X \rightarrow V$ . Set  $V' = \sigma(X)$ . This is an open subspace of  $V$ . Set  $U' = j^{-1}(U)$ . This is another open subspace of  $V$ . Then  $(U' \subset V, V' \rightarrow V)$

is an elementary distinguished square. Observe that  $F|_{U'} = 0$  and  $F|_{V' \cap U'} = 0$  because  $F$  is supported on  $j^{-1}(T)$ . Denote  $j' : V' \rightarrow V$  the open immersion and  $j_{V'} : V' \rightarrow X$  the composition  $V' \rightarrow V \rightarrow X$  which is the inverse of  $\sigma$ . Set  $F' = \sigma^*F$ . The distinguished triangles of Lemmas 10.1 and 10.2 show that  $F = j'_!(F|_{V'})$  and  $F = Rj'_*(F|_{V'})$ . It follows that  $j_!F = j_!j'_!(F|_{V'}) = j_{V'}^!F = F'$  because  $j_{V'} : V' \rightarrow X$  is an isomorphism and the inverse of  $\sigma$ . Similarly,  $Rj_*F = Rj_*Rj'_*F = Rj_{V',*}F = F'$ . This proves (2)(c). To prove (2)(a) and (2)(b) it suffices to show that  $F = F'|_V$ . This is clear because both  $F$  and  $F'|_V$  restrict to zero on  $U'$  and  $U' \cap V'$  and the same object on  $V'$ .  $\square$

We can glue complexes!

**08HB Lemma 10.8.** *Let  $S$  be a scheme. Let  $(U \subset X, V \rightarrow X)$  be an elementary distinguished square of algebraic spaces over  $S$ . Suppose given*

- (1) *an object  $A$  of  $D(\mathcal{O}_U)$ ,*
- (2) *an object  $B$  of  $D(\mathcal{O}_V)$ , and*
- (3) *an isomorphism  $c : A|_{U \times_X V} \rightarrow B|_{U \times_X V}$ .*

*Then there exists an object  $F$  of  $D(\mathcal{O}_X)$  and isomorphisms  $f : F|_U \rightarrow A$ ,  $g : F|_V \rightarrow B$  such that  $c = g|_{U \times_X V} \circ f^{-1}|_{U \times_X V}$ . Moreover, given*

- (1) *an object  $E$  of  $D(\mathcal{O}_X)$ ,*
- (2) *a morphism  $a : A \rightarrow E|_U$  of  $D(\mathcal{O}_U)$ ,*
- (3) *a morphism  $b : B \rightarrow E|_V$  of  $D(\mathcal{O}_V)$ ,*

*such that*

$$a|_{U \times_X V} = b|_{U \times_X V} \circ c.$$

*Then there exists a morphism  $F \rightarrow E$  in  $D(\mathcal{O}_X)$  whose restriction to  $U$  is  $a \circ f$  and whose restriction to  $V$  is  $b \circ g$ .*

**Proof.** Denote  $j_U, j_V, j_{U \times_X V}$  the corresponding morphisms towards  $X$ . Choose a distinguished triangle

$$F \rightarrow Rj_{U,*}A \oplus Rj_{V,*}B \rightarrow Rj_{U \times_X V,*}(B|_{U \times_X V}) \rightarrow F[1]$$

Here the map  $Rj_{V,*}B \rightarrow Rj_{U \times_X V,*}(B|_{U \times_X V})$  is the obvious one. The map  $Rj_{U,*}A \rightarrow Rj_{U \times_X V,*}(B|_{U \times_X V})$  is the composition of  $Rj_{U,*}A \rightarrow Rj_{U \times_X V,*}(A|_{U \times_X V})$  with  $Rj_{U \times_X V,*}c$ . Restricting to  $U$  we obtain

$$F|_U \rightarrow A \oplus (Rj_{V,*}B)|_U \rightarrow (Rj_{U \times_X V,*}(B|_{U \times_X V}))|_U \rightarrow F|_U[1]$$

Denote  $j : U \times_X V \rightarrow U$ . Compatibility of restriction and total direct image (Lemma 3.1) shows that both  $(Rj_{V,*}B)|_U$  and  $(Rj_{U \times_X V,*}(B|_{U \times_X V}))|_U$  are canonically isomorphic to  $Rj_*(B|_{U \times_X V})$ . Hence the second arrow of the last displayed equation has a section, and we conclude that the morphism  $F|_U \rightarrow A$  is an isomorphism.

To see that the morphism  $F|_V \rightarrow B$  is an isomorphism we will use a trick. Namely, choose a distinguished triangle

$$F|_V \rightarrow B \rightarrow B' \rightarrow F[1]|_V$$

in  $D(\mathcal{O}_V)$ . Since  $F|_U \rightarrow A$  is an isomorphism, and since we have the isomorphism  $c : A|_{U \times_X V} \rightarrow B|_{U \times_X V}$  the restriction of  $F|_V \rightarrow B$  is an isomorphism over  $U \times_X V$ .

Thus  $B'$  is supported on  $j_V^{-1}(T)$  where  $T = |X| \setminus |U|$ . On the other hand, there is a morphism of distinguished triangles

$$\begin{array}{ccccccc} F & \longrightarrow & Rj_{U,*}F|_U \oplus Rj_{V,*}F|_V & \longrightarrow & Rj_{U \times_X V,*}F|_{U \times_X V} & \longrightarrow & F[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & Rj_{U,*}A \oplus Rj_{V,*}B & \longrightarrow & Rj_{U \times_X V,*}(B|_{U \times_X V}) & \longrightarrow & F[1] \end{array}$$

The all of the vertical maps in this diagram are isomorphisms, except for the map  $Rj_{V,*}F|_V \rightarrow Rj_{V,*}B$ , hence that is an isomorphism too (Derived Categories, Lemma 4.3). This implies that  $Rj_{V,*}B' = 0$ . Hence  $B' = 0$  by Lemma 10.7.

The existence of the morphism  $F \rightarrow E$  follows from the Mayer-Vietoris sequence for Hom, see Lemma 10.4.  $\square$

### 11. The coherator

08GX Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . The *coherator* is a functor

$$Q_X : \text{Mod}(\mathcal{O}_X) \longrightarrow \text{QCoh}(\mathcal{O}_X)$$

which is right adjoint to the inclusion functor  $\text{QCoh}(\mathcal{O}_X) \rightarrow \text{Mod}(\mathcal{O}_X)$ . It exists for any algebraic space  $X$  and moreover the adjunction mapping  $Q_X(\mathcal{F}) \rightarrow \mathcal{F}$  is an isomorphism for every quasi-coherent module  $\mathcal{F}$ , see Properties of Spaces, Proposition 32.2. Since  $Q_X$  is left exact (as a right adjoint) we can consider its right derived extension

$$RQ_X : D(\mathcal{O}_X) \longrightarrow D(\text{QCoh}(\mathcal{O}_X)).$$

Since  $Q_X$  is right adjoint to the inclusion functor  $\text{QCoh}(\mathcal{O}_X) \rightarrow \text{Mod}(\mathcal{O}_X)$  we see that  $RQ_X$  is right adjoint to the canonical functor  $D(\text{QCoh}(\mathcal{O}_X)) \rightarrow D(\mathcal{O}_X)$  by Derived Categories, Lemma 30.3.

In this section we will study the functor  $RQ_X$ . In Section 19 we will study the (closely related) right adjoint to the inclusion functor  $D_{\text{QCoh}}(\mathcal{O}_X) \rightarrow D(\mathcal{O}_X)$  (when it exists).

08GY **Lemma 11.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be an affine morphism of algebraic spaces over  $S$ . Then  $f_*$  defines a derived functor  $f_* : D(\text{QCoh}(\mathcal{O}_X)) \rightarrow D(\text{QCoh}(\mathcal{O}_Y))$ . This functor has the property that*

$$\begin{array}{ccc} D(\text{QCoh}(\mathcal{O}_X)) & \longrightarrow & D_{\text{QCoh}}(\mathcal{O}_X) \\ f_* \downarrow & & \downarrow Rf_* \\ D(\text{QCoh}(\mathcal{O}_Y)) & \longrightarrow & D_{\text{QCoh}}(\mathcal{O}_Y) \end{array}$$

*commutes.*

**Proof.** The functor  $f_* : \text{QCoh}(\mathcal{O}_X) \rightarrow \text{QCoh}(\mathcal{O}_Y)$  is exact, see Cohomology of Spaces, Lemma 8.2. Hence  $f_*$  defines a derived functor  $f_* : D(\text{QCoh}(\mathcal{O}_X)) \rightarrow D(\text{QCoh}(\mathcal{O}_Y))$  by simply applying  $f_*$  to any representative complex, see Derived Categories, Lemma 16.9. For any complex of  $\mathcal{O}_X$ -modules  $\mathcal{F}^\bullet$  there is a canonical map  $f_*\mathcal{F}^\bullet \rightarrow Rf_*\mathcal{F}^\bullet$ . To finish the proof we show this is a quasi-isomorphism when  $\mathcal{F}^\bullet$  is a complex with each  $\mathcal{F}^n$  quasi-coherent. The statement is étale local on  $Y$  hence we may assume  $Y$  affine. As an affine morphism is representable we reduce

to the case of schemes by the compatibility of Remark 6.3. The case of schemes is Derived Categories of Schemes, Lemma 7.1.  $\square$

08GZ **Lemma 11.2.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Assume  $f$  is quasi-compact, quasi-separated, and flat. Then, denoting*

$$\Phi : D(QCoh(\mathcal{O}_X)) \rightarrow D(QCoh(\mathcal{O}_Y))$$

*the right derived functor of  $f_* : QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_Y)$  we have  $RQ_Y \circ Rf_* = \Phi \circ RQ_X$ .*

**Proof.** We will prove this by showing that  $RQ_Y \circ Rf_*$  and  $\Phi \circ RQ_X$  are right adjoint to the same functor  $D(QCoh(\mathcal{O}_Y)) \rightarrow D(\mathcal{O}_X)$ .

Since  $f$  is quasi-compact and quasi-separated, we see that  $f_*$  preserves quasi-coherence, see Morphisms of Spaces, Lemma 11.2. Recall that  $QCoh(\mathcal{O}_X)$  is a Grothendieck abelian category (Properties of Spaces, Proposition 32.2). Hence any  $K$  in  $D(QCoh(\mathcal{O}_X))$  can be represented by a K-injective complex  $\mathcal{I}^\bullet$  of  $QCoh(\mathcal{O}_X)$ , see Injectives, Theorem 12.6. Then we can define  $\Phi(K) = f_* \mathcal{I}^\bullet$ .

Since  $f$  is flat, the functor  $f^*$  is exact. Hence  $f^*$  defines  $f^* : D(\mathcal{O}_Y) \rightarrow D(\mathcal{O}_X)$  and also  $f^* : D(QCoh(\mathcal{O}_Y)) \rightarrow D(QCoh(\mathcal{O}_X))$ . The functor  $f^* = Lf^* : D(\mathcal{O}_Y) \rightarrow D(\mathcal{O}_X)$  is left adjoint to  $Rf_* : D(\mathcal{O}_X) \rightarrow D(\mathcal{O}_Y)$ , see Cohomology on Sites, Lemma 19.1. Similarly, the functor  $f^* : D(QCoh(\mathcal{O}_Y)) \rightarrow D(QCoh(\mathcal{O}_X))$  is left adjoint to  $\Phi : D(QCoh(\mathcal{O}_X)) \rightarrow D(QCoh(\mathcal{O}_Y))$  by Derived Categories, Lemma 30.3.

Let  $A$  be an object of  $D(QCoh(\mathcal{O}_Y))$  and  $E$  an object of  $D(\mathcal{O}_X)$ . Then

$$\begin{aligned} \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, RQ_Y(Rf_* E)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(A, Rf_* E) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(f^* A, E) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(f^* A, RQ_X(E)) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, \Phi(RQ_X(E))) \end{aligned}$$

This implies what we want.  $\square$

08H0 **Lemma 11.3.** *Let  $S$  be a scheme. Let  $X$  be an affine algebraic space over  $S$ . Set  $A = \Gamma(X, \mathcal{O}_X)$ . Then*

- (1)  $Q_X : \mathrm{Mod}(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_X)$  is the functor which sends  $\mathcal{F}$  to the quasi-coherent  $\mathcal{O}_X$ -module associated to the  $A$ -module  $\Gamma(X, \mathcal{F})$ ,
- (2)  $RQ_X : D(\mathcal{O}_X) \rightarrow D(QCoh(\mathcal{O}_X))$  is the functor which sends  $E$  to the complex of quasi-coherent  $\mathcal{O}_X$ -modules associated to the object  $R\Gamma(X, E)$  of  $D(A)$ ,
- (3) restricted to  $D_{QCoh}(\mathcal{O}_X)$  the functor  $RQ_X$  defines a quasi-inverse to (5.1.1).

**Proof.** Let  $X_0 = \mathrm{Spec}(A)$  be the affine scheme representing  $X$ . Recall that there is a morphism of ringed sites  $\epsilon : X_{\acute{e}tale} \rightarrow X_{0,Zar}$  which induces equivalences

$$QCoh(\mathcal{O}_X) \xrightleftharpoons[\epsilon^*]{\epsilon_*} QCoh(\mathcal{O}_{X_0})$$

see Lemma 4.2. Hence we see that  $Q_X = \epsilon^* \circ Q_{X_0} \circ \epsilon_*$  by uniqueness of adjoint functors. Hence (1) follows from the description of  $Q_{X_0}$  in Derived Categories of Schemes, Lemma 7.3 and the fact that  $\Gamma(X_0, \epsilon_* \mathcal{F}) = \Gamma(X, \mathcal{F})$ . Part (2) follows from (1) and the fact that the functor from  $A$ -modules to quasi-coherent  $\mathcal{O}_X$ -modules

is exact. The third assertion now follows from the result for schemes (Derived Categories of Schemes, Lemma 7.3) and Lemma 4.2.  $\square$

Next, we prove a criterion for when the functor  $D(QCoh(\mathcal{O}_X)) \rightarrow D_{QCoh}(\mathcal{O}_X)$  is an equivalence.

09TG **Lemma 11.4.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Suppose that for every étale morphism  $j : V \rightarrow W$  with  $W \subset X$  quasi-compact open and  $V$  affine the right derived functor*

$$\Phi : D(QCoh(\mathcal{O}_U)) \rightarrow D(QCoh(\mathcal{O}_W))$$

*of the left exact functor  $j_* : QCoh(\mathcal{O}_V) \rightarrow QCoh(\mathcal{O}_W)$  fits into a commutative diagram*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_V)) & \xrightarrow{i_V} & D_{QCoh}(\mathcal{O}_V) \\ \Phi \downarrow & & \downarrow Rj_* \\ D(QCoh(\mathcal{O}_W)) & \xrightarrow{i_W} & D_{QCoh}(\mathcal{O}_W) \end{array}$$

*Then the functor (5.1.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

*is an equivalence with quasi-inverse given by  $RQ_X$ .*

**Proof.** We first use the induction principle to prove  $i_X$  is fully faithful. More precisely, we will use Lemma 9.6. Let  $(U \subset W, V \rightarrow W)$  be an elementary distinguished square with  $V$  affine and  $U, W$  quasi-compact open in  $X$ . Assume that  $i_U$  is fully faithful. We have to show that  $i_W$  is fully faithful. We may replace  $X$  by  $W$ , i.e., we may assume  $W = X$  (we do this just to simplify the notation – observe that the condition in the statement of the lemma is preserved under this operation).

Suppose that  $A, B$  are objects of  $D(QCoh(\mathcal{O}_X))$ . We want to show that

$$\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B))$$

is bijective. Let  $T = |X| \setminus |U|$ .

Assume first  $i_X(B)$  is supported on  $T$ . In this case the map

$$i_X(B) \rightarrow Rj_{V,*}(i_X(B)|_V) = Rj_{V,*}(i_V(B|_V))$$

is a quasi-isomorphism (Lemma 10.7). By assumption we have an isomorphism  $i_X(\Phi(B|_V)) \rightarrow Rj_{V,*}(i_V(B|_V))$  in  $D(\mathcal{O}_X)$ . Moreover,  $\Phi$  and  $-|_V$  are adjoint functors on the derived categories of quasi-coherent modules (by Derived Categories, Lemma 30.3). The adjunction map  $B \rightarrow \Phi(B|_V)$  becomes an isomorphism after applying  $i_X$ , whence is an isomorphism in  $D(QCoh(\mathcal{O}_X))$ . Hence

$$\begin{aligned} \mathrm{Mor}_{D(QCoh(\mathcal{O}_X))}(A, B) &= \mathrm{Mor}_{D(QCoh(\mathcal{O}_X))}(A, \Phi(B|_V)) \\ &= \mathrm{Mor}_{D(QCoh(\mathcal{O}_V))}(A|_V, B|_V) \\ &= \mathrm{Mor}_{D(\mathcal{O}_V)}(i_V(A|_V), i_V(B|_V)) \\ &= \mathrm{Mor}_{D(\mathcal{O}_X)}(i_X(A), Rj_{V,*}(i_V(B|_V))) \\ &= \mathrm{Mor}_{D(\mathcal{O}_X)}(i_X(A), i_X(B)) \end{aligned}$$

as desired. Here we have used that  $i_V$  is fully faithful (Lemma 11.3).

In general, choose any complex  $\mathcal{B}^\bullet$  of quasi-coherent  $\mathcal{O}_X$ -modules representing  $B$ . Next, choose any quasi-isomorphism  $s : \mathcal{B}^\bullet|_U \rightarrow \mathcal{C}^\bullet$  of complexes of quasi-coherent modules on  $U$ . As  $j_U : U \rightarrow X$  is quasi-compact and quasi-separated the functor  $j_{U,*}$  transforms quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 11.2). Thus there is a canonical map  $\mathcal{B}^\bullet \rightarrow j_{U,*}(\mathcal{B}^\bullet|_U) \rightarrow j_{U,*}\mathcal{C}^\bullet$  of complexes of quasi-coherent modules on  $X$ . Set  $B'' = j_{U,*}\mathcal{C}^\bullet$  in  $D(QCoh(\mathcal{O}_X))$  and choose a distinguished triangle

$$B \rightarrow B'' \rightarrow B' \rightarrow B[1]$$

in  $D(QCoh(\mathcal{O}_X))$ . Since the first arrow of the triangle restricts to an isomorphism over  $U$  we see that  $B'$  is supported on  $T$ . Hence in the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'[-1]) & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B')[-1]) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B) & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'') & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B'')) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B') & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B')) \end{array}$$

we have exact columns and the top and bottom horizontal arrows are bijective. Finally, choose a complex  $\mathcal{A}^\bullet$  of quasi-coherent modules representing  $A$ .

Let  $\alpha : i_X(A) \rightarrow i_X(B)$  be a morphism between in  $D(\mathcal{O}_X)$ . The restriction  $\alpha|_U$  comes from a morphism in  $D(QCoh(\mathcal{O}_U))$  as  $i_U$  is fully faithful. Hence there exists a choice of  $s : \mathcal{B}^\bullet|_U \rightarrow \mathcal{C}^\bullet$  as above such that  $\alpha|_U$  is represented by an actual map of complexes  $\mathcal{A}^\bullet|_U \rightarrow \mathcal{C}^\bullet$ . This corresponds to a map of complexes  $\mathcal{A} \rightarrow j_{U,*}\mathcal{C}^\bullet$ . In other words, the image of  $\alpha$  in  $\mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B''))$  comes from an element of  $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'')$ . A diagram chase then shows that  $\alpha$  comes from a morphism  $A \rightarrow B$  in  $D(QCoh(\mathcal{O}_X))$ . Finally, suppose that  $a : A \rightarrow B$  is a morphism of  $D(QCoh(\mathcal{O}_X))$  which becomes zero in  $D(\mathcal{O}_X)$ . After choosing  $\mathcal{B}^\bullet$  suitably, we may assume  $a$  is represented by a morphism of complexes  $a^\bullet : \mathcal{A}^\bullet \rightarrow \mathcal{B}^\bullet$ . Since  $i_U$  is fully faithful the restriction  $a^\bullet|_U$  is zero in  $D(QCoh(\mathcal{O}_U))$ . Thus we can choose  $s$  such that  $s \circ a^\bullet|_U : \mathcal{A}^\bullet|_U \rightarrow \mathcal{C}^\bullet$  is homotopic to zero. Applying the functor  $j_{U,*}$  we conclude that  $\mathcal{A}^\bullet \rightarrow j_{U,*}\mathcal{C}^\bullet$  is homotopic to zero. Thus  $a$  maps to zero in  $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'')$ . Thus we may assume that  $a$  is the image of an element of  $b \in \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'[-1])$ . The image of  $b$  in  $\mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B')[-1])$  comes from a  $\gamma \in \mathrm{Hom}_{D(\mathcal{O}_X)}(A, B''[-1])$  (as  $a$  maps to zero in the group on the right). Since we've seen above the horizontal arrows are surjective, we see that  $\gamma$  comes from a  $c$  in  $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B''[-1])$  which implies  $a = 0$  as desired.

At this point we know that  $i_X$  is fully faithful for our original  $X$ . Since  $RQ_X$  is its right adjoint, we see that  $RQ_X \circ i_X = \mathrm{id}$  (Categories, Lemma 24.4). To finish the proof we show that for any  $E$  in  $D_{QCoh}(\mathcal{O}_X)$  the map  $i_X(RQ_X(E)) \rightarrow E$  is an isomorphism. Choose a distinguished triangle

$$i_X(RQ_X(E)) \rightarrow E \rightarrow E' \rightarrow i_X(RQ_X(E))[1]$$

in  $D_{QCoh}(\mathcal{O}_X)$ . A formal argument using the above shows that  $i_X(RQ_X(E')) = 0$ . Thus it suffices to prove that for  $E \in D_{QCoh}(\mathcal{O}_X)$  the condition  $i_X(RQ_X(E)) = 0$  implies that  $E = 0$ . Consider an étale morphism  $j : V \rightarrow X$  with  $V$  affine. By Lemmas 11.3 and 11.2 and our assumption we have

$$Rj_*(E|_V) = Rj_*(i_V(RQ_V(E|_V))) = i_X(\Phi(RQ_V(E|_V))) = i_X(RQ_X(Rj_*(E|_V)))$$

Choose a distinguished triangle

$$E \rightarrow Rj_*(E|_V) \rightarrow E' \rightarrow E[1]$$

Apply  $RQ_X$  to get a distinguished triangle

$$0 \rightarrow RQ_X(Rj_*(E|_V)) \rightarrow RQ_X(E') \rightarrow 0[1]$$

in other words the map in the middle is an isomorphism. Combined with the string of equalities above we find that our first distinguished triangle becomes a distinguished triangle

$$E \rightarrow i_X(RQ_X(E')) \rightarrow E' \rightarrow E[1]$$

where the middle morphism is the adjunction map. However, the composition  $E \rightarrow E'$  is zero, hence  $E \rightarrow i_X(RQ_X(E'))$  is zero by adjunction! Since this morphism is isomorphic to the morphism  $E \rightarrow Rj_*(E|_V)$  adjoint to  $\text{id} : E|_V \rightarrow E|_V$  we conclude that  $E|_V$  is zero. Since this holds for all affine  $V$  étale over  $X$  we conclude  $E$  is zero as desired.  $\square$

08H1 **Proposition 11.5.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact algebraic space over  $S$  with affine diagonal over  $\mathbf{Z}$  (as in Properties of Spaces, Definition 3.1). Then the functor (5.1.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

*is an equivalence with quasi-inverse given by  $RQ_X$ .*

**Proof.** Let  $V \rightarrow W$  be an étale morphism with  $V$  affine and  $W$  a quasi-compact open subspace of  $X$ . Then the morphism  $V \rightarrow W$  is affine as  $W$  has affine diagonal over  $\mathbf{Z}$  and  $V$  is affine (Morphisms of Spaces, Lemma 20.11). Lemma 11.1 then guarantees that the assumption of Lemma 11.4 holds. Hence we conclude.  $\square$

0CSR **Lemma 11.6.** *Let  $S$  be a scheme and let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Assume  $X$  and  $Y$  are quasi-compact and have affine diagonal over  $\mathbf{Z}$  (as in Properties of Spaces, Definition 3.1). Then, denoting*

$$\Phi : D(QCoh(\mathcal{O}_X)) \rightarrow D(QCoh(\mathcal{O}_Y))$$

*the right derived functor of  $f_* : QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_Y)$  the diagram*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \Phi \downarrow & & \downarrow Rf_* \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

*is commutative.*

**Proof.** Observe that the horizontal arrows in the diagram are equivalences of categories by Proposition 11.5. Hence we can identify these categories (and similarly for other quasi-compact algebraic spaces with affine diagonal) and then the statement of the lemma is that the canonical map  $\Phi(K) \rightarrow Rf_*(K)$  is an isomorphism for all  $K$  in  $D(QCoh(\mathcal{O}_X))$ . Note that if  $K_1 \rightarrow K_2 \rightarrow K_3 \rightarrow K_1[1]$  is a distinguished triangle in  $D(QCoh(\mathcal{O}_X))$  and the statement is true for two-out-of-three, then it is true for the third.

Let  $\mathcal{B} \subset \text{Ob}(X_{\text{spaces}, \text{étale}})$  be the set of objects which are quasi-compact and have affine diagonal. For  $U \in \mathcal{B}$  and any morphism  $g : U \rightarrow Z$  where  $Z$  is a quasi-compact algebraic space over  $S$  with affine diagonal, denote

$$\Phi_g : D(QCoh(\mathcal{O}_U)) \rightarrow D(QCoh(\mathcal{O}_Z))$$

the derived extension of  $g_*$ . Let  $P(U) =$  “for any  $K$  in  $D(QCoh(\mathcal{O}_U))$  and any  $g : U \rightarrow Z$  as above the map  $\Phi_g(K) \rightarrow Rg_*K$  is an isomorphism”. By Remark 9.5 conditions (1), (2), and (3)(a) of Lemma 9.4 hold and we are left with proving (3)(b) and (4).

Checking condition (3)(b). Let  $U$  be an affine scheme étale over  $X$ . Let  $g : U \rightarrow Z$  be as above. Since the diagonal of  $Z$  is affine the morphism  $g : U \rightarrow Z$  is affine (Morphisms of Spaces, Lemma 20.11). Hence  $P(U)$  holds by Lemma 11.1.

Checking condition (4). Let  $(U \subset W, V \rightarrow W)$  be an elementary distinguished square in  $X_{\text{spaces}, \text{étale}}$  with  $U, W, V$  in  $\mathcal{B}$  and  $V$  affine. Assume that  $P$  holds for  $U$ ,  $V$ , and  $U \times_W V$ . We have to show that  $P$  holds for  $W$ . Let  $g : W \rightarrow Z$  be a morphism to a quasi-compact algebraic space with affine diagonal. Let  $K$  be an object of  $D(QCoh(\mathcal{O}_W))$ . Consider the distinguished triangle

$$K \rightarrow Rj_{U,*}K|_U \oplus Rj_{V,*}K|_V \rightarrow Rj_{U \times_W V,*}K|_{U \times_W V} \rightarrow K[1]$$

in  $D(\mathcal{O}_W)$ . By the two-out-of-three property mentioned above, it suffices to show that  $\Phi_g(Rj_{U,*}K|_U) \rightarrow Rg_*(Rj_{U,*}K|_U)$  is an isomorphism and similarly for  $V$  and  $U \times_W V$ . This is discussed in the next paragraph.

Let  $j : U \rightarrow W$  be a morphism  $X_{\text{spaces}, \text{étale}}$  with  $U, W$  in  $\mathcal{B}$  and  $P$  holds for  $U$ . Let  $g : W \rightarrow Z$  be a morphism to a quasi-compact algebraic space with affine diagonal. To finish the proof we have to show that  $\Phi_g(Rj_*K) \rightarrow Rg_*(Rj_*K)$  is an isomorphism for any  $K$  in  $D(QCoh(\mathcal{O}_U))$ . Let  $\mathcal{I}^\bullet$  be a K-injective complex in  $QCoh(\mathcal{O}_U)$  representing  $K$ . From  $P(U)$  applied to  $j$  we see that  $j_*\mathcal{I}^\bullet$  represents  $Rj_*K$ . Since  $j_* : QCoh(\mathcal{O}_U) \rightarrow QCoh(\mathcal{O}_W)$  has an exact left adjoint  $j^* : QCoh(\mathcal{O}_W) \rightarrow QCoh(\mathcal{O}_U)$  we see that  $j_*\mathcal{I}^\bullet$  is a K-injective complex in  $QCoh(\mathcal{O}_W)$ , see Derived Categories, Lemma 31.9. Hence  $\Phi_g(Rj_*K)$  is represented by  $g_*j_*\mathcal{I}^\bullet = (g \circ j)_*\mathcal{I}^\bullet$ . By  $P(U)$  applied to  $g \circ j$  we see that this represents  $R_{g \circ j,*}(K) = Rg_*(Rj_*K)$ . This finishes the proof.  $\square$

## 12. The coherator for Noetherian spaces

09TH We need a little bit more about injective modules to treat the case of a Noetherian algebraic space.

09TI **Lemma 12.1.** *Let  $S$  be a Noetherian affine scheme. Every injective object of  $QCoh(\mathcal{O}_S)$  is a filtered colimit  $\text{colim}_i \mathcal{F}_i$  of quasi-coherent sheaves of the form*

$$\mathcal{F}_i = (Z_i \rightarrow S)_*\mathcal{G}_i$$

where  $Z_i$  is the spectrum of an Artinian ring and  $\mathcal{G}_i$  is a coherent module on  $Z_i$ .

**Proof.** Let  $S = \operatorname{Spec}(A)$ . Let  $\mathcal{J}$  be an injective object of  $QCoh(\mathcal{O}_S)$ . Since  $QCoh(\mathcal{O}_S)$  is equivalent to the category of  $A$ -modules we see that  $\mathcal{J}$  is equal to  $\tilde{J}$  for some injective  $A$ -module  $J$ . By Dualizing Complexes, Proposition 5.9 we can write  $J = \bigoplus E_\alpha$  with  $E_\alpha$  indecomposable and therefore isomorphic to the injective hull of a residue field at a point. Thus (because finite disjoint unions of Artinian schemes are Artinian) we may assume that  $J$  is the injective hull of  $\kappa(\mathfrak{p})$  for some prime  $\mathfrak{p}$  of  $A$ . Then  $J = \bigcup J[\mathfrak{p}^n]$  where  $J[\mathfrak{p}^n]$  is the injective hull of  $\kappa(\mathfrak{p})$  over  $A/\mathfrak{p}^n A_{\mathfrak{p}}$ , see Dualizing Complexes, Lemma 7.3. Thus  $\tilde{J}$  is the colimit of the sheaves  $(Z_n \rightarrow X)_* \mathcal{G}_n$  where  $Z_n = \operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}})$  and  $\mathcal{G}_n$  the coherent sheaf associated to the finite  $A/\mathfrak{p}^n A_{\mathfrak{p}}$ -module  $J[\mathfrak{p}^n]$ . Finiteness follows from Dualizing Complexes, Lemma 6.1.  $\square$

09TJ **Lemma 12.2.** *Let  $S$  be an affine scheme. Let  $X$  be a Noetherian algebraic space over  $S$ . Every injective object of  $QCoh(\mathcal{O}_X)$  is a direct summand of a filtered colimit  $\operatorname{colim}_i \mathcal{F}_i$  of quasi-coherent sheaves of the form*

$$\mathcal{F}_i = (Z_i \rightarrow X)_* \mathcal{G}_i$$

where  $Z_i$  is the spectrum of an Artinian ring and  $\mathcal{G}_i$  is a coherent module on  $Z_i$ .

**Proof.** Choose an affine scheme  $U$  and a surjective étale morphism  $j : U \rightarrow X$  (Properties of Spaces, Lemma 6.3). Then  $U$  is a Noetherian affine scheme. Choose an injective object  $\mathcal{J}'$  of  $QCoh(\mathcal{O}_U)$  such that there exists an injection  $\mathcal{J}|_U \rightarrow \mathcal{J}'$ . Then

$$\mathcal{J} \rightarrow j_* \mathcal{J}'$$

is an injective morphism in  $QCoh(\mathcal{O}_X)$ , hence identifies  $\mathcal{J}$  as a direct summand of  $j_* \mathcal{J}'$ . Thus the result follows from the corresponding result for  $\mathcal{J}'$  proved in Lemma 12.1.  $\square$

09TK **Lemma 12.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a flat, quasi-compact, and quasi-separated morphism of algebraic spaces over  $S$ . If  $\mathcal{J}$  is an injective object of  $QCoh(\mathcal{O}_X)$ , then  $f_* \mathcal{J}$  is an injective object of  $QCoh(\mathcal{O}_Y)$ .*

**Proof.** Since  $f$  is quasi-compact and quasi-separated, the functor  $f_*$  transforms quasi-coherent sheaves into quasi-coherent sheaves (Morphisms of Spaces, Lemma 11.2). The functor  $f^*$  is a left adjoint to  $f_*$  which transforms injections into injections. Hence the result follows from Homology, Lemma 29.1  $\square$

09TL **Lemma 12.4.** *Let  $S$  be a scheme. Let  $X$  be a Noetherian algebraic space over  $S$ . If  $\mathcal{J}$  is an injective object of  $QCoh(\mathcal{O}_X)$ , then*

- (1)  $H^p(U, \mathcal{J}|_U) = 0$  for  $p > 0$  and for every quasi-compact and quasi-separated algebraic space  $U$  étale over  $X$ ,
- (2) for any morphism  $f : X \rightarrow Y$  of algebraic spaces over  $S$  with  $Y$  quasi-separated we have  $R^p f_* \mathcal{J} = 0$  for  $p > 0$ .

**Proof.** Proof of (1). Write  $\mathcal{J}$  as a direct summand of  $\operatorname{colim} \mathcal{F}_i$  with  $\mathcal{F}_i = (Z_i \rightarrow X)_* \mathcal{G}_i$  as in Lemma 12.2. It is clear that it suffices to prove the vanishing for  $\operatorname{colim} \mathcal{F}_i$ . Since pullback commutes with colimits and since  $U$  is quasi-compact and quasi-separated, it suffices to prove  $H^p(U, \mathcal{F}_i|_U) = 0$  for  $p > 0$ , see Cohomology of

Spaces, Lemma 5.1. Observe that  $Z_i \rightarrow X$  is an affine morphism, see Morphisms of Spaces, Lemma 20.12. Thus

$$\mathcal{F}_i|_U = (Z_i \times_X U \rightarrow U)_* \mathcal{G}'_i = R(Z_i \times_X U \rightarrow U)_* \mathcal{G}'_i$$

where  $\mathcal{G}'_i$  is the pullback of  $\mathcal{G}_i$  to  $Z_i \times_X U$ , see Cohomology of Spaces, Lemma 11.1. Since  $Z_i \times_X U$  is affine we conclude that  $\mathcal{G}'_i$  has no higher cohomology on  $Z_i \times_X U$ . By the Leray spectral sequence we conclude the same thing is true for  $\mathcal{F}_i|_U$  (Cohomology on Sites, Lemma 14.6).

Proof of (2). Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $V \rightarrow Y$  be an étale morphism with  $V$  affine. Then  $V \times_Y X \rightarrow X$  is an étale morphism and  $V \times_Y X$  is a quasi-compact and quasi-separated algebraic space étale over  $X$  (details omitted). Hence  $H^p(V \times_Y X, \mathcal{J})$  is zero by part (1). Since  $R^p f_* \mathcal{J}$  is the sheaf associated to the presheaf  $V \mapsto H^p(V \times_Y X, \mathcal{J})$  the result is proved.  $\square$

09TM **Lemma 12.5.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of Noetherian algebraic spaces over  $S$ . Then  $f_*$  on quasi-coherent sheaves has a right derived extension  $\Phi : D(QCoh(\mathcal{O}_X)) \rightarrow D(QCoh(\mathcal{O}_Y))$  such that the diagram*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \Phi \downarrow & & \downarrow Rf_* \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

*commutes.*

**Proof.** Since  $X$  and  $Y$  are Noetherian the morphism is quasi-compact and quasi-separated (see Morphisms of Spaces, Lemma 8.10). Thus  $f_*$  preserve quasi-coherence, see Morphisms of Spaces, Lemma 11.2. Next, let  $K$  be an object of  $D(QCoh(\mathcal{O}_X))$ . Since  $QCoh(\mathcal{O}_X)$  is a Grothendieck abelian category (Properties of Spaces, Proposition 32.2), we can represent  $K$  by a K-injective complex  $\mathcal{I}^\bullet$  such that each  $\mathcal{I}^n$  is an injective object of  $QCoh(\mathcal{O}_X)$ , see Injectives, Theorem 12.6. Thus we see that the functor  $\Phi$  is defined by setting

$$\Phi(K) = f_* \mathcal{I}^\bullet$$

where the right hand side is viewed as an object of  $D(QCoh(\mathcal{O}_Y))$ . To finish the proof of the lemma it suffices to show that the canonical map

$$f_* \mathcal{I}^\bullet \longrightarrow Rf_* \mathcal{I}^\bullet$$

is an isomorphism in  $D(\mathcal{O}_Y)$ . To see this it suffices to prove the map induces an isomorphism on cohomology sheaves. Pick any  $m \in \mathbf{Z}$ . Let  $N = N(X, Y, f)$  be as in Lemma 6.1. Consider the short exact sequence

$$0 \rightarrow \sigma_{\geq m-N-1} \mathcal{I}^\bullet \rightarrow \mathcal{I}^\bullet \rightarrow \sigma_{\leq m-N-2} \mathcal{I}^\bullet \rightarrow 0$$

of complexes of quasi-coherent sheaves on  $X$ . By Lemma 6.1 we see that the cohomology sheaves of  $Rf_* \sigma_{\leq m-N-2} \mathcal{I}^\bullet$  are zero in degrees  $\geq m-1$ . Thus we see that  $R^m f_* \mathcal{I}^\bullet$  is isomorphic to  $R^m f_* \sigma_{\geq m-N-1} \mathcal{I}^\bullet$ . In other words, we may assume that  $\mathcal{I}^\bullet$  is a bounded below complex of injective objects of  $QCoh(\mathcal{O}_X)$ . This case follows from Leray's acyclicity lemma (Derived Categories, Lemma 16.7) with required vanishing because of Lemma 12.4.  $\square$

09TN **Proposition 12.6.** *Let  $S$  be a scheme. Let  $X$  be a Noetherian algebraic space over  $S$ . Then the functor (5.1.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

*is an equivalence with quasi-inverse given by  $RQ_X$ .*

**Proof.** Follows immediately from Lemmas 12.5 and 11.4.  $\square$

### 13. Pseudo-coherent and perfect complexes

08HC In this section we study the general notions defined in Cohomology on Sites, Sections 44, 45, 46, and 47 for the étale site of an algebraic space. In particular we match this with what happens for schemes.

First we compare the notion of a pseudo-coherent complex on a scheme and on its associated small étale site.

08HD **Lemma 13.1.** *Let  $X$  be a scheme. Let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module. The following are equivalent*

- (1)  $\mathcal{F}$  is of finite type as an  $\mathcal{O}_X$ -module, and
- (2)  $\epsilon^*\mathcal{F}$  is of finite type as an  $\mathcal{O}_{\text{étale}}$ -module on the small étale site of  $X$ .

Here  $\epsilon$  is as in (4.0.1).

**Proof.** The implication (1)  $\Rightarrow$  (2) is a general fact, see Modules on Sites, Lemma 23.4. Assume (2). By assumption there exists an étale covering  $\{f_i : X_i \rightarrow X\}$  such that  $\epsilon^*\mathcal{F}|_{(X_i)_{\text{étale}}}$  is generated by finitely many sections. Let  $x \in X$ . We will show that  $\mathcal{F}$  is generated by finitely many sections in a neighbourhood of  $x$ . Say  $x$  is in the image of  $X_i \rightarrow X$  and denote  $X' = X_i$ . Let  $s_1, \dots, s_n \in \Gamma(X', \epsilon^*\mathcal{F}|_{X'_{\text{étale}}})$  be generating sections. As  $\epsilon^*\mathcal{F} = \epsilon^{-1}\mathcal{F} \otimes_{\epsilon^{-1}\mathcal{O}_X} \mathcal{O}_{\text{étale}}$  we can find an étale morphism  $X'' \rightarrow X'$  such that  $x$  is in the image of  $X' \rightarrow X$  and such that  $s_i|_{X''} = \sum s_{ij} \otimes a_{ij}$  for some sections  $s_{ij} \in \epsilon^{-1}\mathcal{F}(X'')$  and  $a_{ij} \in \mathcal{O}_{\text{étale}}(X'')$ . Denote  $U \subset X$  the image of  $X'' \rightarrow X$ . This is an open subscheme as  $f'' : X'' \rightarrow X$  is étale (Morphisms, Lemma 36.13). After possibly shrinking  $X''$  more we may assume  $s_{ij}$  come from elements  $t_{ij} \in \mathcal{F}(U)$  as follows from the construction of the inverse image functor  $\epsilon^{-1}$ . Now we claim that  $t_{ij}$  generate  $\mathcal{F}|_U$  which finishes the proof of the lemma. Namely, the corresponding map  $\mathcal{O}_U^{\oplus N} \rightarrow \mathcal{F}|_U$  has the property that its pullback by  $f''$  to  $X''$  is surjective. Since  $f'' : X'' \rightarrow U$  is a surjective flat morphism of schemes, this implies that  $\mathcal{O}_U^{\oplus N} \rightarrow \mathcal{F}|_U$  is surjective by looking at stalks and using that  $\mathcal{O}_{U, f''(z)} \rightarrow \mathcal{O}_{X'', z}$  is faithfully flat for all  $z \in X''$ .  $\square$

In the situation above the morphism of sites  $\epsilon$  is flat hence defines a pullback on complexes of modules.

08HE **Lemma 13.2.** *Let  $X$  be a scheme. Let  $E$  be an object of  $D(\mathcal{O}_X)$ . The following are equivalent*

- (1)  $E$  is  $m$ -pseudo-coherent, and
- (2)  $\epsilon^*E$  is  $m$ -pseudo-coherent on the small étale site of  $X$ .

Here  $\epsilon$  is as in (4.0.1).

**Proof.** The implication (1)  $\Rightarrow$  (2) is a general fact, see Cohomology on Sites, Lemma 45.3. Assume  $\epsilon^*E$  is  $m$ -pseudo-coherent. We will use without further mention that  $\epsilon^*$  is an exact functor and that therefore

$$\epsilon^*H^i(E) = H^i(\epsilon^*E).$$

To show that  $E$  is  $m$ -pseudo-coherent we may work locally on  $X$ , hence we may assume that  $X$  is quasi-compact (for example affine). Since  $X$  is quasi-compact every étale covering  $\{U_i \rightarrow X\}$  has a finite refinement. Thus we see that  $\epsilon^*E$  is an object of  $D^-(\mathcal{O}_{\text{étale}})$ , see comments following Cohomology on Sites, Definition 45.1. By Lemma 4.1 it follows that  $E$  is an object of  $D^-(\mathcal{O}_X)$ .

Let  $n \in \mathbf{Z}$  be the largest integer such that  $H^n(E)$  is nonzero; then  $n$  is also the largest integer such that  $H^n(\epsilon^*E)$  is nonzero. We will prove the lemma by induction on  $n - m$ . If  $n < m$ , then the lemma is clearly true. If  $n \geq m$ , then  $H^n(\epsilon^*E)$  is a finite  $\mathcal{O}_{\text{étale}}$ -module, see Cohomology on Sites, Lemma 45.7. Hence  $H^n(E)$  is a finite  $\mathcal{O}_X$ -module, see Lemma 13.1. After replacing  $X$  by the members of an open covering, we may assume there exists a surjection  $\mathcal{O}_X^{\oplus t} \rightarrow H^n(E)$ . We may locally on  $X$  lift this to a map of complexes  $\alpha : \mathcal{O}_X^{\oplus t}[-n] \rightarrow E$  (details omitted). Choose a distinguished triangle

$$\mathcal{O}_X^{\oplus t}[-n] \rightarrow E \rightarrow C \rightarrow \mathcal{O}_X^{\oplus t}[-n+1]$$

Then  $C$  has vanishing cohomology in degrees  $\geq n$ . On the other hand, the complex  $\epsilon^*C$  is  $m$ -pseudo-coherent, see Cohomology on Sites, Lemma 45.4. Hence by induction we see that  $C$  is  $m$ -pseudo-coherent. Applying Cohomology on Sites, Lemma 45.4 once more we conclude.  $\square$

08HF **Lemma 13.3.** *Let  $X$  be a scheme. Let  $E$  be an object of  $D(\mathcal{O}_X)$ . Then*

- (1)  *$E$  has tor amplitude in  $[a, b]$  if and only if  $\epsilon^*E$  has tor amplitude in  $[a, b]$ .*
- (2)  *$E$  has finite tor dimension if and only if  $\epsilon^*E$  has finite tor dimension.*

*Here  $\epsilon$  is as in (4.0.1).*

**Proof.** The easy implication follows from Cohomology on Sites, Lemma 46.5. For the converse, assume that  $\epsilon^*E$  has tor amplitude in  $[a, b]$ . Let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module. As  $\epsilon$  is a flat morphism of ringed sites (Lemma 4.1) we have

$$\epsilon^*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}) = \epsilon^*E \otimes_{\mathcal{O}_{\text{étale}}}^{\mathbf{L}} \epsilon^*\mathcal{F}$$

Thus the (assumed) vanishing of cohomology sheaves on the right hand side implies the desired vanishing of the cohomology sheaves of  $E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}$  via Lemma 4.1.  $\square$

0DK7 **Lemma 13.4.** *Let  $f : X \rightarrow Y$  be a morphism of schemes. Let  $E$  be an object of  $D(\mathcal{O}_X)$ . Then*

- (1)  *$E$  as an object of  $D(f^{-1}\mathcal{O}_Y)$  has tor amplitude in  $[a, b]$  if and only if  $\epsilon^*E$  has tor amplitude in  $[a, b]$  as an object of  $D(f_{\text{small}}^{-1}\mathcal{O}_{Y_{\text{étale}}})$ .*
- (2)  *$E$  locally has finite tor dimension as an object of  $D(f^{-1}\mathcal{O}_Y)$  if and only if  $\epsilon^*E$  locally has finite tor dimension as an object of  $D(f_{\text{small}}^{-1}\mathcal{O}_{Y_{\text{étale}}})$ .*

*Here  $\epsilon$  is as in (4.0.1).*

**Proof.** The easy direction in (1) follows from Cohomology on Sites, Lemma 46.5. Let  $x \in X$  be a point and let  $\bar{x}$  be a geometric point lying over  $x$ . Let  $y = f(x)$  and denote  $\bar{y}$  the geometric point of  $Y$  coming from  $\bar{x}$ . Then  $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y, \bar{y}}$  (Sheaves, Lemma 21.5) and

$$(f_{\text{small}}^{-1}\mathcal{O}_{Y_{\text{étale}}})_{\bar{x}} = \mathcal{O}_{Y_{\text{étale}}, \bar{y}} = \mathcal{O}_{Y, y}^{sh}$$

is the strict henselization (by Étale Cohomology, Lemmas 36.2 and 33.1). Since the stalk of  $\mathcal{O}_{X_{\text{étale}}}$  at  $X$  is  $\mathcal{O}_{X, x}^{sh}$  we obtain

$$(\epsilon^*E)_{\bar{x}} = E_x \otimes_{\mathcal{O}_{X, x}}^{\mathbf{L}} \mathcal{O}_{X, x}^{sh}$$

by transitivity of pullbacks. If  $\epsilon^*E$  has tor amplitude in  $[a, b]$  as a complex of  $f_{small}^{-1}\mathcal{O}_{Y_{\acute{e}tale}}$ -modules, then  $(\epsilon^*E)_{\overline{x}}$  has tor amplitude in  $[a, b]$  as a complex of  $\mathcal{O}_{Y,y}^{sh}$ -modules (because taking stalks is a pullback and lemma cited above). By More on Flatness, Lemma 2.6 we find the tor amplitude of  $(\epsilon^*E)_{\overline{x}}$  as a complex of  $\mathcal{O}_{Y,y}$ -modules is in  $[a, b]$ . Since  $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}^{sh}$  is faithfully flat (More on Algebra, Lemma 45.1) and since  $(\epsilon^*E)_{\overline{x}} = E_x \otimes_{\mathcal{O}_{X,x}}^{\mathbf{L}} \mathcal{O}_{X,x}^{sh}$  we may apply More on Algebra, Lemma 66.18 to conclude the tor amplitude of  $E_x$  as a complex of  $\mathcal{O}_{Y,y}$ -modules is in  $[a, b]$ . By Cohomology, Lemma 48.5 we conclude that  $E$  as an object of  $D(f^{-1}\mathcal{O}_Y)$  has tor amplitude in  $[a, b]$ . This gives the reverse implication in (1). Part (2) follows formally from (1).  $\square$

08HG **Lemma 13.5.** *Let  $X$  be a scheme. Let  $E$  be an object of  $D(\mathcal{O}_X)$ . Then  $E$  is a perfect object of  $D(\mathcal{O}_X)$  if and only if  $\epsilon^*E$  is a perfect object of  $D(\mathcal{O}_{\acute{e}tale})$ . Here  $\epsilon$  is as in (4.0.1).*

**Proof.** The easy implication follows from the general result contained in Cohomology on Sites, Lemma 47.5. For the converse, we can use the equivalence of Cohomology on Sites, Lemma 47.4 and the corresponding results for pseudo-coherent and complexes of finite tor dimension, namely Lemmas 13.2 and 13.3. Some details omitted.  $\square$

08JL **Lemma 13.6.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . If  $E$  is an  $m$ -pseudo-coherent object of  $D(\mathcal{O}_X)$ , then  $H^i(E)$  is a quasi-coherent  $\mathcal{O}_X$ -module for  $i > m$ . If  $E$  is pseudo-coherent, then  $E$  is an object of  $D_{QCoH}(\mathcal{O}_X)$ .*

**Proof.** Locally  $H^i(E)$  is isomorphic to  $H^i(\mathcal{E}^\bullet)$  with  $\mathcal{E}^\bullet$  strictly perfect. The sheaves  $\mathcal{E}^i$  are direct summands of finite free modules, hence quasi-coherent. The lemma follows.  $\square$

08IK **Lemma 13.7.** *Let  $S$  be a scheme. Let  $X$  be a Noetherian algebraic space over  $S$ . Let  $E$  be an object of  $D_{QCoH}(\mathcal{O}_X)$ . For  $m \in \mathbf{Z}$  the following are equivalent*

- (1)  $H^i(E)$  is coherent for  $i \geq m$  and zero for  $i \gg 0$ , and
- (2)  $E$  is  $m$ -pseudo-coherent.

*In particular,  $E$  is pseudo-coherent if and only if  $E$  is an object of  $D_{CoH}^-(\mathcal{O}_X)$ .*

**Proof.** As  $X$  is quasi-compact we can find an affine scheme  $U$  and a surjective étale morphism  $U \rightarrow X$  (Properties of Spaces, Lemma 6.3). Observe that  $U$  is Noetherian. Note that  $E$  is  $m$ -pseudo-coherent if and only if  $E|_U$  is  $m$ -pseudo-coherent (follows from the definition or from Cohomology on Sites, Lemma 45.2). Similarly,  $H^i(E)$  is coherent if and only if  $H^i(E)|_U = H^i(E|_U)$  is coherent (see Cohomology of Spaces, Lemma 12.2). Thus we may assume that  $X$  is representable.

If  $X$  is representable by a scheme  $X_0$  then (Lemma 4.2) we can write  $E = \epsilon^*E_0$  where  $E_0$  is an object of  $D_{QCoH}(\mathcal{O}_{X_0})$  and  $\epsilon : X_{\acute{e}tale} \rightarrow (X_0)_{Zar}$  is as in (4.0.1). In this case  $E$  is  $m$ -pseudo-coherent if and only if  $E_0$  is by Lemma 13.2. Similarly,  $H^i(E_0)$  is of finite type (i.e., coherent) if and only if  $H^i(E)$  is by Lemma 13.1. Finally,  $H^i(E_0) = 0$  if and only if  $H^i(E) = 0$  by Lemma 4.1. Thus we reduce to the case of schemes which is Derived Categories of Schemes, Lemma 10.3.  $\square$

08IL **Lemma 13.8.** *Let  $S$  be a scheme. Let  $X$  be a quasi-separated algebraic space over  $S$ . Let  $E$  be an object of  $D_{QCoH}(\mathcal{O}_X)$ . Let  $a \leq b$ . The following are equivalent*

- (1)  $E$  has tor amplitude in  $[a, b]$ , and

(2) for all  $\mathcal{F}$  in  $QCoh(\mathcal{O}_X)$  we have  $H^i(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}) = 0$  for  $i \notin [a, b]$ .

**Proof.** It is clear that (1) implies (2). Assume (2). Let  $j : U \rightarrow X$  be an étale morphism with  $U$  affine. As  $X$  is quasi-separated  $j : U \rightarrow X$  is quasi-compact and separated, hence  $j_*$  transforms quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 11.2). Thus the functor  $QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_U)$  is essentially surjective. It follows that condition (2) implies the vanishing of  $H^i(E|_U \otimes_{\mathcal{O}_U}^{\mathbf{L}} \mathcal{G})$  for  $i \notin [a, b]$  for all quasi-coherent  $\mathcal{O}_U$ -modules  $\mathcal{G}$ . Since it suffices to prove that  $E|_U$  has tor amplitude in  $[a, b]$  we reduce to the case where  $X$  is representable.

If  $X$  is representable by a scheme  $X_0$  then (Lemma 4.2) we can write  $E = \epsilon^* E_0$  where  $E_0$  is an object of  $D_{QCoh}(\mathcal{O}_{X_0})$  and  $\epsilon : X_{\acute{e}tale} \rightarrow (X_0)_{Zar}$  is as in (4.0.1). For every quasi-coherent module  $\mathcal{F}_0$  on  $X_0$  the module  $\epsilon^* \mathcal{F}_0$  is quasi-coherent on  $X$  and

$$H^i(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \epsilon^* \mathcal{F}_0) = \epsilon^* H^i(E_0 \otimes_{\mathcal{O}_{X_0}}^{\mathbf{L}} \mathcal{F}_0)$$

as  $\epsilon$  is flat (Lemma 4.1). Moreover, the vanishing of these sheaves for  $i \notin [a, b]$  implies the same thing for  $H^i(E_0 \otimes_{\mathcal{O}_{X_0}}^{\mathbf{L}} \mathcal{F}_0)$  by the same lemma. Thus we've reduced the problem to the case of schemes which is treated in Derived Categories of Schemes, Lemma 10.6.  $\square$

08JP **Lemma 13.9.** *Let  $X$  be a scheme. Let  $E, F$  be objects of  $D(\mathcal{O}_X)$ . Assume either*

- (1)  *$E$  is pseudo-coherent and  $F$  lies in  $D^+(\mathcal{O}_X)$ , or*
- (2)  *$E$  is perfect and  $F$  arbitrary,*

*then there is a canonical isomorphism*

$$\epsilon^* R\mathcal{H}om(E, F) \longrightarrow R\mathcal{H}om(\epsilon^* E, \epsilon^* F)$$

*Here  $\epsilon$  is as in (4.0.1).*

**Proof.** Recall that  $\epsilon$  is flat (Lemma 4.1) and hence  $\epsilon^* = L\epsilon^*$ . There is a canonical map from left to right by Cohomology on Sites, Remark 35.11. To see this is an isomorphism we can work locally, i.e., we may assume  $X$  is an affine scheme.

In case (1) we can represent  $E$  by a bounded above complex  $\mathcal{E}^\bullet$  of finite free  $\mathcal{O}_X$ -modules, see Derived Categories of Schemes, Lemma 13.3. We may also represent  $F$  by a bounded below complex  $\mathcal{F}^\bullet$  of  $\mathcal{O}_X$ -modules. Applying Cohomology, Lemma 46.11 we see that  $R\mathcal{H}om(E, F)$  is represented by the complex with terms

$$\bigoplus_{n=-p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{F}^q)$$

Applying Cohomology on Sites, Lemma 44.10 we see that  $R\mathcal{H}om(\epsilon^* E, \epsilon^* F)$  is represented by the complex with terms

$$\bigoplus_{n=-p+q} \mathcal{H}om_{\mathcal{O}_{\acute{e}tale}}(\epsilon^* \mathcal{E}^p, \epsilon^* \mathcal{F}^q)$$

Thus the statement of the lemma boils down to the true fact that the canonical map

$$\epsilon^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{\acute{e}tale}}(\epsilon^* \mathcal{E}, \epsilon^* \mathcal{F})$$

is an isomorphism for any  $\mathcal{O}_X$ -module  $\mathcal{F}$  and finite free  $\mathcal{O}_X$ -module  $\mathcal{E}$ .

In case (2) we can represent  $E$  by a strictly perfect complex  $\mathcal{E}^\bullet$  of  $\mathcal{O}_X$ -modules, use Derived Categories of Schemes, Lemmas 3.5 and 10.7 and the fact that a perfect

complex of modules is represented by a finite complex of finite projective modules. Thus we can do the exact same proof as above, replacing the reference to Cohomology, Lemma 46.11 by a reference to Cohomology, Lemma 46.9.  $\square$

0A8A **Lemma 13.10.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $L, K$  be objects of  $D(\mathcal{O}_X)$ . If either*

- (1)  $L$  in  $D_{Q\text{Coh}}^+(\mathcal{O}_X)$  and  $K$  is pseudo-coherent,
- (2)  $L$  in  $D_{Q\text{Coh}}(\mathcal{O}_X)$  and  $K$  is perfect,

*then  $R\mathcal{H}om(K, L)$  is in  $D_{Q\text{Coh}}(\mathcal{O}_X)$ .*

**Proof.** This follows from the analogue for schemes (Derived Categories of Schemes, Lemma 10.8) via the criterion of Lemma 5.2, the criterion of Lemmas 13.2 and 13.5, and the result of Lemma 13.9.  $\square$

0E4Q **Lemma 13.11.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $K, L, M$  be objects of  $D_{Q\text{Coh}}(\mathcal{O}_X)$ . The map*

$$K \otimes_{\mathcal{O}_X}^{\mathbf{L}} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$$

*of Cohomology on Sites, Lemma 35.7 is an isomorphism in the following cases*

- (1)  $M$  perfect, or
- (2)  $K$  is perfect, or
- (3)  $M$  is pseudo-coherent,  $L \in D^+(\mathcal{O}_X)$ , and  $K$  has finite tor dimension.

**Proof.** Checking whether or not the map is an isomorphism can be done étale locally hence we may assume  $X$  is an affine scheme. Then we can write  $K, L, M$  as  $\epsilon^*K_0, \epsilon^*L_0, \epsilon^*M_0$  for some  $K_0, L_0, M_0$  in  $D_{Q\text{Coh}}(\mathcal{O}_X)$  by Lemma 4.2. Then we see that Lemma 13.9 reduces cases (1) and (3) to the case of schemes which is Derived Categories of Schemes, Lemma 10.9. If  $K$  is perfect but no other assumptions are made, then we do not know that either side of the arrow is in  $D_{Q\text{Coh}}(\mathcal{O}_X)$  but the result is still true because  $K$  will be represented (after localizing further) by a finite complex of finite free modules in which case it is clear.  $\square$

## 14. Approximation by perfect complexes

08HH In this section we continue the discussion started in Derived Categories of Schemes, Section 14.

08HI **Definition 14.1.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Consider triples  $(T, E, m)$  where

- (1)  $T \subset |X|$  is a closed subset,
- (2)  $E$  is an object of  $D_{Q\text{Coh}}(\mathcal{O}_X)$ , and
- (3)  $m \in \mathbf{Z}$ .

We say *approximation holds for the triple  $(T, E, m)$*  if there exists a perfect object  $P$  of  $D(\mathcal{O}_X)$  supported on  $T$  and a map  $\alpha : P \rightarrow E$  which induces isomorphisms  $H^i(P) \rightarrow H^i(E)$  for  $i > m$  and a surjection  $H^m(P) \rightarrow H^m(E)$ .

Approximation cannot hold for every triple. Please read the remarks following Derived Categories of Schemes, Definition 14.1 to see why.

08HJ **Definition 14.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . We say *approximation by perfect complexes holds on  $X$*  if for any closed subset  $T \subset |X|$  such that the morphism  $X \setminus T \rightarrow X$  is quasi-compact there exists an integer  $r$  such that for every triple  $(T, E, m)$  as in Definition 14.1 with

- (1)  $E$  is  $(m - r)$ -pseudo-coherent, and
- (2)  $H^i(E)$  is supported on  $T$  for  $i \geq m - r$

approximation holds.

08HK **Lemma 14.3.** *Let  $S$  be a scheme. Let  $(U \subset X, j : V \rightarrow X)$  be an elementary distinguished square of algebraic space over  $S$ . Let  $E$  be a perfect object of  $D(\mathcal{O}_V)$  supported on  $j^{-1}(T)$  where  $T = |X| \setminus |U|$ . Then  $Rj_*E$  is a perfect object of  $D(\mathcal{O}_X)$ .*

**Proof.** Being perfect is local on  $X_{\text{étale}}$ . Thus it suffices to check that  $Rj_*E$  is perfect when restricted to  $U$  and  $V$ . We have  $Rj_*E|_V = E$  by Lemma 10.7 which is perfect. We have  $Rj_*E|_U = 0$  because  $E|_{V \setminus j^{-1}(T)} = 0$  (use Lemma 3.1).  $\square$

08HL **Lemma 14.4.** *Let  $S$  be a scheme. Let  $(U \subset X, j : V \rightarrow X)$  be an elementary distinguished square of algebraic spaces over  $S$ . Let  $T$  be a closed subset of  $|X| \setminus |U|$  and let  $(T, E, m)$  be a triple as in Definition 14.1. If*

- (1) *approximation holds for  $(j^{-1}T, E|_V, m)$ , and*
- (2) *the sheaves  $H^i(E)$  for  $i \geq m$  are supported on  $T$ ,*

*then approximation holds for  $(T, E, m)$ .*

**Proof.** Let  $P \rightarrow E|_V$  be an approximation of the triple  $(j^{-1}T, E|_V, m)$  over  $V$ . Then  $Rj_*P$  is a perfect object of  $D(\mathcal{O}_X)$  by Lemma 14.3. On the other hand,  $Rj_*P = j_!P$  by Lemma 10.7. We see that  $j_!P$  is supported on  $T$  for example by (10.0.2). Hence we obtain an approximation  $Rj_*P = j_!P \rightarrow j_!(E|_V) \rightarrow E$ .  $\square$

08HM **Lemma 14.5.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$  which is representable by an affine scheme. Then approximation holds for every triple  $(T, E, m)$  as in Definition 14.1 such that there exists an integer  $r \geq 0$  with*

- (1)  *$E$  is  $m$ -pseudo-coherent,*
- (2)  *$H^i(E)$  is supported on  $T$  for  $i \geq m - r + 1$ ,*
- (3)  *$X \setminus T$  is the union of  $r$  affine opens.*

*In particular, approximation by perfect complexes holds for affine schemes.*

**Proof.** Let  $X_0$  be an affine scheme representing  $X$ . Let  $T_0 \subset X_0$  be the closed subset corresponding to  $T$ . Let  $\epsilon : X_{\text{étale}} \rightarrow X_{0, \text{Zar}}$  be the morphism (4.0.1). We may write  $E = \epsilon^*E_0$  for some object  $E_0$  of  $D_{Q\text{Coh}}(\mathcal{O}_{X_0})$ , see Lemma 4.2. Then  $E_0$  is  $m$ -pseudo-coherent, see Lemma 13.2. Comparing stalks of cohomology sheaves (see proof of Lemma 4.1) we see that  $H^i(E_0)$  is supported on  $T_0$  for  $i \geq m - r + 1$ . By Derived Categories of Schemes, Lemma 14.4 there exists an approximation  $P_0 \rightarrow E_0$  of  $(T_0, E_0, m)$ . By Lemma 13.5 we see that  $P = \epsilon^*P_0$  is a perfect object of  $D(\mathcal{O}_X)$ . Pulling back we obtain an approximation  $P = \epsilon^*P_0 \rightarrow \epsilon^*E_0 = E$  as desired.  $\square$

08HN **Lemma 14.6.** *Let  $S$  be a scheme. Let  $(U \subset X, j : V \rightarrow X)$  be an elementary distinguished square of algebraic spaces over  $S$ . Assume  $U$  quasi-compact,  $V$  affine, and  $U \times_X V$  quasi-compact. If approximation by perfect complexes holds on  $U$ , then approximation by perfect complexes holds on  $X$ .*

**Proof.** Let  $T \subset |X|$  be a closed subset with  $X \setminus T \rightarrow X$  quasi-compact. Let  $r_U$  be the integer of Definition 14.2 adapted to the pair  $(U, T \cap |U|)$ . Set  $T' = T \setminus |U|$ . Endow  $T'$  with the induced reduced subspace structure. Since  $|T'|$  is contained in  $|X| \setminus |U|$  we see that  $j^{-1}(T') \rightarrow T'$  is an isomorphism. Moreover,  $V \setminus j^{-1}(T')$  is quasi-compact as it is the fibre product of  $U \times_X V$  with  $X \setminus T$  over  $X$  and we've assumed  $U \times_X V$  quasi-compact and  $X \setminus T \rightarrow X$  quasi-compact. Let  $r'$  be the

number of affines needed to cover  $V \setminus j^{-1}(T')$ . We claim that  $r = \max(r_U, r')$  works for the pair  $(X, T)$ .

To see this choose a triple  $(T, E, m)$  such that  $E$  is  $(m - r)$ -pseudo-coherent and  $H^i(E)$  is supported on  $T$  for  $i \geq m - r$ . Let  $t$  be the largest integer such that  $H^t(E)|_U$  is nonzero. (Such an integer exists as  $U$  is quasi-compact and  $E|_U$  is  $(m - r)$ -pseudo-coherent.) We will prove that  $E$  can be approximated by induction on  $t$ .

Base case:  $t \leq m - r'$ . This means that  $H^i(E)$  is supported on  $T'$  for  $i \geq m - r'$ . Hence Lemma 14.5 guarantees the existence of an approximation  $P \rightarrow E|_V$  of  $(T', E|_V, m)$  on  $V$ . Applying Lemma 14.4 we see that  $(T', E, m)$  can be approximated. Such an approximation is also an approximation of  $(T, E, m)$ .

Induction step. Choose an approximation  $P \rightarrow E|_U$  of  $(T \cap |U|, E|_U, m)$ . This in particular gives a surjection  $H^t(P) \rightarrow H^t(E|_U)$ . In the rest of the proof we will use the equivalence of Lemma 4.2 (and the compatibilities of Remark 6.3) for the representable algebraic spaces  $V$  and  $U \times_X V$ . We will also use the fact that  $(m - r)$ -pseudo-coherence, resp. perfectness on the Zariski site and étale site agree, see Lemmas 13.2 and 13.5. Thus we can use the results of Derived Categories of Schemes, Section 13 for the open immersion  $U \times_X V \subset V$ . In this way Derived Categories of Schemes, Lemma 13.10 implies there exists a perfect object  $Q$  in  $D(\mathcal{O}_V)$  supported on  $j^{-1}(T)$  and an isomorphism  $Q|_{U \times_X V} \rightarrow (P \oplus P[1])|_{U \times_X V}$ . By Derived Categories of Schemes, Lemma 13.7 we can replace  $Q$  by  $Q \otimes^{\mathbf{L}} I$  and assume that the map

$$Q|_{U \times_X V} \longrightarrow (P \oplus P[1])|_{U \times_X V} \longrightarrow P|_{U \times_X V} \longrightarrow E|_{U \times_X V}$$

lifts to  $Q \rightarrow E|_V$ . By Lemma 10.8 we find an morphism  $a : R \rightarrow E$  of  $D(\mathcal{O}_X)$  such that  $a|_U$  is isomorphic to  $P \oplus P[1] \rightarrow E|_U$  and  $a|_V$  isomorphic to  $Q \rightarrow E|_V$ . Thus  $R$  is perfect and supported on  $T$  and the map  $H^t(R) \rightarrow H^t(E)$  is surjective on restriction to  $U$ . Choose a distinguished triangle

$$R \rightarrow E \rightarrow E' \rightarrow R[1]$$

Then  $E'$  is  $(m - r)$ -pseudo-coherent (Cohomology on Sites, Lemma 45.4),  $H^i(E')|_U = 0$  for  $i \geq t$ , and  $H^i(E')$  is supported on  $T$  for  $i \geq m - r$ . By induction we find an approximation  $R' \rightarrow E'$  of  $(T, E', m)$ . Fit the composition  $R' \rightarrow E' \rightarrow R[1]$  into a distinguished triangle  $R \rightarrow R'' \rightarrow R' \rightarrow R[1]$  and extend the morphisms  $R' \rightarrow E'$  and  $R[1] \rightarrow R[1]$  into a morphism of distinguished triangles

$$\begin{array}{ccccccc} R & \longrightarrow & R'' & \longrightarrow & R' & \longrightarrow & R[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & R[1] \end{array}$$

using TR3. Then  $R''$  is a perfect complex (Cohomology on Sites, Lemma 47.6) supported on  $T$ . An easy diagram chase shows that  $R'' \rightarrow E$  is the desired approximation.  $\square$

08HP **Theorem 14.7.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Then approximation by perfect complexes holds on  $X$ .*

**Proof.** This follows from the induction principle of Lemma 9.3 and Lemmas 14.6 and 14.5.  $\square$

### 15. Generating derived categories

09IU This section is the analogue of Derived Categories of Schemes, Section 15. However, we first prove the following lemma which is the analogue of Derived Categories of Schemes, Lemma 13.11.

09IV **Lemma 15.1.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $W \subset X$  be a quasi-compact open. Let  $T \subset |X|$  be a closed subset such that  $X \setminus T \rightarrow X$  is a quasi-compact morphism. Let  $E$  be an object of  $D_{Q\text{Coh}}(\mathcal{O}_X)$ . Let  $\alpha : P \rightarrow E|_W$  be a map where  $P$  is a perfect object of  $D(\mathcal{O}_W)$  supported on  $T \cap W$ . Then there exists a map  $\beta : R \rightarrow E$  where  $R$  is a perfect object of  $D(\mathcal{O}_X)$  supported on  $T$  such that  $P$  is a direct summand of  $R|_W$  in  $D(\mathcal{O}_W)$  compatible  $\alpha$  and  $\beta|_W$ .*

**Proof.** We will use the induction principle of Lemma 9.6 to prove this. Thus we immediately reduce to the case where we have an elementary distinguished square  $(W \subset X, f : V \rightarrow X)$  with  $V$  affine and  $P \rightarrow E|_W$  as in the statement of the lemma. In the rest of the proof we will use Lemma 4.2 (and the compatibilities of Remark 6.3) for the representable algebraic spaces  $V$  and  $W \times_X V$ . We will also use the fact that perfectness on the Zariski site and étale site agree, see Lemma 13.5.

By Derived Categories of Schemes, Lemma 13.10 we can choose a perfect object  $Q$  in  $D(\mathcal{O}_V)$  supported on  $f^{-1}T$  and an isomorphism  $Q|_{W \times_X V} \rightarrow (P \oplus P[1])|_{W \times_X V}$ . By Derived Categories of Schemes, Lemma 13.7 we can replace  $Q$  by  $Q \otimes^{\mathbf{L}} I$  (still supported on  $f^{-1}T$ ) and assume that the map

$$Q|_{W \times_X V} \rightarrow (P \oplus P[1])|_{W \times_X V} \longrightarrow P|_{W \times_X V} \longrightarrow E|_{W \times_X V}$$

lifts to  $Q \rightarrow E|_V$ . By Lemma 10.8 we find an morphism  $a : R \rightarrow E$  of  $D(\mathcal{O}_X)$  such that  $a|_W$  is isomorphic to  $P \oplus P[1] \rightarrow E|_W$  and  $a|_V$  isomorphic to  $Q \rightarrow E|_V$ . Thus  $R$  is perfect and supported on  $T$  as desired.  $\square$

09IW **Remark 15.2.** The proof of Lemma 15.1 shows that

$$R|_W = P \oplus P^{\oplus n_1}[1] \oplus \dots \oplus P^{\oplus n_m}[m]$$

for some  $m \geq 0$  and  $n_j \geq 0$ . Thus the highest degree cohomology sheaf of  $R|_W$  equals that of  $P$ . By repeating the construction for the map  $P^{\oplus n_1}[1] \oplus \dots \oplus P^{\oplus n_m}[m] \rightarrow R|_W$ , taking cones, and using induction we can achieve equality of cohomology sheaves of  $R|_W$  and  $P$  above any given degree.

09IX **Lemma 15.3.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $W$  be a quasi-compact open subspace of  $X$ . Let  $P$  be a perfect object of  $D(\mathcal{O}_W)$ . Then  $P$  is a direct summand of the restriction of a perfect object of  $D(\mathcal{O}_X)$ .*

**Proof.** Special case of Lemma 15.1.  $\square$

09IY **Theorem 15.4.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . The category  $D_{Q\text{Coh}}(\mathcal{O}_X)$  can be generated by a single perfect object. More precisely, there exists a perfect object  $P$  of  $D(\mathcal{O}_X)$  such that for  $E \in D_{Q\text{Coh}}(\mathcal{O}_X)$  the following are equivalent*

- (1)  $E = 0$ , and
- (2)  $\text{Hom}_{D(\mathcal{O}_X)}(P[n], E) = 0$  for all  $n \in \mathbf{Z}$ .

**Proof.** We will prove this using the induction principle of Lemma 9.3.

If  $X$  is affine, then  $\mathcal{O}_X$  is a perfect generator. This follows from Lemma 4.2 and Derived Categories of Schemes, Lemma 3.5.

Assume that  $(U \subset X, f : V \rightarrow X)$  is an elementary distinguished square with  $U$  quasi-compact such that the theorem holds for  $U$  and  $V$  is an affine scheme. Let  $P$  be a perfect object of  $D(\mathcal{O}_U)$  which is a generator for  $D_{Qcoh}(\mathcal{O}_U)$ . Using Lemma 15.3 we may choose a perfect object  $Q$  of  $D(\mathcal{O}_X)$  whose restriction to  $U$  is a direct sum one of whose summands is  $P$ . Say  $V = \text{Spec}(A)$ . Let  $Z \subset V$  be the reduced closed subscheme which is the inverse image of  $X \setminus U$  and maps isomorphically to it (see Definition 9.1). This is a retrocompact closed subset of  $V$ . Choose  $f_1, \dots, f_r \in A$  such that  $Z = V(f_1, \dots, f_r)$ . Let  $K \in D(\mathcal{O}_V)$  be the perfect object corresponding to the Koszul complex on  $f_1, \dots, f_r$  over  $A$ . Note that since  $K$  is supported on  $Z$ , the pushforward  $K' = Rf_*K$  is a perfect object of  $D(\mathcal{O}_X)$  whose restriction to  $V$  is  $K$  (see Lemmas 14.3 and 10.7). We claim that  $Q \oplus K'$  is a generator for  $D_{Qcoh}(\mathcal{O}_X)$ .

Let  $E$  be an object of  $D_{Qcoh}(\mathcal{O}_X)$  such that there are no nontrivial maps from any shift of  $Q \oplus K'$  into  $E$ . By Lemma 10.7 we have  $K' = f_!K$  and hence

$$\text{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \text{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Derived Categories of Schemes, Lemma 15.2 (using also Lemma 4.2) the vanishing of these groups implies that  $E|_V$  is isomorphic to  $R(U \times_X V \rightarrow V)_*E|_{U \times_X V}$ . This implies that  $E = R(U \rightarrow X)_*E|_U$  (small detail omitted). If this is the case then

$$\text{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \text{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains  $\text{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$  as a direct summand. Thus by our choice of  $P$  the vanishing of these groups implies that  $E|_U$  is zero. Whence  $E$  is zero.  $\square$

0E4R **Remark 15.5.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of quasi-compact and quasi-separated algebraic spaces over  $S$ . Let  $E \in D_{Qcoh}(\mathcal{O}_Y)$  be a generator (see Theorem 15.4). Then the following are equivalent

- (1) for  $K \in D_{Qcoh}(\mathcal{O}_X)$  we have  $Rf_*K = 0$  if and only if  $K = 0$ ,
- (2)  $Rf_* : D_{Qcoh}(\mathcal{O}_X) \rightarrow D_{Qcoh}(\mathcal{O}_Y)$  reflects isomorphisms, and
- (3)  $Lf^*E$  is a generator for  $D_{Qcoh}(\mathcal{O}_X)$ .

The equivalence between (1) and (2) is a formal consequence of the fact that  $Rf_* : D_{Qcoh}(\mathcal{O}_X) \rightarrow D_{Qcoh}(\mathcal{O}_Y)$  is an exact functor of triangulated categories. Similarly, the equivalence between (1) and (3) follows formally from the fact that  $Lf^*$  is the left adjoint to  $Rf_*$ . These conditions hold if  $f$  is affine (Lemma 6.4) or if  $f$  is an open immersion, or if  $f$  is a composition of such.

The following result is a strengthening of Theorem 15.4 proved using exactly the same methods. Let  $T \subset |X|$  be a closed subset where  $X$  is an algebraic space. Let's denote  $D_T(\mathcal{O}_X)$  the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on  $T$ .

0AEC **Lemma 15.6.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $T \subset |X|$  be a closed subset such that  $|X| \setminus T$  is quasi-compact. With notation as above, the category  $D_{Qcoh,T}(\mathcal{O}_X)$  is generated by a single perfect object.*

**Proof.** We will prove this using the induction principle of Lemma 9.3. The property is true for representable quasi-compact and quasi-separated objects of the site  $X_{spaces, \acute{e}tale}$  by Derived Categories of Schemes, Lemma 15.4.

Assume that  $(U \subset X, f : V \rightarrow X)$  is an elementary distinguished square such that the lemma holds for  $U$  and  $V$  is affine. To finish the proof we have to show that the result holds for  $X$ . Let  $P$  be a perfect object of  $D(\mathcal{O}_U)$  supported on  $T \cap U$  which is a generator for  $D_{QCoh, T \cap U}(\mathcal{O}_U)$ . Using Lemma 15.1 we may choose a perfect object  $Q$  of  $D(\mathcal{O}_X)$  supported on  $T$  whose restriction to  $U$  is a direct sum one of whose summands is  $P$ . Write  $V = \text{Spec}(B)$ . Let  $Z = X \setminus U$ . Then  $f^{-1}Z$  is a closed subset of  $V$  such that  $V \setminus f^{-1}Z$  is quasi-compact. As  $X$  is quasi-separated, it follows that  $f^{-1}Z \cap f^{-1}T = f^{-1}(Z \cap T)$  is a closed subset of  $V$  such that  $W = V \setminus f^{-1}(Z \cap T)$  is quasi-compact. Thus we can choose  $g_1, \dots, g_s \in B$  such that  $f^{-1}(Z \cap T) = V(g_1, \dots, g_s)$ . Let  $K \in D(\mathcal{O}_V)$  be the perfect object corresponding to the Koszul complex on  $g_1, \dots, g_s$  over  $B$ . Note that since  $K$  is supported on  $f^{-1}(Z \cap T) \subset V$  closed, the pushforward  $K' = R(V \rightarrow X)_* K$  is a perfect object of  $D(\mathcal{O}_X)$  whose restriction to  $V$  is  $K$  (see Lemmas 14.3 and 10.7). We claim that  $Q \oplus K'$  is a generator for  $D_{QCoh, T}(\mathcal{O}_X)$ .

Let  $E$  be an object of  $D_{QCoh, T}(\mathcal{O}_X)$  such that there are no nontrivial maps from any shift of  $Q \oplus K'$  into  $E$ . By Lemma 10.7 we have  $K' = R(V \rightarrow X)_! K$  and hence

$$\text{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \text{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Derived Categories of Schemes, Lemma 15.2 we have  $E|_V = Rj_* E|_W$  where  $j : W \rightarrow V$  is the inclusion. Picture

$$\begin{array}{ccccc} W & \xrightarrow{\quad} & V & \xleftarrow{\quad} & Z \cap T \\ & \nearrow j & & \nwarrow & \downarrow \\ V \setminus f^{-1}Z & & & & Z \end{array}$$

Since  $E$  is supported on  $T$  we see that  $E|_W$  is supported on  $f^{-1}T \cap W = f^{-1}T \cap (V \setminus f^{-1}Z)$  which is closed in  $W$ . We conclude that

$$E|_V = Rj_*(E|_W) = Rj_*(Rj'_*(E|_{U \cap V})) = Rj''_*(E|_{U \cap V})$$

Here the second equality is part (1) of Cohomology, Lemma 33.6 which applies because  $V$  is a scheme and  $E$  has quasi-coherent cohomology sheaves hence pushforward along the quasi-compact open immersion  $j'$  agrees with pushforward on the underlying schemes, see Remark 6.3. This implies that  $E = R(U \rightarrow X)_* E|_U$  (small detail omitted). If this is the case then

$$\text{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \text{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains  $\text{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$  as a direct summand. Thus by our choice of  $P$  the vanishing of these groups implies that  $E|_U$  is zero. Whence  $E$  is zero.  $\square$

## 16. Compact and perfect objects

09M7 This section is the analogue of Derived Categories of Schemes, Section 17.

09M8 **Proposition 16.1.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . An object of  $D_{QCoh}(\mathcal{O}_X)$  is compact if and only if it is perfect.*

**Proof.** If  $K$  is a perfect object of  $D(\mathcal{O}_X)$  with dual  $K^\vee$  (Cohomology on Sites, Lemma 48.4) we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$$

functorially in  $M$ . Since  $K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} -$  commutes with direct sums and since  $H^0(X, -)$  commutes with direct sums on  $D_{QCoh}(\mathcal{O}_X)$  by Lemma 6.2 we conclude that  $K$  is compact in  $D_{QCoh}(\mathcal{O}_X)$ .

Conversely, let  $K$  be a compact object of  $D_{QCoh}(\mathcal{O}_X)$ . To show that  $K$  is perfect, it suffices to show that  $K|_U$  is perfect for every affine scheme  $U$  étale over  $X$ , see Cohomology on Sites, Lemma 47.2. Observe that  $j : U \rightarrow X$  is a quasi-compact and separated morphism. Hence  $Rj_* : D_{QCoh}(\mathcal{O}_U) \rightarrow D_{QCoh}(\mathcal{O}_X)$  commutes with direct sums, see Lemma 6.2. Thus the adjointness of restriction to  $U$  and  $Rj_*$  implies that  $K|_U$  is a perfect object of  $D_{QCoh}(\mathcal{O}_U)$ . Hence we reduce to the case that  $X$  is affine, in particular a quasi-compact and quasi-separated scheme. Via Lemma 4.2 and 13.5 we reduce to the case of schemes, i.e., to Derived Categories of Schemes, Proposition 17.1.  $\square$

**OGFC Remark 16.2.** Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $G$  be a perfect object of  $D(\mathcal{O}_X)$  which is a generator for  $D_{QCoh}(\mathcal{O}_X)$ . By Theorem 15.4 there is at least one of these. Combining Lemma 5.3 with Proposition 16.1 and with Derived Categories, Proposition 37.6 we see that  $G$  is a classical generator for  $D_{perf}(\mathcal{O}_X)$ .

The following result is a strengthening of Proposition 16.1. Let  $T \subset |X|$  be a closed subset where  $X$  is an algebraic space. As before  $D_T(\mathcal{O}_X)$  denotes the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on  $T$ . Since taking direct sums commutes with taking cohomology sheaves, it follows that  $D_T(\mathcal{O}_X)$  has direct sums and that they are equal to direct sums in  $D(\mathcal{O}_X)$ .

**0AED Lemma 16.3.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $T \subset |X|$  be a closed subset such that  $|X| \setminus T$  is quasi-compact. An object of  $D_{QCoh,T}(\mathcal{O}_X)$  is compact if and only if it is perfect as an object of  $D(\mathcal{O}_X)$ .*

**Proof.** We observe that  $D_{QCoh,T}(\mathcal{O}_X)$  is a triangulated category with direct sums by the remark preceding the lemma. By Proposition 16.1 the perfect objects define compact objects of  $D(\mathcal{O}_X)$  hence a fortiori of any subcategory preserved under taking direct sums. For the converse we will use there exists a generator  $E \in D_{QCoh,T}(\mathcal{O}_X)$  which is a perfect complex of  $\mathcal{O}_X$ -modules, see Lemma 15.6. Hence by the above,  $E$  is compact. Then it follows from Derived Categories, Proposition 37.6 that  $E$  is a classical generator of the full subcategory of compact objects of  $D_{QCoh,T}(\mathcal{O}_X)$ . Thus any compact object can be constructed out of  $E$  by a finite sequence of operations consisting of (a) taking shifts, (b) taking finite direct sums, (c) taking cones, and (d) taking direct summands. Each of these operations preserves the property of being perfect and the result follows.  $\square$

**0GFD Remark 16.4.** Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $T \subset |X|$  be a closed subset such that  $|X| \setminus T$  is quasi-compact. Let  $G$  be a perfect object of  $D_{QCoh,T}(\mathcal{O}_X)$  which is a generator for  $D_{QCoh,T}(\mathcal{O}_X)$ . By Lemma 15.6 there is at least one of these. Combining the fact

that  $D_{QCoh,T}(\mathcal{O}_X)$  has direct sums with Lemma 16.3 and with Derived Categories, Proposition 37.6 we see that  $G$  is a classical generator for  $D_{perf,T}(\mathcal{O}_X)$ .

The following lemma is an application of the ideas that go into the proof of the preceding lemma.

0AEE **Lemma 16.5.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $T \subset |X|$  be a closed subset such that the complement  $U \subset X$  is quasi-compact. Let  $\alpha : P \rightarrow E$  be a morphism of  $D_{QCoh}(\mathcal{O}_X)$  with either*

- (1)  *$P$  is perfect and  $E$  supported on  $T$ , or*
- (2)  *$P$  pseudo-coherent,  $E$  supported on  $T$ , and  $E$  bounded below.*

*Then there exists a perfect complex of  $\mathcal{O}_X$ -modules  $I$  and a map  $I \rightarrow \mathcal{O}_X[0]$  such that  $I \otimes^{\mathbf{L}} P \rightarrow E$  is zero and such that  $I|_U \rightarrow \mathcal{O}_U[0]$  is an isomorphism.*

**Proof.** Set  $\mathcal{D} = D_{QCoh,T}(\mathcal{O}_X)$ . In both cases the complex  $K = R\mathcal{H}om(P, E)$  is an object of  $\mathcal{D}$ . See Lemma 13.10 for quasi-coherence. It is clear that  $K$  is supported on  $T$  as formation of  $R\mathcal{H}om$  commutes with restriction to opens. The map  $\alpha$  defines an element of  $H^0(K) = \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[0], K)$ . Then it suffices to prove the result for the map  $\alpha : \mathcal{O}_X[0] \rightarrow K$ .

Let  $E \in \mathcal{D}$  be a perfect generator, see Lemma 15.6. Write

$$K = \text{hocolim} K_n$$

as in Derived Categories, Lemma 37.3 using the generator  $E$ . Since the functor  $\mathcal{D} \rightarrow D(\mathcal{O}_X)$  commutes with direct sums, we see that  $K = \text{hocolim} K_n$  holds in  $D(\mathcal{O}_X)$ . Since  $\mathcal{O}_X$  is a compact object of  $D(\mathcal{O}_X)$  we find an  $n$  and a morphism  $\alpha_n : \mathcal{O}_X \rightarrow K_n$  which gives rise to  $\alpha$ , see Derived Categories, Lemma 33.9. By Derived Categories, Lemma 37.4 applied to the morphism  $\mathcal{O}_X[0] \rightarrow K_n$  in the ambient category  $D(\mathcal{O}_X)$  we see that  $\alpha_n$  factors as  $\mathcal{O}_X[0] \rightarrow Q \rightarrow K_n$  where  $Q$  is an object of  $\langle E \rangle$ . We conclude that  $Q$  is a perfect complex supported on  $T$ .

Choose a distinguished triangle

$$I \rightarrow \mathcal{O}_X[0] \rightarrow Q \rightarrow I[1]$$

By construction  $I$  is perfect, the map  $I \rightarrow \mathcal{O}_X[0]$  restricts to an isomorphism over  $U$ , and the composition  $I \rightarrow K$  is zero as  $\alpha$  factors through  $Q$ . This proves the lemma.  $\square$

## 17. Derived categories as module categories

09M9 The section is the analogue of Derived Categories of Schemes, Section 18.

09MA **Lemma 17.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $K^\bullet$  be a complex of  $\mathcal{O}_X$ -modules whose cohomology sheaves are quasi-coherent. Let  $(E, d) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$  be the endomorphism differential graded algebra. Then the functor*

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, d) \longrightarrow D(\mathcal{O}_X)$$

*of Differential Graded Algebra, Lemma 35.3 has image contained in  $D_{QCoh}(\mathcal{O}_X)$ .*

**Proof.** Let  $P$  be a differential graded  $E$ -module with property  $P$ . Let  $F_\bullet$  be a filtration on  $P$  as in Differential Graded Algebra, Section 20. Then we have

$$P \otimes_E K^\bullet = \text{hocolim} F_i P \otimes_E K^\bullet$$

Each of the  $F_i P$  has a finite filtration whose graded pieces are direct sums of  $E[k]$ . The result follows easily.  $\square$

The following lemma can be strengthened (there is a uniformity in the vanishing over all  $L$  with nonzero cohomology sheaves only in a fixed range).

09MB **Lemma 17.2.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $K$  be a perfect object of  $D(\mathcal{O}_X)$ . Then*

- (1) *there exist integers  $a \leq b$  such that  $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L) = 0$  for  $L \in D_{Q\mathrm{Coh}}(\mathcal{O}_X)$  with  $H^i(L) = 0$  for  $i \in [a, b]$ , and*
- (2) *if  $L$  is bounded, then  $\mathrm{Ext}_{D(\mathcal{O}_X)}^n(K, L)$  is zero for all but finitely many  $n$ .*

**Proof.** Part (2) follows from (1) as  $\mathrm{Ext}_{D(\mathcal{O}_X)}^n(K, L) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K, L[n])$ . We prove (1). Since  $K$  is perfect we have

$$\mathrm{Ext}_{D(\mathcal{O}_X)}^i(K, L) = H^i(X, K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$$

where  $K^\vee$  is the “dual” perfect complex to  $K$ , see Cohomology on Sites, Lemma 48.4. Note that  $P = K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L$  is in  $D_{Q\mathrm{Coh}}(X)$  by Lemmas 5.6 and 13.6 (to see that a perfect complex has quasi-coherent cohomology sheaves). Say  $K^\vee$  has tor amplitude in  $[a, b]$ . Then the spectral sequence

$$E_1^{p,q} = H^p(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} H^q(L)) \Rightarrow H^{p+q}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$$

shows that  $H^j(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$  is zero if  $H^q(L) = 0$  for  $q \in [j-b, j-a]$ . Let  $N$  be the integer  $\max(d_p + p)$  of Cohomology of Spaces, Lemma 7.3. Then  $H^0(X, K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$  vanishes if the cohomology sheaves

$$H^{-N}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L), H^{-N+1}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L), \dots, H^0(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L)$$

are zero. Namely, by the lemma cited and Lemma 5.8, we have

$$H^0(X, K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L) = H^0(X, \tau_{\geq -N}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L))$$

and by the vanishing of cohomology sheaves, this is equal to  $H^0(X, \tau_{\geq 1}(K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} L))$  which is zero by Derived Categories, Lemma 16.1. It follows that  $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L)$  is zero if  $H^i(L) = 0$  for  $i \in [-b-N, -a]$ .  $\square$

The following is the analogue of Derived Categories of Schemes, Theorem 18.2.

09MC **Theorem 17.3.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Then there exist a differential graded algebra  $(E, d)$  with only a finite number of nonzero cohomology groups  $H^i(E)$  such that  $D_{Q\mathrm{Coh}}(\mathcal{O}_X)$  is equivalent to  $D(E, d)$ .*

**Proof.** Let  $K^\bullet$  be a K-injective complex of  $\mathcal{O}$ -modules which is perfect and generates  $D_{Q\mathrm{Coh}}(\mathcal{O}_X)$ . Such a thing exists by Theorem 15.4 and the existence of K-injective resolutions. We will show the theorem holds with

$$(E, d) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

where  $\mathrm{Comp}^{dg}(\mathcal{O}_X)$  is the differential graded category of complexes of  $\mathcal{O}$ -modules. Please see Differential Graded Algebra, Section 35. Since  $K^\bullet$  is K-injective we have

09MD (17.3.1) 
$$H^n(E) = \mathrm{Ext}_{D(\mathcal{O}_X)}^n(K^\bullet, K^\bullet)$$

for all  $n \in \mathbf{Z}$ . Only a finite number of these Exts are nonzero by Lemma 17.2. Consider the functor

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, d) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 35.3. Since  $K^\bullet$  is perfect, it defines a compact object of  $D(\mathcal{O}_X)$ , see Proposition 16.1. Combined with (17.3.1) the functor above is fully faithful as follows from Differential Graded Algebra, Lemmas 35.6. It has a right adjoint

$$R\mathrm{Hom}(K^\bullet, -) : D(\mathcal{O}_X) \longrightarrow D(E, d)$$

by Differential Graded Algebra, Lemmas 35.5 which is a left quasi-inverse functor by generalities on adjoint functors. On the other hand, it follows from Lemma 17.1 that we obtain

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, d) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

and by our choice of  $K^\bullet$  as a generator of  $D_{QCoh}(\mathcal{O}_X)$  the kernel of the adjoint restricted to  $D_{QCoh}(\mathcal{O}_X)$  is zero. A formal argument shows that we obtain the desired equivalence, see Derived Categories, Lemma 7.2.  $\square$

0DK8 **Remark 17.4** (Variant with support). Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space. Let  $T \subset |X|$  be a closed subset such that  $|X| \setminus T$  is quasi-compact. The analogue of Theorem 17.3 holds for  $D_{QCoh, T}(\mathcal{O}_X)$ . This follows from the exact same argument as in the proof of the theorem, using Lemmas 15.6 and 16.3 and a variant of Lemma 17.1 with supports. If we ever need this, we will precisely state the result here and give a detailed proof.

0DK9 **Remark 17.5** (Uniqueness of dga). Let  $X$  be a quasi-compact and quasi-separated algebraic space over a ring  $R$ . By the construction of the proof of Theorem 17.3 there exists a differential graded algebra  $(A, d)$  over  $R$  such that  $D_{QCoh}(X)$  is  $R$ -linearly equivalent to  $D(A, d)$  as a triangulated category. One may ask: how unique is  $(A, d)$ ? The answer is (only) slightly better than just saying that  $(A, d)$  is well defined up to derived equivalence. Namely, suppose that  $(B, d)$  is a second such pair. Then we have

$$(A, d) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

and

$$(B, d) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, L^\bullet)$$

for some K-injective complexes  $K^\bullet$  and  $L^\bullet$  of  $\mathcal{O}_X$ -modules corresponding to perfect generators of  $D_{QCoh}(\mathcal{O}_X)$ . Set

$$\Omega = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, L^\bullet) \quad \Omega' = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, K^\bullet)$$

Then  $\Omega$  is a differential graded  $B^{opp} \otimes_R A$ -module and  $\Omega'$  is a differential graded  $A^{opp} \otimes_R B$ -module. Moreover, the equivalence

$$D(A, d) \rightarrow D_{QCoh}(\mathcal{O}_X) \rightarrow D(B, d)$$

is given by the functor  $- \otimes_A^{\mathbf{L}} \Omega'$  and similarly for the quasi-inverse. Thus we are in the situation of Differential Graded Algebra, Remark 37.10. If we ever need this remark we will provide a precise statement with a detailed proof here.

### 18. Characterizing pseudo-coherent complexes, I

0DKA This material will be continued in More on Morphisms of Spaces, Section 51. We can characterize pseudo-coherent objects as derived homotopy limits of approximations by perfect objects.

0DKB **Lemma 18.1.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $K \in D(\mathcal{O}_X)$ . The following are equivalent*

- (1)  *$K$  is pseudo-coherent, and*
- (2)  *$K = \text{hocolim} K_n$  where  $K_n$  is perfect and  $\tau_{\geq -n} K_n \rightarrow \tau_{\geq -n} K$  is an isomorphism for all  $n$ .*

**Proof.** The implication (2)  $\Rightarrow$  (1) is true on any ringed site. Namely, assume (2) holds. Recall that a perfect object of the derived category is pseudo-coherent, see Cohomology on Sites, Lemma 47.4. Then it follows from the definitions that  $\tau_{\geq -n} K_n$  is  $(-n+1)$ -pseudo-coherent and hence  $\tau_{\geq -n} K$  is  $(-n+1)$ -pseudo-coherent, hence  $K$  is  $(-n+1)$ -pseudo-coherent. This is true for all  $n$ , hence  $K$  is pseudo-coherent, see Cohomology on Sites, Definition 45.1.

Assume (1). We start by choosing an approximation  $K_1 \rightarrow K$  of  $(X, K, -2)$  by a perfect complex  $K_1$ , see Definitions 14.1 and 14.2 and Theorem 14.7. Suppose by induction we have

$$K_1 \rightarrow K_2 \rightarrow \dots \rightarrow K_n \rightarrow K$$

with  $K_i$  perfect such that  $\tau_{\geq -i} K_i \rightarrow \tau_{\geq -i} K$  is an isomorphism for all  $1 \leq i \leq n$ . Then we pick  $a \leq b$  as in Lemma 17.2 for the perfect object  $K_n$ . Choose an approximation  $K_{n+1} \rightarrow K$  of  $(X, K, \min(a-1, -n-1))$ . Choose a distinguished triangle

$$K_{n+1} \rightarrow K \rightarrow C \rightarrow K_{n+1}[1]$$

Then we see that  $C \in D_{QCoh}(\mathcal{O}_X)$  has  $H^i(C) = 0$  for  $i \geq a$ . Thus by our choice of  $a, b$  we see that  $\text{Hom}_{D(\mathcal{O}_X)}(K_n, C) = 0$ . Hence the composition  $K_n \rightarrow K \rightarrow C$  is zero. Hence by Derived Categories, Lemma 4.2 we can factor  $K_n \rightarrow K$  through  $K_{n+1}$  proving the induction step.

We still have to prove that  $K = \text{hocolim} K_n$ . This follows by an application of Derived Categories, Lemma 33.8 to the functors  $H^i(-) : D(\mathcal{O}_X) \rightarrow \text{Mod}(\mathcal{O}_X)$  and our choice of  $K_n$ .  $\square$

0DKC **Lemma 18.2.** *Let  $X$  be a quasi-compact and quasi-separated scheme. Let  $T \subset X$  be a closed subset such that  $X \setminus T$  is quasi-compact. Let  $K \in D(\mathcal{O}_X)$  supported on  $T$ . The following are equivalent*

- (1)  *$K$  is pseudo-coherent, and*
- (2)  *$K = \text{hocolim} K_n$  where  $K_n$  is perfect, supported on  $T$ , and  $\tau_{\geq -n} K_n \rightarrow \tau_{\geq -n} K$  is an isomorphism for all  $n$ .*

**Proof.** The proof of this lemma is exactly the same as the proof of Lemma 18.1 except that in the choice of the approximations we use the triples  $(T, K, m)$ .  $\square$

### 19. The coherator revisited

0CR3 In Section 11 we constructed and studied the right adjoint  $RQ_X$  to the canonical functor  $D(QCoh(\mathcal{O}_X)) \rightarrow D(\mathcal{O}_X)$ . It was constructed as the right derived extension

of the coherator  $Q_X : \text{Mod}(\mathcal{O}_X) \rightarrow Q\text{Coh}(\mathcal{O}_X)$ . In this section, we study when the inclusion functor

$$D_{Q\text{Coh}}(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

has a right adjoint. If this right adjoint exists, we will denote<sup>2</sup> it

$$DQ_X : D(\mathcal{O}_X) \longrightarrow D_{Q\text{Coh}}(\mathcal{O}_X)$$

It turns out that quasi-compact and quasi-separated algebraic spaces have such a right adjoint.

0CR4 **Lemma 19.1.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . The inclusion functor  $D_{Q\text{Coh}}(\mathcal{O}_X) \rightarrow D(\mathcal{O}_X)$  has a right adjoint.*

**First proof.** We will use the induction principle in Lemma 9.3 to prove this. If  $D(Q\text{Coh}(\mathcal{O}_X)) \rightarrow D_{Q\text{Coh}}(\mathcal{O}_X)$  is an equivalence, then the lemma is true because the functor  $RQ_X$  of Section 11 is a right adjoint to the functor  $D(Q\text{Coh}(\mathcal{O}_X)) \rightarrow D(\mathcal{O}_X)$ . In particular, our lemma is true for affine algebraic spaces, see Lemma 11.3. Thus we see that it suffices to show: if  $(U \subset X, f : V \rightarrow X)$  is an elementary distinguished square with  $U$  quasi-compact and  $V$  affine and the lemma holds for  $U$ ,  $V$ , and  $U \times_X V$ , then the lemma holds for  $X$ .

The adjoint exists if and only if for every object  $K$  of  $D(\mathcal{O}_X)$  we can find a distinguished triangle

$$E' \rightarrow E \rightarrow K \rightarrow E'[1]$$

in  $D(\mathcal{O}_X)$  such that  $E'$  is in  $D_{Q\text{Coh}}(\mathcal{O}_X)$  and such that  $\text{Hom}(M, K) = 0$  for all  $M$  in  $D_{Q\text{Coh}}(\mathcal{O}_X)$ . See Derived Categories, Lemma 40.7. Consider the distinguished triangle

$$E \rightarrow Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \rightarrow Rj_{U \times_X V,*}E|_{U \times_X V} \rightarrow E[1]$$

in  $D(\mathcal{O}_X)$  of Lemma 10.2. By Derived Categories, Lemma 40.5 it suffices to construct the desired distinguished triangles for  $Rj_{U,*}E|_U$ ,  $Rj_{V,*}E|_V$ , and  $Rj_{U \times_X V,*}E|_{U \times_X V}$ . This reduces us to the statement discussed in the next paragraph.

Let  $j : U \rightarrow X$  be an étale morphism corresponding with  $U$  quasi-compact and quasi-separated and the lemma is true for  $U$ . Let  $L$  be an object of  $D(\mathcal{O}_U)$ . Then there exists a distinguished triangle

$$E' \rightarrow Rj_*L \rightarrow K \rightarrow E'[1]$$

in  $D(\mathcal{O}_X)$  such that  $E'$  is in  $D_{Q\text{Coh}}(\mathcal{O}_X)$  and such that  $\text{Hom}(M, K) = 0$  for all  $M$  in  $D_{Q\text{Coh}}(\mathcal{O}_X)$ . To see this we choose a distinguished triangle

$$L' \rightarrow L \rightarrow Q \rightarrow L'[1]$$

in  $D(\mathcal{O}_U)$  such that  $L'$  is in  $D_{Q\text{Coh}}(\mathcal{O}_U)$  and such that  $\text{Hom}(N, Q) = 0$  for all  $N$  in  $D_{Q\text{Coh}}(\mathcal{O}_U)$ . This is possible because the statement in Derived Categories, Lemma 40.7 is an if and only if. We obtain a distinguished triangle

$$Rj_*L' \rightarrow Rj_*L \rightarrow Rj_*Q \rightarrow Rj_*L'[1]$$

in  $D(\mathcal{O}_X)$ . Observe that  $Rj_*L'$  is in  $D_{Q\text{Coh}}(\mathcal{O}_X)$  by Lemma 6.1. On the other hand, if  $M$  in  $D_{Q\text{Coh}}(\mathcal{O}_X)$ , then

$$\text{Hom}(M, Rj_*Q) = \text{Hom}(Lj^*M, Q) = 0$$

---

<sup>2</sup>This is probably nonstandard notation. However, we have already used  $Q_X$  for the coherator and  $RQ_X$  for its derived extension.

because  $Lj^*M$  is in  $D_{QCoh}(\mathcal{O}_U)$  by Lemma 5.5. This finishes the proof.  $\square$

**Second proof.** The adjoint exists by Derived Categories, Proposition 38.2. The hypotheses are satisfied: First, note that  $D_{QCoh}(\mathcal{O}_X)$  has direct sums and direct sums commute with the inclusion functor (Lemma 5.3). On the other hand,  $D_{QCoh}(\mathcal{O}_X)$  is compactly generated because it has a perfect generator Theorem 15.4 and because perfect objects are compact by Proposition 16.1.  $\square$

0CR5 **Lemma 19.2.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a quasi-compact and quasi-separated morphism of algebraic spaces over  $S$ . If the right adjoints  $DQ_X$  and  $DQ_Y$  of the inclusion functors  $D_{QCoh} \rightarrow D$  exist for  $X$  and  $Y$ , then*

$$Rf_* \circ DQ_X = DQ_Y \circ Rf_*$$

**Proof.** The statement makes sense because  $Rf_*$  sends  $D_{QCoh}(\mathcal{O}_X)$  into  $D_{QCoh}(\mathcal{O}_Y)$  by Lemma 6.1. The statement is true because  $Lf^*$  similarly maps  $D_{QCoh}(\mathcal{O}_Y)$  into  $D_{QCoh}(\mathcal{O}_X)$  (Lemma 5.5) and hence both  $Rf_* \circ DQ_X$  and  $DQ_Y \circ Rf_*$  are right adjoint to  $Lf^* : D_{QCoh}(\mathcal{O}_Y) \rightarrow D(\mathcal{O}_X)$ .  $\square$

0CR6 **Remark 19.3.** Let  $S$  be a scheme. Let  $(U \subset X, f : V \rightarrow X)$  be an elementary distinguished square of algebraic spaces over  $S$ . Assume  $X, U, V$  are quasi-compact and quasi-separated. By Lemma 19.1 the functors  $DQ_X, DQ_U, DQ_V, DQ_{U \times_X V}$  exist. Moreover, there is a canonical distinguished triangle

$$DQ_X(K) \rightarrow Rj_{U,*}DQ_U(K|_U) \oplus Rj_{V,*}DQ_V(K|_V) \rightarrow Rj_{U \times_X V,*}DQ_{U \times_X V}(K|_{U \times_X V}) \rightarrow$$

for any  $K \in D(\mathcal{O}_X)$ . This follows by applying the exact functor  $DQ_X$  to the distinguished triangle of Lemma 10.2 and using Lemma 19.2 three times.

0CSS **Lemma 19.4.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . The functor  $DQ_X$  of Lemma 19.1 has the following boundedness property: there exists an integer  $N = N(X)$  such that, if  $K$  in  $D(\mathcal{O}_X)$  with  $H^i(U, K) = 0$  for  $U$  affine étale over  $X$  and  $i \notin [a, b]$ , then the cohomology sheaves  $H^i(DQ_X(K))$  are zero for  $i \notin [a, b + N]$ .*

**Proof.** We will prove this using the induction principle of Lemma 9.3.

If  $X$  is affine, then the lemma is true with  $N = 0$  because then  $RQ_X = DQ_X$  is given by taking the complex of quasi-coherent sheaves associated to  $R\Gamma(X, K)$ . See Lemma 11.3.

Let  $(U \subset W, f : V \rightarrow W)$  be an elementary distinguished square with  $W$  quasi-compact and quasi-separated,  $U \subset W$  quasi-compact open,  $V$  affine such that the lemma holds for  $U, V$ , and  $U \times_W V$ . Say with integers  $N(U), N(V)$ , and  $N(U \times_W V)$ . Now suppose  $K$  is in  $D(\mathcal{O}_X)$  with  $H^i(W, K) = 0$  for all affine  $W$  étale over  $X$  and all  $i \notin [a, b]$ . Then  $K|_U, K|_V, K|_{U \times_W V}$  have the same property. Hence we see that  $RQ_U(K|_U)$  and  $RQ_V(K|_V)$  and  $RQ_{U \cap V}(K|_{U \times_W V})$  have vanishing cohomology sheaves outside the interval  $[a, b + \max(N(U), N(V), N(U \times_W V))]$ . Since the functors  $Rj_{U,*}, Rj_{V,*}, Rj_{U \times_W V,*}$  have finite cohomological dimension on  $D_{QCoh}$  by Lemma 6.1 we see that there exists an  $N$  such that  $Rj_{U,*}DQ_U(K|_U), Rj_{V,*}DQ_V(K|_V)$ , and  $Rj_{U \cap V,*}DQ_{U \times_W V}(K|_{U \times_W V})$  have vanishing cohomology sheaves outside the interval  $[a, b + N]$ . Then finally we conclude by the distinguished triangle of Remark 19.3.  $\square$

0CST **Example 19.5.** Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $(\mathcal{F}_n)$  be an inverse system of quasi-coherent sheaves on  $X$ . Since  $DQ_X$  is a right adjoint it commutes with products and therefore with derived limits. Hence we see that

$$DQ_X(R\lim \mathcal{F}_n) = (R\lim \text{ in } D_{Q\text{Coh}}(\mathcal{O}_X))(\mathcal{F}_n)$$

where the first  $R\lim$  is taken in  $D(\mathcal{O}_X)$ . In fact, let's write  $K = R\lim \mathcal{F}_n$  for this. For any affine  $U$  étale over  $X$  we have

$$H^i(U, K) = H^i(R\Gamma(U, R\lim \mathcal{F}_n)) = H^i(R\lim R\Gamma(U, \mathcal{F}_n)) = H^i(R\lim \Gamma(U, \mathcal{F}_n))$$

since cohomology commutes with derived limits and since the quasi-coherent sheaves  $\mathcal{F}_n$  have no higher cohomology on affines. By the computation of  $R\lim$  in the category of abelian groups, we see that  $H^i(U, K) = 0$  unless  $i \in [0, 1]$ . Then finally we conclude that the  $R\lim$  in  $D_{Q\text{Coh}}(\mathcal{O}_X)$ , which is  $DQ_X(K)$  by the above, is in  $D_{Q\text{Coh}}^b(\mathcal{O}_X)$  and has vanishing cohomology sheaves in negative degrees by Lemma 19.4.

## 20. Cohomology and base change, IV

08IM This section is the analogue of Derived Categories of Schemes, Section 22.

08IN **Lemma 20.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a quasi-compact and quasi-separated morphism of algebraic spaces over  $S$ . For  $E$  in  $D_{Q\text{Coh}}(\mathcal{O}_X)$  and  $K$  in  $D_{Q\text{Coh}}(\mathcal{O}_Y)$  we have*

$$Rf_*(E) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} K = Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*K)$$

**Proof.** Without any assumptions there is a map  $Rf_*(E) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} K \rightarrow Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*K)$ . Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*(E) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} K) = Lf^*(Rf_*(E)) \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*K \longrightarrow E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*K$$

coming from the map  $Lf^*Rf_*E \rightarrow E$ . See Cohomology on Sites, Lemmas 18.4 and 19.1. To check it is an isomorphism we may work étale locally on  $Y$ . Hence we reduce to the case that  $Y$  is an affine scheme.

Suppose that  $K = \bigoplus K_i$  is a direct sum of some complexes  $K_i \in D_{Q\text{Coh}}(\mathcal{O}_Y)$ . If the statement holds for each  $K_i$ , then it holds for  $K$ . Namely, the functors  $Lf^*$  and  $\otimes^{\mathbf{L}}$  preserve direct sums by construction and  $Rf_*$  commutes with direct sums (for complexes with quasi-coherent cohomology sheaves) by Lemma 6.2. Moreover, suppose that  $K \rightarrow L \rightarrow M \rightarrow K[1]$  is a distinguished triangle in  $D_{Q\text{Coh}}(Y)$ . Then if the statement of the lemma holds for two of  $K, L, M$ , then it holds for the third (as the functors involved are exact functors of triangulated categories).

Assume  $Y$  affine, say  $Y = \text{Spec}(A)$ . The functor  $\sim : D(A) \rightarrow D_{Q\text{Coh}}(\mathcal{O}_Y)$  is an equivalence by Lemma 4.2 and Derived Categories of Schemes, Lemma 3.5. Let  $T$  be the property for  $K \in D(A)$  that the statement of the lemma holds for  $\tilde{K}$ . The discussion above and More on Algebra, Remark 59.11 shows that it suffices to prove  $T$  holds for  $A[k]$ . This finishes the proof, as the statement of the lemma is clear for shifts of the structure sheaf.  $\square$

08IP **Definition 20.2.** Let  $S$  be a scheme. Let  $B$  be an algebraic space over  $S$ . Let  $X, Y$  be algebraic spaces over  $B$ . We say  $X$  and  $Y$  are *Tor independent over  $B$*  if and only if for every commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\quad \bar{x} \quad} & X \\ \bar{y} \downarrow & \searrow \bar{b} & \downarrow \\ Y & \xrightarrow{\quad} & B \end{array}$$

of geometric points the rings  $\mathcal{O}_{X,\bar{x}}$  and  $\mathcal{O}_{Y,\bar{y}}$  are Tor independent over  $\mathcal{O}_{B,\bar{b}}$  (see More on Algebra, Definition 61.1).

The following lemma shows in particular that this definition agrees with our definition in the case of representable algebraic spaces.

08IQ **Lemma 20.3.** *Let  $S$  be a scheme. Let  $B$  be an algebraic space over  $S$ . Let  $X, Y$  be algebraic spaces over  $B$ . The following are equivalent*

- (1)  *$X$  and  $Y$  are Tor independent over  $B$ ,*
- (2) *for every commutative diagram*

$$\begin{array}{ccccc} U & \longrightarrow & W & \longleftarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & B & \longleftarrow & Y \end{array}$$

*with étale vertical arrows  $U$  and  $V$  are Tor independent over  $W$ ,*

- (3) *for some commutative diagram as in (2) with (a)  $W \rightarrow B$  étale surjective, (b)  $U \rightarrow X \times_B W$  étale surjective, (c)  $V \rightarrow Y \times_B W$  étale surjective, the spaces  $U$  and  $V$  are Tor independent over  $W$ , and*
- (4) *for some commutative diagram as in (3) with  $U, V, W$  schemes, the schemes  $U$  and  $V$  are Tor independent over  $W$  in the sense of Derived Categories of Schemes, Definition 22.2.*

**Proof.** For an étale morphism  $\varphi : U \rightarrow X$  of algebraic spaces and geometric point  $\bar{u}$  the map of local rings  $\mathcal{O}_{X,\varphi(\bar{u})} \rightarrow \mathcal{O}_{U,\bar{u}}$  is an isomorphism. Hence the equivalence of (1) and (2) follows. So does the implication (1)  $\Rightarrow$  (3). Assume (3) and pick a diagram of geometric points as in Definition 20.2. The assumptions imply that we can first lift  $\bar{b}$  to a geometric point  $\bar{w}$  of  $W$ , then lift the geometric point  $(\bar{x}, \bar{b})$  to a geometric point  $\bar{u}$  of  $U$ , and finally lift the geometric point  $(\bar{y}, \bar{b})$  to a geometric point  $\bar{v}$  of  $V$ . Use Properties of Spaces, Lemma 19.4 to find the lifts. Using the remark on local rings above we conclude that the condition of the definition is satisfied for the given diagram.

Having made these initial points, it is clear that (4) comes down to the statement that Definition 20.2 agrees with Derived Categories of Schemes, Definition 22.2 when  $X, Y$ , and  $B$  are schemes.

Let  $\bar{x}, \bar{b}, \bar{y}$  be as in Definition 20.2 lying over the points  $x, y, b$ . Recall that  $\mathcal{O}_{X,\bar{x}} = \mathcal{O}_{X,x}^{sh}$  (Properties of Spaces, Lemma 22.1) and similarly for the other two. By Algebra, Lemma 155.12 we see that  $\mathcal{O}_{X,\bar{x}}$  is a strict henselization of  $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{B,\bar{b}}$ . In particular, the ring map

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{B,\bar{b}} \longrightarrow \mathcal{O}_{X,\bar{x}}$$

is flat (More on Algebra, Lemma 45.1). By More on Algebra, Lemma 61.3 we see that

$$\mathrm{Tor}_i^{\mathcal{O}_{B,b}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y}) \otimes_{\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y}} (\mathcal{O}_{X,\bar{x}} \otimes_{\mathcal{O}_{B,\bar{b}}} \mathcal{O}_{Y,\bar{y}}) = \mathrm{Tor}_i^{\mathcal{O}_{B,\bar{b}}}(\mathcal{O}_{X,\bar{x}}, \mathcal{O}_{Y,\bar{y}})$$

Hence it follows that if  $X$  and  $Y$  are Tor independent over  $B$  as schemes, then  $X$  and  $Y$  are Tor independent as algebraic spaces over  $B$ .

For the converse, we may assume  $X$ ,  $Y$ , and  $B$  are affine. Observe that the ring map

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,\bar{x}} \otimes_{\mathcal{O}_{B,\bar{b}}} \mathcal{O}_{Y,\bar{y}}$$

is flat by the observations given above. Moreover, the image of the map on spectra includes all primes  $\mathfrak{s} \subset \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y}$  lying over  $\mathfrak{m}_x$  and  $\mathfrak{m}_y$ . Hence from this and the displayed formula of Tor's above we see that if  $X$  and  $Y$  are Tor independent over  $B$  as algebraic spaces, then

$$\mathrm{Tor}_i^{\mathcal{O}_{B,b}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y})_{\mathfrak{s}} = 0$$

for all  $i > 0$  and all  $\mathfrak{s}$  as above. By More on Algebra, Lemma 61.6 applied to the ring maps  $\Gamma(B, \mathcal{O}_B) \rightarrow \Gamma(X, \mathcal{O}_X)$  and  $\Gamma(B, \mathcal{O}_B) \rightarrow \Gamma(X, \mathcal{O}_X)$  this implies that  $X$  and  $Y$  are Tor independent over  $B$ .  $\square$

**08IR Lemma 20.4.** *Let  $S$  be a scheme. Let  $g : Y' \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $f : X \rightarrow Y$  be a quasi-compact and quasi-separated morphism of algebraic spaces over  $S$ . Consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

*If  $X$  and  $Y'$  are Tor independent over  $Y$ , then for all  $E \in D_{Qcoh}(\mathcal{O}_X)$  we have  $Rf'_*L(g')^*E = Lg^*Rf_*E$ .*

**Proof.** For any object  $E$  of  $D(\mathcal{O}_X)$  we can use Cohomology on Sites, Remark 19.3 to get a canonical base change map  $Lg^*Rf_*E \rightarrow Rf'_*L(g')^*E$ . To check this is an isomorphism we may work étale locally on  $Y'$ . Hence we may assume  $g : Y' \rightarrow Y$  is a morphism of affine schemes. In particular,  $g$  is affine and it suffices to show that

$$Rg_*Lg^*Rf_*E \rightarrow Rg_*Rf'_*L(g')^*E = Rf_*(Rg'_*L(g')^*E)$$

is an isomorphism, see Lemma 6.4 (and use Lemmas 5.5, 5.6, and 6.1 to see that the objects  $Rf'_*L(g')^*E$  and  $Lg^*Rf_*E$  have quasi-coherent cohomology sheaves). Note that  $g'$  is affine as well (Morphisms of Spaces, Lemma 20.5). By Lemma 6.5 the map becomes a map

$$Rf_*E \otimes_{\mathcal{O}_Y}^L g_*\mathcal{O}_{Y'} \longrightarrow Rf_*(E \otimes_{\mathcal{O}_X}^L g'_*\mathcal{O}_{X'})$$

Observe that  $g'_*\mathcal{O}_{X'} = f^*g_*\mathcal{O}_{Y'}$ . Thus by Lemma 20.1 it suffices to prove that  $Lf^*g_*\mathcal{O}_{Y'} = f^*g_*\mathcal{O}_{Y'}$ . This follows from our assumption that  $X$  and  $Y'$  are Tor independent over  $Y$ . Namely, to check it we may work étale locally on  $X$ , hence we may also assume  $X$  is affine. Say  $X = \mathrm{Spec}(A)$ ,  $Y = \mathrm{Spec}(R)$  and  $Y' = \mathrm{Spec}(R')$ . Our assumption implies that  $A$  and  $R'$  are Tor independent over  $R$  (see Lemma 20.3 and More on Algebra, Lemma 61.6), i.e.,  $\mathrm{Tor}_i^R(A, R') = 0$  for  $i > 0$ . In other words  $A \otimes_R^L R' = A \otimes_R R'$  which exactly means that  $Lf^*g_*\mathcal{O}_{Y'} = f^*g_*\mathcal{O}_{Y'}$ .  $\square$

The following lemma will be used in the chapter on dualizing complexes.

0E4S **Lemma 20.5.** *Let  $g : S' \rightarrow S$  be a morphism of affine schemes. Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

*of quasi-compact and quasi-separated algebraic spaces. Assume  $g$  and  $f$  Tor independent. Write  $S = \operatorname{Spec}(R)$  and  $S' = \operatorname{Spec}(R')$ . For  $M, K \in D(\mathcal{O}_X)$  the canonical map*

$$R\operatorname{Hom}_X(M, K) \otimes_R^{\mathbf{L}} R' \longrightarrow R\operatorname{Hom}_{X'}(L(g')^* M, L(g')^* K)$$

*in  $D(R')$  is an isomorphism in the following two cases*

- (1)  $M \in D(\mathcal{O}_X)$  is perfect and  $K \in D_{Q\operatorname{Coh}}(X)$ , or
- (2)  $M \in D(\mathcal{O}_X)$  is pseudo-coherent,  $K \in D_{Q\operatorname{Coh}}^+(X)$ , and  $R'$  has finite tor dimension over  $R$ .

**Proof.** There is a canonical map  $R\operatorname{Hom}_X(M, K) \rightarrow R\operatorname{Hom}_{X'}(L(g')^* M, L(g')^* K)$  in  $D(\Gamma(X, \mathcal{O}_X))$  of global hom complexes, see Cohomology on Sites, Section 36. Restricting scalars we can view this as a map in  $D(R)$ . Then we can use the adjointness of restriction and  $- \otimes_R^{\mathbf{L}} R'$  to get the displayed map of the lemma. Having defined the map it suffices to prove it is an isomorphism in the derived category of abelian groups.

The right hand side is equal to

$$R\operatorname{Hom}_X(M, R(g')_* L(g')^* K) = R\operatorname{Hom}_X(M, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_* \mathcal{O}_{X'})$$

by Lemma 6.5. In both cases the complex  $R\operatorname{Hom}(M, K)$  is an object of  $D_{Q\operatorname{Coh}}(\mathcal{O}_X)$  by Lemma 13.10. There is a natural map

$$R\operatorname{Hom}(M, K) \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_* \mathcal{O}_{X'} \longrightarrow R\operatorname{Hom}(M, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_* \mathcal{O}_{X'})$$

which is an isomorphism in both cases Lemma 13.11. To see that this lemma applies in case (2) we note that  $g'_* \mathcal{O}_{X'} = Rg'_* \mathcal{O}_{X'} = Lf^* g_* \mathcal{O}_X$  the second equality by Lemma 20.4. Using Derived Categories of Schemes, Lemma 10.4, Lemma 13.3, and Cohomology on Sites, Lemma 46.5 we conclude that  $g'_* \mathcal{O}_{X'}$  has finite Tor dimension. Hence, in both cases by replacing  $K$  by  $R\operatorname{Hom}(M, K)$  we reduce to proving

$$R\Gamma(X, K) \otimes_A^{\mathbf{L}} A' \longrightarrow R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_* \mathcal{O}_{X'})$$

is an isomorphism. Note that the left hand side is equal to  $R\Gamma(X', L(g')^* K)$  by Lemma 6.5. Hence the result follows from Lemma 20.4.  $\square$

0E4T **Remark 20.6.** With notation as in Lemma 20.5. The diagram

$$\begin{array}{ccc} R\operatorname{Hom}_X(M, Rg'_* L) \otimes_R^{\mathbf{L}} R' & \longrightarrow & R\operatorname{Hom}_{X'}(L(g')^* M, L(g')^* Rg'_* L) \\ \mu \downarrow & & \downarrow a \\ R\operatorname{Hom}_X(M, R(g')_* L) & \xlongequal{\quad} & R\operatorname{Hom}_{X'}(L(g')^* M, L) \end{array}$$

is commutative where the top horizontal arrow is the map from the lemma,  $\mu$  is the multiplication map, and  $a$  comes from the adjunction map  $L(g')^* Rg'_* L \rightarrow L$ . The multiplication map is the adjunction map  $K' \otimes_R^{\mathbf{L}} R' \rightarrow K'$  for any  $K' \in D(R')$ .

0DKD **Lemma 20.7.** *Let  $S$  be a scheme. Consider a cartesian square of algebraic spaces*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

over  $S$ . Assume  $g$  and  $f$  Tor independent.

- (1) *If  $E \in D(\mathcal{O}_X)$  has tor amplitude in  $[a, b]$  as a complex of  $f^{-1}\mathcal{O}_Y$ -modules, then  $L(g')^*E$  has tor amplitude in  $[a, b]$  as a complex of  $f'^{-1}\mathcal{O}_{Y'}$ -modules.*
- (2) *If  $\mathcal{G}$  is an  $\mathcal{O}_X$ -module flat over  $Y$ , then  $L(g')^*\mathcal{G} = (g')^*\mathcal{G}$ .*

**Proof.** We can compute tor dimension at stalks, see Cohomology on Sites, Lemma 46.10 and Properties of Spaces, Theorem 19.12. If  $\bar{x}'$  is a geometric point of  $X'$  with image  $\bar{x}$  in  $X$ , then

$$(L(g')^*E)_{\bar{x}'} = E_{\bar{x}} \otimes_{\mathcal{O}_{X, \bar{x}}}^{\mathbf{L}} \mathcal{O}_{X', \bar{x}'}$$

Let  $\bar{y}'$  in  $Y'$  and  $\bar{y}$  in  $Y$  be the image of  $\bar{x}'$  and  $\bar{x}$ . Since  $X$  and  $Y'$  are tor independent over  $Y$ , we can apply More on Algebra, Lemma 61.2 to see that the right hand side of the displayed formula is equal to  $E_{\bar{x}} \otimes_{\mathcal{O}_{Y, \bar{y}}}^{\mathbf{L}} \mathcal{O}_{Y', \bar{y}'}$  in  $D(\mathcal{O}_{Y', \bar{y}'})$ . Thus (1) follows from More on Algebra, Lemma 66.13. To see (2) observe that flatness of  $\mathcal{G}$  is equivalent to the condition that  $\mathcal{G}[0]$  has tor amplitude in  $[0, 0]$ . Applying (1) we conclude.  $\square$

## 21. Cohomology and base change, V

0DKD This section is the analogue of Derived Categories of Schemes, Section 26. In Section 20 we saw a base change theorem holds when the morphisms are tor independent. Even in the affine case there cannot be a base change theorem without such a condition, see More on Algebra, Section 61. In this section we analyze when one can get a base change result “one complex at a time”.

To make this work, let  $S$  be a base scheme and suppose we have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

of algebraic spaces over  $S$  (usually we will assume it is cartesian). Let  $K \in D_{Qcoh}(\mathcal{O}_X)$  and let  $L(g')^*K \rightarrow K'$  be a map in  $D_{Qcoh}(\mathcal{O}_{X'})$ . For a geometric point  $\bar{x}'$  of  $X'$  consider the geometric points  $\bar{x} = g'(\bar{x}')$ ,  $\bar{y}' = f'(\bar{x}')$ ,  $\bar{y} = f(\bar{x}) = g(\bar{y}')$  of  $X, Y', Y$ . Then we can consider the maps

$$K_{\bar{x}} \otimes_{\mathcal{O}_{Y, \bar{y}}}^{\mathbf{L}} \mathcal{O}_{Y', \bar{y}'} \rightarrow K_{\bar{x}} \otimes_{\mathcal{O}_{X, \bar{x}}}^{\mathbf{L}} \mathcal{O}_{X', \bar{x}'} \rightarrow K'_{\bar{x}'}$$

where the first arrow is More on Algebra, Equation (61.0.1) and the second comes from  $(L(g')^*K)_{\bar{x}'} = K_{\bar{x}} \otimes_{\mathcal{O}_{X, \bar{x}}}^{\mathbf{L}} \mathcal{O}_{X', \bar{x}'}$  and the given map  $L(g')^*K \rightarrow K'$ . For each  $i \in \mathbf{Z}$  we obtain a  $\mathcal{O}_{X, \bar{x}} \otimes_{\mathcal{O}_{Y, \bar{y}}} \mathcal{O}_{Y', \bar{y}'}$ -module structure on  $H^i(K_{\bar{x}} \otimes_{\mathcal{O}_{Y, \bar{y}}}^{\mathbf{L}} \mathcal{O}_{Y', \bar{y}'})$ . Putting everything together we obtain canonical maps

0DKF (21.0.1)  $H^i(K_{\bar{x}} \otimes_{\mathcal{O}_{Y, \bar{y}}}^{\mathbf{L}} \mathcal{O}_{Y', \bar{y}'}) \otimes_{(\mathcal{O}_{X, \bar{x}} \otimes_{\mathcal{O}_{Y, \bar{y}}} \mathcal{O}_{Y', \bar{y}'})} \mathcal{O}_{X', \bar{x}'} \longrightarrow H^i(K'_{\bar{x}'})$

of  $\mathcal{O}_{X', \bar{x}'}$ -modules.

ODKG **Lemma 21.1.** *Let  $S$  be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

*be a cartesian diagram of algebraic spaces over  $S$ . Let  $K \in D_{QCoh}(\mathcal{O}_X)$  and let  $L(g')^*K \rightarrow K'$  be a map in  $D_{QCoh}(\mathcal{O}_{X'})$ . The following are equivalent*

- (1) *for any  $x' \in X'$  and  $i \in \mathbf{Z}$  the map (21.0.1) is an isomorphism,*
- (2) *for any commutative diagram*

$$\begin{array}{ccccc} & & U & & \\ & & \downarrow & \searrow a & \\ V' & \xrightarrow{\quad} & V & & X \\ & \searrow c & \downarrow b & & \downarrow f \\ & & Y' & \xrightarrow{g} & Y \end{array}$$

*with  $a, b, c$  étale,  $U, V, V'$  schemes, and with  $U' = V' \times_V U$  the equivalent conditions of Derived Categories of Schemes, Lemma 26.1 hold for  $(U \rightarrow X)^*K$  and  $(U' \rightarrow X')^*K'$ , and*

- (3) *there is some diagram as in (2) with  $U' \rightarrow X'$  surjective.*

**Proof.** Observe that (1) is étale local on  $X'$ . Working through formal implications of what is known, we see that it suffices to prove condition (1) of this lemma is equivalent to condition (1) of Derived Categories of Schemes, Lemma 26.1 if  $X, Y, Y', X'$  are representable by schemes  $X_0, Y_0, Y'_0, X'_0$ . Denote  $f_0, g_0, g'_0, f'_0$  the morphisms between these schemes corresponding to  $f, g, g', f'$ . We may assume  $K = \epsilon^*K_0$  and  $K' = \epsilon'^*K'_0$  for some objects  $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$  and  $K'_0 \in D_{QCoh}(\mathcal{O}_{X'_0})$ , see Lemma 4.2. Moreover, the map  $Lg^*K \rightarrow K'$  is the pullback of a map  $L(g_0)^*K_0 \rightarrow K'_0$  with notation as in Remark 6.3. Recall that  $\mathcal{O}_{X, \bar{x}}$  is the strict henselization of  $\mathcal{O}_{X, x}$  (Properties of Spaces, Lemma 22.1) and that we have

$$K_{\bar{x}} = K_{0, x} \otimes_{\mathcal{O}_{X, x}}^{\mathbf{L}} \mathcal{O}_{X, \bar{x}} \quad \text{and} \quad K'_{\bar{x}'} = K'_{0, x'} \otimes_{\mathcal{O}_{X', x'}}^{\mathbf{L}} \mathcal{O}_{X', \bar{x}'}$$

(akin to Properties of Spaces, Lemma 29.4). Consider the commutative diagram

$$\begin{array}{ccc} H^i(K_{\bar{x}} \otimes_{\mathcal{O}_{Y, \bar{y}}}^{\mathbf{L}} \mathcal{O}_{Y', \bar{y}'}) \otimes_{(\mathcal{O}_{X, \bar{x}} \otimes_{\mathcal{O}_{Y, \bar{y}}} \mathcal{O}_{Y', \bar{y}'})} \mathcal{O}_{X', \bar{x}'} & \longrightarrow & H^i(K'_{\bar{x}'}) \\ \uparrow & & \uparrow \\ H^i(K_{0, x} \otimes_{\mathcal{O}_{Y, y}}^{\mathbf{L}} \mathcal{O}_{Y', y'}) \otimes_{(\mathcal{O}_{X, x} \otimes_{\mathcal{O}_{Y, y}} \mathcal{O}_{Y', y'})} \mathcal{O}_{X', x'} & \longrightarrow & H^i(K'_{0, x'}) \end{array}$$

We have to show that the lower horizontal arrow is an isomorphism if and only if the upper horizontal arrow is an isomorphism. Since  $\mathcal{O}_{X', x'} \rightarrow \mathcal{O}_{X', \bar{x}'}$  is faithfully flat (More on Algebra, Lemma 45.1) it suffices to show that the top arrow is the base change of the bottom arrow by this map. This follows immediately from the relationships between stalks given above for the objects on the right. For the objects

on the left it suffices to show that

$$\begin{aligned} & H^i \left( (K_{0,x} \otimes_{\mathcal{O}_{X,x}}^{\mathbf{L}} \mathcal{O}_{X,\bar{x}}) \otimes_{\mathcal{O}_{Y,\bar{y}}}^{\mathbf{L}} \mathcal{O}_{Y',\bar{y}'} \right) \\ &= H^i (K_{0,x} \otimes_{\mathcal{O}_{Y,y}}^{\mathbf{L}} \mathcal{O}_{Y',y'}) \otimes_{(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y',y'})} (\mathcal{O}_{X,\bar{x}} \otimes_{\mathcal{O}_{Y,\bar{y}}} \mathcal{O}_{Y',\bar{y}'} ) \end{aligned}$$

This follows from More on Algebra, Lemma 61.5. The flatness assumptions of this lemma hold by what was said above as well as Algebra, Lemma 155.12 implying that  $\mathcal{O}_{X,\bar{x}}$  is the strict henselization of  $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,\bar{y}}$  and that  $\mathcal{O}_{Y',\bar{y}'}$  is the strict henselization of  $\mathcal{O}_{Y',y'} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,\bar{y}}$ .  $\square$

ODKH **Lemma 21.2.** *Let  $S$  be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

*be a cartesian diagram of algebraic spaces over  $S$ . Let  $K \in D_{QCoh}(\mathcal{O}_X)$  and let  $L(g')^* K \rightarrow K'$  be a map in  $D_{QCoh}(\mathcal{O}_{X'})$ . If*

- (1) *the equivalent conditions of Lemma 21.1 hold, and*
- (2)  *$f$  is quasi-compact and quasi-separated,*

*then the composition  $Lg^* Rf_* K \rightarrow Rf'_* L(g')^* K \rightarrow Rf'_* K'$  is an isomorphism.*

**Proof.** To check the map is an isomorphism we may work étale locally on  $Y'$ . Hence we may assume  $g : Y' \rightarrow Y$  is a morphism of affine schemes. In this case, we will use the induction principle of Lemma 9.3 to prove that for a quasi-compact and quasi-separated algebraic space  $U$  étale over  $X$  the similarly constructed map  $Lg^* R(U \rightarrow Y)_* K|_U \rightarrow R(U' \rightarrow Y')_* K'|_{U'}$  is an isomorphism. Here  $U' = X' \times_{g',X} U = Y' \times_{g,Y} U$ .

If  $U$  is a scheme (for example affine), then the result holds. Namely, then  $Y, Y', U, U'$  are schemes,  $K$  and  $K'$  come from objects of the derived category of the underlying schemes by Lemma 4.2 and the condition of Derived Categories of Schemes, Lemma 26.1 holds for these complexes by Lemma 21.1. Thus (by the compatibilities explained in Remark 6.3) we can apply the result in the case of schemes which is Derived Categories of Schemes, Lemma 26.2.

The induction step. Let  $(U \subset W, V \rightarrow W)$  be an elementary distinguished square with  $W$  a quasi-compact and quasi-separated algebraic space étale over  $X$ , with  $U$  quasi-compact,  $V$  affine and the result holds for  $U$ ,  $V$ , and  $U \times_W V$ . To easy notation we replace  $W$  by  $X$  (this is permissible at this point). Denote  $a : U \rightarrow Y$ ,  $b : V \rightarrow Y$ , and  $c : U \times_X V \rightarrow Y$  the obvious morphisms. Let  $a' : U' \rightarrow Y'$ ,  $b' : V' \rightarrow Y'$  and  $c' : U' \times_{X'} V' \rightarrow Y'$  be the base changes of  $a$ ,  $b$ , and  $c$ . Using the distinguished triangles from relative Mayer-Vietoris (Lemma 10.3) we obtain a

commutative diagram

$$\begin{array}{ccc}
Lg^*Rf_*K & \xrightarrow{\quad\quad\quad} & Rf'_*K' \\
\downarrow & & \downarrow \\
Lg^*Ra_*K|_U \oplus Lg^*Rb_*K|_V & \xrightarrow{\quad\quad\quad} & Ra'_*K'|_{U'} \oplus Rb'_*K'|_{V'} \\
\downarrow & & \downarrow \\
Lg^*Rc_*K|_{U \times_X V} & \xrightarrow{\quad\quad\quad} & Rc'_*K'|_{U' \times_{X'} V'} \\
\downarrow & & \downarrow \\
Lg^*Rf_*K[1] & \xrightarrow{\quad\quad\quad} & Rf'_*K'[1]
\end{array}$$

Since the 2nd and 3rd horizontal arrows are isomorphisms so is the first (Derived Categories, Lemma 4.3) and the proof of the lemma is finished.  $\square$

0DKI **Lemma 21.3.** *Let  $S$  be a scheme. Let*

$$\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
f' \downarrow & & \downarrow f \\
S' & \xrightarrow{g} & S
\end{array}$$

*be a cartesian diagram of algebraic spaces over  $S$ . Let  $K \in D_{Q\text{Coh}}(\mathcal{O}_X)$  and let  $L(g')^*K \rightarrow K'$  be a map in  $D_{Q\text{Coh}}(\mathcal{O}_{X'})$ . If the equivalent conditions of Lemma 21.1 hold, then*

- (1) *for  $E \in D_{Q\text{Coh}}(\mathcal{O}_X)$  the equivalent conditions of Lemma 21.1 hold for  $L(g')^*(E \otimes^{\mathbf{L}} K) \rightarrow L(g')^*E \otimes^{\mathbf{L}} K'$ ,*
- (2) *if  $E$  in  $D(\mathcal{O}_X)$  is perfect the equivalent conditions of Lemma 21.1 hold for  $L(g')^*R\mathcal{H}om(E, K) \rightarrow R\mathcal{H}om(L(g')^*E, K')$ , and*
- (3) *if  $K$  is bounded below and  $E$  in  $D(\mathcal{O}_X)$  pseudo-coherent the equivalent conditions of Lemma 21.1 hold for  $L(g')^*R\mathcal{H}om(E, K) \rightarrow R\mathcal{H}om(L(g')^*E, K')$ .*

**Proof.** The statement makes sense as the complexes involved have quasi-coherent cohomology sheaves by Lemmas 5.5, 5.6, and 13.10 and Cohomology on Sites, Lemmas 45.3 and 47.5. Having said this, we can check the maps (21.0.1) are isomorphisms in case (1) by computing the source and target of (21.0.1) using the transitive property of tensor product, see More on Algebra, Lemma 59.15. The map in (2) and (3) is the composition

$$L(g')^*R\mathcal{H}om(E, K) \rightarrow R\mathcal{H}om(L(g')^*E, L(g')^*K) \rightarrow R\mathcal{H}om(L(g')^*E, K')$$

where the first arrow is Cohomology on Sites, Remark 35.11 and the second arrow comes from the given map  $L(g')^*K \rightarrow K'$ . To prove the maps (21.0.1) are isomorphisms one represents  $E_x$  by a bounded complex of finite projective  $\mathcal{O}_{X,x}$ -modules in case (2) or by a bounded above complex of finite free modules in case (3) and computes the source and target of the arrow. Some details omitted.  $\square$

0A1K **Lemma 21.4.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a quasi-compact and quasi-separated morphism of algebraic spaces over  $S$ . Let  $E \in D_{Q\text{Coh}}(\mathcal{O}_X)$ . Let  $\mathcal{G}^\bullet$  be a*

bounded above complex of quasi-coherent  $\mathcal{O}_X$ -modules flat over  $Y$ . Then formation of

$$Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

**Proof.** The statement means the following. Let  $g : Y' \rightarrow Y$  be a morphism of algebraic spaces and consider the base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{\quad g \quad} & Y \end{array}$$

in other words  $X' = Y' \times_Y X$ . The lemma asserts that

$$Lg^* Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet) \longrightarrow Rf'_*(L(g')^* E \otimes_{\mathcal{O}_{X'}}^{\mathbf{L}} (g')^* \mathcal{G}^\bullet)$$

is an isomorphism. Observe that on the right hand side we do **not** use derived pullback on  $\mathcal{G}^\bullet$ . To prove this, we apply Lemmas 21.2 and 21.3 to see that it suffices to prove the canonical map

$$L(g')^* \mathcal{G}^\bullet \rightarrow (g')^* \mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 21.1. This follows by checking the condition on stalks, where it immediately follows from the fact that  $\mathcal{G}_{\bar{x}}^\bullet \otimes_{\mathcal{O}_{Y, \bar{y}}} \mathcal{O}_{Y', \bar{y}'}$  computes the derived tensor product by our assumptions on the complex  $\mathcal{G}^\bullet$ .  $\square$

08JQ **Lemma 21.5.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a quasi-compact and quasi-separated morphism of algebraic spaces over  $S$ . Let  $E$  be an object of  $D(\mathcal{O}_X)$ . Let  $\mathcal{G}^\bullet$  be a complex of quasi-coherent  $\mathcal{O}_X$ -modules. If*

- (1)  *$E$  is perfect,  $\mathcal{G}^\bullet$  is a bounded above, and  $\mathcal{G}^n$  is flat over  $Y$ , or*
- (2)  *$E$  is pseudo-coherent,  $\mathcal{G}^\bullet$  is bounded, and  $\mathcal{G}^n$  is flat over  $Y$ ,*

*then formation of*

$$Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

*commutes with arbitrary base change (see proof for precise statement).*

**Proof.** The statement means the following. Let  $g : Y' \rightarrow Y$  be a morphism of algebraic spaces and consider the base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad h \quad} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{\quad g \quad} & Y \end{array}$$

in other words  $X' = Y' \times_Y X$ . The lemma asserts that

$$Lg^* Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet) \longrightarrow R(f')_* R\mathcal{H}om(L(g')^* E, (g')^* \mathcal{G}^\bullet)$$

is an isomorphism. Observe that on the right hand side we do **not** use the derived pullback on  $\mathcal{G}^\bullet$ . To prove this, we apply Lemmas 21.2 and 21.3 to see that it suffices to prove the canonical map

$$L(g')^* \mathcal{G}^\bullet \rightarrow (g')^* \mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 21.1. This was shown in the proof of Lemma 21.4.  $\square$

## 22. Producing perfect complexes

0A1L The following lemma is our main technical tool for producing perfect complexes. Later versions of this result will reduce to this by Noetherian approximation.

08IS **Lemma 22.1.** *Let  $S$  be a scheme. Let  $Y$  be a Noetherian algebraic space over  $S$ . Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let  $E \in D(\mathcal{O}_X)$  such that*

- (1)  $E \in D_{Coh}^b(\mathcal{O}_X)$ ,
- (2) *the support of  $H^i(E)$  is proper over  $Y$  for all  $i$ ,*
- (3)  $E$  *has finite tor dimension as an object of  $D(f^{-1}\mathcal{O}_Y)$ .*

*Then  $Rf_*E$  is a perfect object of  $D(\mathcal{O}_Y)$ .*

**Proof.** By Lemma 8.1 we see that  $Rf_*E$  is an object of  $D_{Coh}^b(\mathcal{O}_Y)$ . Hence  $Rf_*E$  is pseudo-coherent (Lemma 13.7). Hence it suffices to show that  $Rf_*E$  has finite tor dimension, see Cohomology on Sites, Lemma 47.4. By Lemma 13.8 it suffices to check that  $Rf_*(E) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{F}$  has universally bounded cohomology for all quasi-coherent sheaves  $\mathcal{F}$  on  $Y$ . Bounded from above is clear as  $Rf_*(E)$  is bounded from above. Let  $T \subset |X|$  be the union of the supports of  $H^i(E)$  for all  $i$ . Then  $T$  is proper over  $Y$  by assumptions (1) and (2) and Lemma 7.6. In particular there exists a quasi-compact open subspace  $X' \subset X$  containing  $T$ . Setting  $f' = f|_{X'}$  we have  $Rf_*(E) = Rf'_*(E|_{X'})$  because  $E$  restricts to zero on  $X \setminus T$ . Thus we may replace  $X$  by  $X'$  and assume  $f$  is quasi-compact. We have assumed  $f$  is quasi-separated. Thus

$$Rf_*(E) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{F} = Rf_* (E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^* \mathcal{F}) = Rf_* (E \otimes_{f^{-1}\mathcal{O}_Y}^{\mathbf{L}} f^{-1} \mathcal{F})$$

by Lemma 20.1 and Cohomology on Sites, Lemma 18.5. By assumption (3) the complex  $E \otimes_{f^{-1}\mathcal{O}_Y}^{\mathbf{L}} f^{-1} \mathcal{F}$  has cohomology sheaves in a given finite range, say  $[a, b]$ . Then  $Rf_*$  of it has cohomology in the range  $[a, \infty)$  and we win.  $\square$

0DKJ **Lemma 22.2.** *Let  $S$  be a scheme. Let  $B$  be a Noetherian algebraic space over  $S$ . Let  $f : X \rightarrow B$  be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let  $E \in D(\mathcal{O}_X)$  be perfect. Let  $\mathcal{G}^\bullet$  be a bounded complex of coherent  $\mathcal{O}_X$ -modules flat over  $B$  with support proper over  $B$ . Then  $K = Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$  is a perfect object of  $D(\mathcal{O}_B)$ .*

**Proof.** The object  $K$  is perfect by Lemma 22.1. We check the lemma applies: Locally  $E$  is isomorphic to a finite complex of finite free  $\mathcal{O}_X$ -modules. Hence locally  $E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet$  is isomorphic to a finite complex whose terms are of the form

$$\bigoplus_{i=a, \dots, b} (\mathcal{G}^i)^{\oplus r_i}$$

for some integers  $a, b, r_a, \dots, r_b$ . This immediately implies the cohomology sheaves  $H^i(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$  are coherent. The hypothesis on the tor dimension also follows as  $\mathcal{G}^i$  is flat over  $f^{-1}\mathcal{O}_Y$ .  $\square$

0DKK **Lemma 22.3.** *Let  $S$  be a scheme. Let  $B$  be a Noetherian algebraic space over  $S$ . Let  $f : X \rightarrow B$  be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let  $E \in D(\mathcal{O}_X)$  be perfect. Let  $\mathcal{G}^\bullet$  be a bounded complex of coherent  $\mathcal{O}_X$ -modules flat over  $B$  with support proper over  $B$ . Then  $K = Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet)$  is a perfect object of  $D(\mathcal{O}_B)$ .*

**Proof.** Since  $E$  is a perfect complex there exists a dual perfect complex  $E^\vee$ , see Cohomology on Sites, Lemma 48.4. Observe that  $R\mathcal{H}om(E, \mathcal{G}^\bullet) = E^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet$ . Thus the perfectness of  $K$  follows from Lemma 22.2.  $\square$

### 23. A projection formula for Ext

08JM Lemma 23.3 (or similar results in the literature) is sometimes useful to verify properties of an obstruction theory needed to verify one of Artin's criteria for Quot functors, Hilbert schemes, and other moduli problems. Suppose that  $f : X \rightarrow Y$  is a proper, flat, finitely presented morphism of algebraic spaces and  $E \in D(\mathcal{O}_X)$  is perfect. Here the lemma says

$$\mathrm{Ext}_X^i(E, f^* \mathcal{F}) = \mathrm{Ext}_Y^i((Rf_* E^\vee)^\vee, \mathcal{F})$$

for  $\mathcal{F}$  quasi-coherent on  $Y$ . Writing it this way makes it look like a projection formula for Ext and indeed the result follows rather easily from Lemma 20.1.

0A1M **Lemma 23.1.** *Assumptions and notation as in Lemma 22.2. Then there are functorial isomorphisms*

$$H^i(B, K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}) \longrightarrow H^i(X, E \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}))$$

for  $\mathcal{F}$  quasi-coherent on  $B$  compatible with boundary maps (see proof).

**Proof.** We have

$$\mathcal{G}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^* \mathcal{F} = \mathcal{G}^\bullet \otimes_{f^{-1}\mathcal{O}_B}^{\mathbf{L}} f^{-1} \mathcal{F} = \mathcal{G}^\bullet \otimes_{f^{-1}\mathcal{O}_B} f^{-1} \mathcal{F} = \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}$$

the first equality by Cohomology on Sites, Lemma 18.5, the second as  $\mathcal{G}^n$  is a flat  $f^{-1}\mathcal{O}_B$ -module, and the third by definition of pullbacks. Hence we obtain

$$\begin{aligned} H^i(X, E \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F})) &= H^i(X, E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^* \mathcal{F}) \\ &= H^i(B, Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^* \mathcal{F})) \\ &= H^i(B, Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet) \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}) \\ &= H^i(B, K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}) \end{aligned}$$

The first equality by the above, the second by Leray (Cohomology on Sites, Remark 14.4), and the third equality by Lemma 20.1. The statement on boundary maps means the following: Given a short exact sequence  $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$  then the isomorphisms fit into commutative diagrams

$$\begin{array}{ccc} H^i(B, K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}_3) & \longrightarrow & H^i(X, E \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_3)) \\ \delta \downarrow & & \downarrow \delta \\ H^{i+1}(B, K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}_1) & \longrightarrow & H^{i+1}(X, E \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_1)) \end{array}$$

where the boundary maps come from the distinguished triangle

$$K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}_1 \rightarrow K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}_2 \rightarrow K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}_3 \rightarrow K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}_1[1]$$

and the distinguished triangle in  $D(\mathcal{O}_X)$  associated to the short exact sequence

$$0 \rightarrow \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_1 \rightarrow \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_2 \rightarrow \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_3 \rightarrow 0$$

of complexes. This sequence is exact because  $\mathcal{G}^n$  is flat over  $B$ . We omit the verification of the commutativity of the displayed diagram.  $\square$

08JN **Lemma 23.2.** *Assumption and notation as in Lemma 22.3. Then there are functorial isomorphisms*

$$H^i(B, K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}) \longrightarrow \mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F})$$

for  $\mathcal{F}$  quasi-coherent on  $B$  compatible with boundary maps (see proof).

**Proof.** As in the proof of Lemma 22.3 let  $E^\vee$  be the dual perfect complex and recall that  $K = Rf_*(E^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$ . Since we also have

$$\mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}) = H^i(X, E^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}))$$

by construction of  $E^\vee$ , the existence of the isomorphisms follows from Lemma 23.1 applied to  $E^\vee$  and  $\mathcal{G}^\bullet$ . The statement on boundary maps means the following: Given a short exact sequence  $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$  then the isomorphisms fit into commutative diagrams

$$\begin{array}{ccc} H^i(B, K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}_3) & \longrightarrow & \mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_3) \\ \delta \downarrow & & \downarrow \delta \\ H^{i+1}(B, K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}_1) & \longrightarrow & \mathrm{Ext}_{\mathcal{O}_X}^{i+1}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_1) \end{array}$$

where the boundary maps come from the distinguished triangle

$$K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}_1 \rightarrow K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}_2 \rightarrow K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}_3 \rightarrow K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F}_1[1]$$

and the distinguished triangle in  $D(\mathcal{O}_X)$  associated to the short exact sequence

$$0 \rightarrow \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_1 \rightarrow \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_2 \rightarrow \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}_3 \rightarrow 0$$

of complexes. This sequence is exact because  $\mathcal{G}^n$  is flat over  $B$ . We omit the verification of the commutativity of the displayed diagram.  $\square$

08JR **Lemma 23.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow B$  be a morphism of algebraic spaces over  $S$ ,  $E \in D(\mathcal{O}_X)$ , and  $\mathcal{F}^\bullet$  a complex of  $\mathcal{O}_X$ -modules. Assume*

- (1)  $B$  is Noetherian,
- (2)  $f$  is locally of finite type and quasi-separated,
- (3)  $E \in D_{\mathrm{Coh}}^-(\mathcal{O}_X)$ ,
- (4)  $\mathcal{G}^\bullet$  is a bounded complex of coherent  $\mathcal{O}_X$ -module flat over  $B$  with support proper over  $B$ .

Then the following two statements are true

- (A) *for every  $m \in \mathbf{Z}$  there exists a perfect object  $K$  of  $D(\mathcal{O}_B)$  and functorial maps*

$$\alpha_{\mathcal{F}}^i : \mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}) \longrightarrow H^i(B, K \otimes_{\mathcal{O}_B}^{\mathbf{L}} \mathcal{F})$$

*for  $\mathcal{F}$  quasi-coherent on  $B$  compatible with boundary maps (see proof) such that  $\alpha_{\mathcal{F}}^i$  is an isomorphism for  $i \leq m$ , and*

- (B) *there exists a pseudo-coherent  $L \in D(\mathcal{O}_B)$  and functorial isomorphisms*

$$\mathrm{Ext}_{\mathcal{O}_B}^i(L, \mathcal{F}) \longrightarrow \mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F})$$

*for  $\mathcal{F}$  quasi-coherent on  $B$  compatible with boundary maps.*

**Proof.** Proof of (A). Suppose  $\mathcal{G}^i$  is nonzero only for  $i \in [a, b]$ . We may replace  $X$  by a quasi-compact open neighbourhood of the union of the supports of  $\mathcal{G}^i$ . Hence we may assume  $X$  is Noetherian. In this case  $X$  and  $f$  are quasi-compact and quasi-separated. Choose an approximation  $P \rightarrow E$  by a perfect complex  $P$  of  $(X, E, -m-1+a)$  (possible by Theorem 14.7). Then the induced map

$$\mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}) \longrightarrow \mathrm{Ext}_{\mathcal{O}_X}^i(P, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F})$$

is an isomorphism for  $i \leq m$ . Namely, the kernel, resp. cokernel of this map is a quotient, resp. submodule of

$$\mathrm{Ext}_{\mathcal{O}_X}^i(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}) \quad \text{resp.} \quad \mathrm{Ext}_{\mathcal{O}_X}^{i+1}(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F})$$

where  $C$  is the cone of  $P \rightarrow E$ . Since  $C$  has vanishing cohomology sheaves in degrees  $\geq -m-1+a$  these Ext-groups are zero for  $i \leq m+1$  by Derived Categories, Lemma 27.3. This reduces us to the case that  $E$  is a perfect complex which is Lemma 23.2. The statement on boundaries is explained in the proof of Lemma 23.2.

Proof of (B). As in the proof of (A) we may assume  $X$  is Noetherian. Observe that  $E$  is pseudo-coherent by Lemma 13.7. By Lemma 18.1 we can write  $E = \mathrm{hocolim} E_n$  with  $E_n$  perfect and  $E_n \rightarrow E$  inducing an isomorphism on truncations  $\tau_{\geq -n}$ . Let  $E_n^\vee$  be the dual perfect complex (Cohomology on Sites, Lemma 48.4). We obtain an inverse system  $\dots \rightarrow E_3^\vee \rightarrow E_2^\vee \rightarrow E_1^\vee$  of perfect objects. This in turn gives rise to an inverse system

$$\dots \rightarrow K_3 \rightarrow K_2 \rightarrow K_1 \quad \text{with} \quad K_n = Rf_*(E_n^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$$

perfect on  $Y$ , see Lemma 22.2. By Lemma 23.2 and its proof and by the arguments in the previous paragraph (with  $P = E_n$ ) for any quasi-coherent  $\mathcal{F}$  on  $Y$  we have functorial canonical maps

$$\begin{array}{ccc} & \mathrm{Ext}_{\mathcal{O}_X}^i(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^* \mathcal{F}) & \\ \swarrow & & \searrow \\ H^i(Y, K_{n+1} \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{F}) & \xrightarrow{\quad} & H^i(Y, K_n \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{F}) \end{array}$$

which are isomorphisms for  $i \leq n+a$ . Let  $L_n = K_n^\vee$  be the dual perfect complex. Then we see that  $L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow \dots$  is a system of perfect objects in  $D(\mathcal{O}_Y)$  such that for any quasi-coherent  $\mathcal{F}$  on  $Y$  the maps

$$\mathrm{Ext}_{\mathcal{O}_Y}^i(L_{n+1}, \mathcal{F}) \longrightarrow \mathrm{Ext}_{\mathcal{O}_Y}^i(L_n, \mathcal{F})$$

are isomorphisms for  $i \leq n+a-1$ . This implies that  $L_n \rightarrow L_{n+1}$  induces an isomorphism on truncations  $\tau_{\geq -n-a+2}$  (hint: take cone of  $L_n \rightarrow L_{n+1}$  and look at its last nonvanishing cohomology sheaf). Thus  $L = \mathrm{hocolim} L_n$  is pseudo-coherent, see Lemma 18.1. The mapping property of homotopy colimits gives that  $\mathrm{Ext}_{\mathcal{O}_Y}^i(L, \mathcal{F}) = \mathrm{Ext}_{\mathcal{O}_Y}^i(L_n, \mathcal{F})$  for  $i \leq n+a-3$  which finishes the proof.  $\square$

ODKL **Remark 23.4.** The pseudo-coherent complex  $L$  of part (B) of Lemma 23.3 is canonically associated to the situation. For example, formation of  $L$  as in (B) is

compatible with base change. In other words, given a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ f' \downarrow & g' \searrow & \downarrow f \\ Y' & \xrightarrow{\quad} & Y \end{array}$$

of schemes we have canonical functorial isomorphisms

$$\mathrm{Ext}_{\mathcal{O}_{Y'}}^i(Lg^*L, \mathcal{F}') \longrightarrow \mathrm{Ext}_{\mathcal{O}_X}^i(L(g')^*E, (g')^*\mathcal{G}^\bullet \otimes_{\mathcal{O}_{X'}} (f')^*\mathcal{F}')$$

for  $\mathcal{F}'$  quasi-coherent on  $Y'$ . Observe that we do **not** use derived pullback on  $\mathcal{G}^\bullet$  on the right hand side. If we ever need this, we will formulate a precise result here and give a detailed proof.

## 24. Limits and derived categories

09RG In this section we collect some results about the derived category of an algebraic space which is the limit of an inverse system of algebraic spaces. More precisely, we will work in the following setting.

09RH **Situation 24.1.** Let  $S$  be a scheme. Let  $X = \lim_{i \in I} X_i$  be a limit of a directed system of algebraic spaces over  $S$  with affine transition morphisms  $f_{i'i} : X_{i'} \rightarrow X_i$ . We denote  $f_i : X \rightarrow X_i$  the projection. We assume that  $X_i$  is quasi-compact and quasi-separated for all  $i \in I$ . We also choose an element  $0 \in I$ .

09RI **Lemma 24.2.** *In Situation 24.1. Let  $E_0$  and  $K_0$  be objects of  $D(\mathcal{O}_{X_0})$ . Set  $E_i = Lf_{i0}^*E_0$  and  $K_i = Lf_{i0}^*K_0$  for  $i \geq 0$  and set  $E = Lf_0^*E_0$  and  $K = Lf_0^*K_0$ . Then the map*

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{X_i})}(E_i, K_i) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(E, K)$$

*is an isomorphism if either*

- (1)  $E_0$  is perfect and  $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ , or
- (2)  $E_0$  is pseudo-coherent and  $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$  has finite tor dimension.

**Proof.** For every quasi-compact and quasi-separated object  $U_0$  of  $(X_0)_{\text{spaces}, \text{étale}}$  consider the condition  $P$  that the canonical map

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{U_i})}(E_i|_{U_i}, K_i|_{U_i}) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_U)}(E|_U, K|_U)$$

is an isomorphism, where  $U = X \times_{X_0} U_0$  and  $U_i = X_i \times_{X_0} U_0$ . We will prove  $P$  holds for each  $U_0$  by the induction principle of Lemma 9.3. Condition (2) of this lemma follows immediately from Mayer-Vietoris for hom in the derived category, see Lemma 10.4. Thus it suffices to prove the lemma when  $X_0$  is affine.

If  $X_0$  is affine, then the result follows from the case of schemes, see Derived Categories of Schemes, Lemma 29.2. To see this use the equivalence of Lemma 4.2 and use the translation of properties explained in Lemmas 13.2, 13.3, and 13.5.  $\square$

09RJ **Lemma 24.3.** *In Situation 24.1 the category of perfect objects of  $D(\mathcal{O}_X)$  is the colimit of the categories of perfect objects of  $D(\mathcal{O}_{X_i})$ .*

**Proof.** For every quasi-compact and quasi-separated object  $U_0$  of  $(X_0)_{\text{spaces}, \text{étale}}$  consider the condition  $P$  that the functor

$$\mathrm{colim}_{i \geq 0} D_{\text{perf}}(\mathcal{O}_{U_i}) \longrightarrow D_{\text{perf}}(\mathcal{O}_U)$$

is an equivalence where  $_{perf}$  indicates the full subcategory of perfect objects and where  $U = X \times_{X_0} U_0$  and  $U_i = X_i \times_{X_0} U_0$ . We will prove  $P$  holds for every  $U_0$  by the induction principle of Lemma 9.3. First, we observe that we already know the functor is fully faithful by Lemma 24.2. Thus it suffices to prove essential surjectivity.

We first check condition (2) of the induction principle. Thus suppose that we have an elementary distinguished square  $(U_0 \subset X_0, V_0 \rightarrow X_0)$  and that  $P$  holds for  $U_0$ ,  $V_0$ , and  $U_0 \times_{X_0} V_0$ . Let  $E$  be a perfect object of  $D(\mathcal{O}_X)$ . We can find  $i \geq 0$  and  $E_{U,i}$  perfect on  $U_i$  and  $E_{V,i}$  perfect on  $V_i$  whose pullback to  $U$  and  $V$  are isomorphic to  $E|_U$  and  $E|_V$ . Denote

$$a : E_{U,i} \rightarrow (R(X \rightarrow X_i)_* E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \rightarrow (R(X \rightarrow X_i)_* E)|_{V_i}$$

the maps adjoint to the isomorphisms  $L(U \rightarrow U_i)^* E_{U,i} \rightarrow E|_U$  and  $L(V \rightarrow V_i)^* E_{V,i} \rightarrow E|_V$ . By fully faithfulness, after increasing  $i$ , we can find an isomorphism  $c : E_{U,i}|_{U_i \times_{X_i} V_i} \rightarrow E_{V,i}|_{U_i \times_{X_i} V_i}$  which pulls back to the identifications

$$L(U \rightarrow U_i)^* E_{U,i}|_{U \times_X V} \rightarrow E|_{U \times_X V} \rightarrow L(V \rightarrow V_i)^* E_{V,i}|_{U \times_X V}.$$

Apply Lemma 10.8 to get an object  $E_i$  on  $X_i$  and a map  $d : E_i \rightarrow R(X \rightarrow X_i)_* E$  which restricts to the maps  $a$  and  $b$  over  $U_i$  and  $V_i$ . Then it is clear that  $E_i$  is perfect and that  $d$  is adjoint to an isomorphism  $L(X \rightarrow X_i)^* E_i \rightarrow E$ .

Finally, we check condition (1) of the induction principle, in other words, we check the lemma holds when  $X_0$  is affine. This follows from the case of schemes, see Derived Categories of Schemes, Lemma 29.3. To see this use the equivalence of Lemma 4.2 and use the translation of Lemma 13.5.  $\square$

## 25. Cohomology and base change, VI

0A1N A final section on cohomology and base change continuing the discussion of Sections 20, 21, and 22. An easy to grok special case is given in Remark 25.2.

0A1P **Lemma 25.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of finite presentation between algebraic spaces over  $S$ . Let  $E \in D(\mathcal{O}_X)$  be a perfect object. Let  $\mathcal{G}^\bullet$  be a bounded complex of finitely presented  $\mathcal{O}_X$ -modules, flat over  $Y$ , with support proper over  $Y$ . Then*

$$K = Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$$

*is a perfect object of  $D(\mathcal{O}_Y)$  and its formation commutes with arbitrary base change.*

**Proof.** The statement on base change is Lemma 21.4. Thus it suffices to show that  $K$  is a perfect object. If  $Y$  is Noetherian, then this follows from Lemma 22.2. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on  $Y$ , hence we may assume  $Y$  is affine. Say  $Y = \text{Spec}(R)$ . We write  $R = \text{colim } R_i$  as a filtered colimit of Noetherian rings  $R_i$ . By Limits of Spaces, Lemma 7.1 there exists an  $i$  and an algebraic space  $X_i$  of finite presentation over  $R_i$  whose base change to  $R$  is  $X$ . By Limits of Spaces, Lemma 7.2 we may assume after increasing  $i$ , that there exists a bounded complex of finitely presented  $\mathcal{O}_{X_i}$ -modules  $\mathcal{G}_i^\bullet$  whose pullback to  $X$  is  $\mathcal{G}^\bullet$ . After increasing  $i$  we may assume  $\mathcal{G}_i^n$  is flat over  $R_i$ , see Limits of Spaces, Lemma 6.12. After increasing  $i$  we may assume the support of  $\mathcal{G}_i^n$  is proper over  $R_i$ , see Limits of Spaces, Lemma 12.3.

Finally, by Lemma 24.3 we may, after increasing  $i$ , assume there exists a perfect object  $E_i$  of  $D(\mathcal{O}_{X_i})$  whose pullback to  $X$  is  $E$ . By Lemma 22.2 we have that  $K_i = Rf_{i,*}(E_i \otimes_{\mathcal{O}_{X_i}}^{\mathbf{L}} \mathcal{G}^\bullet)$  is perfect on  $\mathrm{Spec}(R_i)$  where  $f_i : X_i \rightarrow \mathrm{Spec}(R_i)$  is the structure morphism. By the base change result (Lemma 21.4) the pullback of  $K_i$  to  $Y = \mathrm{Spec}(R)$  is  $K$  and we conclude.  $\square$

0A1Q **Remark 25.2.** Let  $R$  be a ring. Let  $X$  be an algebraic space of finite presentation over  $R$ . Let  $\mathcal{G}$  be a finitely presented  $\mathcal{O}_X$ -module flat over  $R$  with support proper over  $R$ . By Lemma 25.1 there exists a finite complex of finite projective  $R$ -modules  $M^\bullet$  such that we have

$$R\Gamma(X_{R'}, \mathcal{G}_{R'}) = M^\bullet \otimes_R R'$$

functorially in the  $R$ -algebra  $R'$ .

OCTL **Lemma 25.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of finite presentation between algebraic spaces over  $S$ . Let  $E \in D(\mathcal{O}_X)$  be a pseudo-coherent object. Let  $\mathcal{G}^\bullet$  be a bounded above complex of finitely presented  $\mathcal{O}_X$ -modules, flat over  $Y$ , with support proper over  $Y$ . Then*

$$K = Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet)$$

*is a pseudo-coherent object of  $D(\mathcal{O}_Y)$  and its formation commutes with arbitrary base change.*

**Proof.** The statement on base change is Lemma 21.4. Thus it suffices to show that  $K$  is a pseudo-coherent object. This will follow from Lemma 25.1 by approximation by perfect complexes. We encourage the reader to skip the rest of the proof.

The question is étale local on  $Y$ , hence we may assume  $Y$  is affine. Then  $X$  is quasi-compact and quasi-separated. Moreover, there exists an integer  $N$  such that total direct image  $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$  has cohomological dimension  $N$  as explained in Lemma 6.1. Choose an integer  $b$  such that  $\mathcal{G}^i = 0$  for  $i > b$ . It suffices to show that  $K$  is  $m$ -pseudo-coherent for every  $m$ . Choose an approximation  $P \rightarrow E$  by a perfect complex  $P$  of  $(X, E, m - N - 1 - b)$ . This is possible by Theorem 14.7. Choose a distinguished triangle

$$P \rightarrow E \rightarrow C \rightarrow P[1]$$

in  $D_{QCoh}(\mathcal{O}_X)$ . The cohomology sheaves of  $C$  are zero in degrees  $\geq m - N - 1 - b$ . Hence the cohomology sheaves of  $C \otimes^{\mathbf{L}} \mathcal{G}^\bullet$  are zero in degrees  $\geq m - N - 1$ . Thus the cohomology sheaves of  $Rf_*(C \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$  are zero in degrees  $\geq m - 1$ . Hence

$$Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}^\bullet) \rightarrow Rf_*(E \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$$

is an isomorphism on cohomology sheaves in degrees  $\geq m$ . Next, suppose that  $H^i(P) = 0$  for  $i > a$ . Then  $P \otimes^{\mathbf{L}} \sigma_{\geq m-N-1-a} \mathcal{G}^\bullet \rightarrow P \otimes^{\mathbf{L}} \mathcal{G}^\bullet$  is an isomorphism on cohomology sheaves in degrees  $\geq m - N - 1$ . Thus again we find that

$$Rf_*(P \otimes^{\mathbf{L}} \sigma_{\geq m-N-1-a} \mathcal{G}^\bullet) \rightarrow Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$$

is an isomorphism on cohomology sheaves in degrees  $\geq m$ . By Lemma 25.1 the source is a perfect complex. We conclude that  $K$  is  $m$ -pseudo-coherent as desired.  $\square$

OCTM **Lemma 25.4.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a proper morphism of finite presentation of algebraic spaces over  $S$ .*

- (1) Let  $E \in D(\mathcal{O}_X)$  be perfect and  $f$  flat. Then  $Rf_*E$  is a perfect object of  $D(\mathcal{O}_Y)$  and its formation commutes with arbitrary base change.
- (2) Let  $\mathcal{G}$  be an  $\mathcal{O}_X$ -module of finite presentation, flat over  $S$ . Then  $Rf_*\mathcal{G}$  is a perfect object of  $D(\mathcal{O}_Y)$  and its formation commutes with arbitrary base change.

**Proof.** Special cases of Lemma 25.1 applied with (1)  $\mathcal{G}^\bullet$  equal to  $\mathcal{O}_X$  in degree 0 and (2)  $E = \mathcal{O}_X$  and  $\mathcal{G}^\bullet$  consisting of  $\mathcal{G}$  sitting in degree 0.  $\square$

0CTN **Lemma 25.5.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a flat proper morphism of finite presentation of algebraic spaces over  $S$ . Let  $E \in D(\mathcal{O}_X)$  be pseudo-coherent. Then  $Rf_*E$  is a pseudo-coherent object of  $D(\mathcal{O}_Y)$  and its formation commutes with arbitrary base change.*

More generally, if  $f : X \rightarrow Y$  is proper and  $E$  on  $X$  is pseudo-coherent relative to  $Y$  (More on Morphisms of Spaces, Definition 45.3), then  $Rf_*E$  is pseudo-coherent (but formation does not commute with base change in this generality). The case of this for schemes is proved in [Kie72].

**Proof.** Special case of Lemma 25.3 applied with  $\mathcal{G} = \mathcal{O}_X$ .  $\square$

0D3F **Lemma 25.6.** *Let  $R$  be a ring. Let  $X$  be an algebraic space and let  $f : X \rightarrow \text{Spec}(R)$  be proper, flat, and of finite presentation. Let  $(M_n)$  be an inverse system of  $R$ -modules with surjective transition maps. Then the canonical map*

$$\mathcal{O}_X \otimes_R (\lim M_n) \longrightarrow \lim \mathcal{O}_X \otimes_R M_n$$

*induces an isomorphism from the source to  $DQ_X$  applied to the target.*

**Proof.** The statement means that for any object  $E$  of  $D_{Q\text{Coh}}(\mathcal{O}_X)$  the induced map

$$\text{Hom}(E, \mathcal{O}_X \otimes_R (\lim M_n)) \longrightarrow \text{Hom}(E, \lim \mathcal{O}_X \otimes_R M_n)$$

is an isomorphism. Since  $D_{Q\text{Coh}}(\mathcal{O}_X)$  has a perfect generator (Theorem 15.4) it suffices to check this for perfect  $E$ . By Lemma 5.4 we have  $\lim \mathcal{O}_X \otimes_R M_n = R \lim \mathcal{O}_X \otimes_R M_n$ . The exact functor  $R\text{Hom}_X(E, -) : D_{Q\text{Coh}}(\mathcal{O}_X) \rightarrow D(R)$  of Cohomology on Sites, Section 36 commutes with products and hence with derived limits, whence

$$R\text{Hom}_X(E, \lim \mathcal{O}_X \otimes_R M_n) = R \lim R\text{Hom}_X(E, \mathcal{O}_X \otimes_R M_n)$$

Let  $E^\vee$  be the dual perfect complex, see Cohomology on Sites, Lemma 48.4. We have

$$R\text{Hom}_X(E, \mathcal{O}_X \otimes_R M_n) = R\Gamma(X, E^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^* M_n) = R\Gamma(X, E^\vee) \otimes_R^{\mathbf{L}} M_n$$

by Lemma 20.1. From Lemma 25.4 we see  $R\Gamma(X, E^\vee)$  is a perfect complex of  $R$ -modules. In particular it is a pseudo-coherent complex and by More on Algebra, Lemma 102.3 we obtain

$$R \lim R\Gamma(X, E^\vee) \otimes_R^{\mathbf{L}} M_n = R\Gamma(X, E^\vee) \otimes_R^{\mathbf{L}} \lim M_n$$

as desired.  $\square$

0CWH **Lemma 25.7.** *Let  $A$  be a ring. Let  $X$  be an algebraic space over  $A$  which is quasi-compact and quasi-separated. Let  $K \in D_{Q\text{Coh}}^-(\mathcal{O}_X)$ . If  $R\Gamma(X, E \otimes^{\mathbf{L}} K)$  is pseudo-coherent in  $D(A)$  for every perfect  $E$  in  $D(\mathcal{O}_X)$ , then  $R\Gamma(X, E \otimes^{\mathbf{L}} K)$  is pseudo-coherent in  $D(A)$  for every pseudo-coherent  $E$  in  $D(\mathcal{O}_X)$ .*

**Proof.** There exists an integer  $N$  such that  $R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \rightarrow D(A)$  has cohomological dimension  $N$  as explained in Lemma 6.1. Let  $b \in \mathbf{Z}$  be such that  $H^i(K) = 0$  for  $i > b$ . Let  $E$  be pseudo-coherent on  $X$ . It suffices to show that  $R\Gamma(X, E \otimes^{\mathbf{L}} K)$  is  $m$ -pseudo-coherent for every  $m$ . Choose an approximation  $P \rightarrow E$  by a perfect complex  $P$  of  $(X, E, m - N - 1 - b)$ . This is possible by Theorem 14.7. Choose a distinguished triangle

$$P \rightarrow E \rightarrow C \rightarrow P[1]$$

in  $D_{QCoh}(\mathcal{O}_X)$ . The cohomology sheaves of  $C$  are zero in degrees  $\geq m - N - 1 - b$ . Hence the cohomology sheaves of  $C \otimes^{\mathbf{L}} K$  are zero in degrees  $\geq m - N - 1$ . Thus the cohomology of  $R\Gamma(X, C \otimes^{\mathbf{L}} K)$  are zero in degrees  $\geq m - 1$ . Hence

$$R\Gamma(X, P \otimes^{\mathbf{L}} K) \rightarrow R\Gamma(X, E \otimes^{\mathbf{L}} K)$$

is an isomorphism on cohomology in degrees  $\geq m$ . By assumption the source is pseudo-coherent. We conclude that  $R\Gamma(X, E \otimes^{\mathbf{L}} K)$  is  $m$ -pseudo-coherent as desired.  $\square$

0A1R **Lemma 25.8.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of finite presentation between algebraic spaces over  $S$ . Let  $E \in D(\mathcal{O}_X)$  be a perfect object. Let  $\mathcal{G}^\bullet$  be a bounded complex of finitely presented  $\mathcal{O}_X$ -modules, flat over  $Y$ , with support proper over  $Y$ . Then*

$$K = Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

*is a perfect object of  $D(\mathcal{O}_Y)$  and its formation commutes with arbitrary base change.*

**Proof.** The statement on base change is Lemma 21.5. Thus it suffices to show that  $K$  is a perfect object. If  $Y$  is Noetherian, then this follows from Lemma 22.3. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on  $Y$ , hence we may assume  $Y$  is affine. Say  $Y = \text{Spec}(R)$ . We write  $R = \text{colim } R_i$  as a filtered colimit of Noetherian rings  $R_i$ . By Limits of Spaces, Lemma 7.1 there exists an  $i$  and an algebraic space  $X_i$  of finite presentation over  $R_i$  whose base change to  $R$  is  $X$ . By Limits of Spaces, Lemma 7.2 we may assume after increasing  $i$ , that there exists a bounded complex of finitely presented  $\mathcal{O}_{X_i}$ -module  $\mathcal{G}_i^\bullet$  whose pullback to  $X$  is  $\mathcal{G}$ . After increasing  $i$  we may assume  $\mathcal{G}_i^n$  is flat over  $R_i$ , see Limits of Spaces, Lemma 6.12. After increasing  $i$  we may assume the support of  $\mathcal{G}_i^n$  is proper over  $R_i$ , see Limits of Spaces, Lemma 12.3. Finally, by Lemma 13.5 we may, after increasing  $i$ , assume there exists a perfect object  $E_i$  of  $D(\mathcal{O}_{X_i})$  whose pullback to  $X$  is  $E$ . Applying Lemma 23.2 to  $X_i \rightarrow \text{Spec}(R_i)$ ,  $E_i$ ,  $\mathcal{G}_i^\bullet$  and using the base change property already shown we obtain the result.  $\square$

## 26. Perfect complexes

0D1X We first talk about jumping loci for betti numbers of perfect complexes. First we have to define betti numbers.

Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $E$  be an object of  $D(\mathcal{O}_X)$ . Let  $x \in |X|$ . We want to define  $\beta_i(x) \in \{0, 1, 2, \dots\} \cup \{\infty\}$ . To do this, choose a morphism  $f : \text{Spec}(k) \rightarrow X$  in the equivalence class of  $x$ . Then  $Lf^*E$  is an object of  $D(\text{Spec}(k)_{\acute{e}tale}, \mathcal{O})$ . By Étale Cohomology, Lemma 59.4 and Theorem 17.4 we find that  $D(\text{Spec}(k)_{\acute{e}tale}, \mathcal{O}) = D(k)$  is the derived category of

$k$ -vector spaces. Hence  $Lf^*E$  is a complex of  $k$ -vector spaces and we can take  $\beta_i(x) = \dim_k H^i(Lf^*E)$ . It is easy to see that this does not depend on the choice of the representative in  $x$ . Moreover, if  $X$  is a scheme, this is the same as the notion used in Derived Categories of Schemes, Section 31.

0D1Y **Lemma 26.1.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $E \in D(\mathcal{O}_X)$  be pseudo-coherent (for example perfect). For any  $i \in \mathbf{Z}$  consider the function*

$$\beta_i : |X| \longrightarrow \{0, 1, 2, \dots\}$$

*defined above. Then we have*

- (1) *formation of  $\beta_i$  commutes with arbitrary base change,*
- (2) *the functions  $\beta_i$  are upper semi-continuous, and*
- (3) *the level sets of  $\beta_i$  are étale locally constructible.*

**Proof.** Choose a scheme  $U$  and a surjective étale morphism  $\varphi : U \rightarrow X$ . Then  $L\varphi^*E$  is a pseudo-coherent complex on the scheme  $U$  (use Lemma 13.2) and we can apply the result for schemes, see Derived Categories of Schemes, Lemma 31.1. The meaning of part (3) is that the inverse image of the level sets to  $U$  are locally constructible, see Properties of Spaces, Definition 8.2.  $\square$

0E0R **Lemma 26.2.** *Let  $Y$  be a scheme and let  $X$  be an algebraic space over  $Y$  such that the structure morphism  $f : X \rightarrow Y$  is flat, proper, and of finite presentation. Let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module of finite presentation, flat over  $Y$ . For fixed  $i \in \mathbf{Z}$  consider the function*

$$\beta_i : |Y| \rightarrow \{0, 1, 2, \dots\}, \quad y \longmapsto \dim_{\kappa(y)} H^i(X_y, \mathcal{F}_y)$$

*Then we have*

- (1) *formation of  $\beta_i$  commutes with arbitrary base change,*
- (2) *the functions  $\beta_i$  are upper semi-continuous, and*
- (3) *the level sets of  $\beta_i$  are locally constructible in  $Y$ .*

**Proof.** By cohomology and base change (more precisely by Lemma 25.4) the object  $K = Rf_*\mathcal{F}$  is a perfect object of the derived category of  $Y$  whose formation commutes with arbitrary base change. In particular we have

$$H^i(X_y, \mathcal{F}_y) = H^i(K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \kappa(y))$$

Thus the lemma follows from Lemma 26.1.  $\square$

0D1Z **Lemma 26.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $E \in D(\mathcal{O}_X)$  be perfect. The function*

$$\chi_E : |X| \longrightarrow \mathbf{Z}, \quad x \longmapsto \sum (-1)^i \beta_i(x)$$

*is locally constant on  $X$ .*

**Proof.** Omitted. Hints: Follows from the case of schemes by étale localization. See Derived Categories of Schemes, Lemma 31.2.  $\square$

0D20 **Lemma 26.4.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $E \in D(\mathcal{O}_X)$  be perfect. Given  $i, r \in \mathbf{Z}$ , there exists an open subspace  $U \subset X$  characterized by the following*

- (1)  *$E|_U \cong H^i(E|_U)[-i]$  and  $H^i(E|_U)$  is a locally free  $\mathcal{O}_U$ -module of rank  $r$ ,*

- (2) a morphism  $f : Y \rightarrow X$  factors through  $U$  if and only if  $Lf^*E$  is isomorphic to a locally free module of rank  $r$  placed in degree  $i$ .

**Proof.** Omitted. Hints: Follows from the case of schemes by étale localization. See Derived Categories of Schemes, Lemma 31.3.  $\square$

0E6A **Lemma 26.5.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is proper, flat, and of finite presentation. Let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module of finite presentation, flat over  $Y$ . Fix  $i, r \in \mathbf{Z}$ . Then there exists an open subspace  $V \subset Y$  with the following property: A morphism  $T \rightarrow Y$  factors through  $V$  if and only if  $Rf_{T,*}\mathcal{F}_T$  is isomorphic to a finite locally free module of rank  $r$  placed in degree  $i$ .*

**Proof.** By cohomology and base change (Lemma 25.4) the object  $K = Rf_*\mathcal{F}$  is a perfect object of the derived category of  $Y$  whose formation commutes with arbitrary base change. Thus this lemma follows immediately from Lemma 26.4.  $\square$

0D21 **Lemma 26.6.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $E \in D(\mathcal{O}_X)$  be perfect of tor-amplitude in  $[a, b]$  for some  $a, b \in \mathbf{Z}$ . Let  $r \geq 0$ . Then there exists a locally closed subspace  $j : Z \rightarrow X$  characterized by the following*

- (1)  $H^a(Lj^*E)$  is a locally free  $\mathcal{O}_Z$ -module of rank  $r$ , and
- (2) a morphism  $f : Y \rightarrow X$  factors through  $Z$  if and only if for all morphisms  $g : Y' \rightarrow Y$  the  $\mathcal{O}_{Y'}$ -module  $H^a(L(f \circ g)^*E)$  is locally free of rank  $r$ .

Moreover,  $j : Z \rightarrow X$  is of finite presentation and we have

- (3) if  $f : Y \rightarrow X$  factors as  $Y \xrightarrow{g} Z \rightarrow X$ , then  $H^a(Lf^*E) = g^*H^a(Lj^*E)$ ,
- (4) if  $\beta_a(x) \leq r$  for all  $x \in |X|$ , then  $j$  is a closed immersion and given  $f : Y \rightarrow X$  the following are equivalent
  - (a)  $f : Y \rightarrow X$  factors through  $Z$ ,
  - (b)  $H^0(Lf^*E)$  is a locally free  $\mathcal{O}_Y$ -module of rank  $r$ ,
 and if  $r = 1$  these are also equivalent to
  - (c)  $\mathcal{O}_Y \rightarrow \text{Hom}_{\mathcal{O}_Y}(H^0(Lf^*E), H^0(Lf^*E))$  is injective.

**Proof.** Omitted. Hints: Follows from the case of schemes by étale localization. See Derived Categories of Schemes, Lemma 31.4.  $\square$

0E6B **Lemma 26.7.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Assume*

- (1)  $f$  is proper, flat, and of finite presentation, and
- (2) for a morphism  $\text{Spec}(k) \rightarrow Y$  where  $k$  is a field, we have  $k = H^0(X_k, \mathcal{O}_{X_k})$ .

Then we have

- (a)  $f_*\mathcal{O}_X = \mathcal{O}_Y$  and this holds after any base change,
- (b) étale locally on  $Y$  we have

$$Rf_*\mathcal{O}_X = \mathcal{O}_Y \oplus P$$

in  $D(\mathcal{O}_Y)$  where  $P$  is perfect of tor amplitude in  $[1, \infty)$ .

**Proof.** It suffices to prove (a) and (b) étale locally on  $Y$ , thus we may and do assume  $Y$  is an affine scheme. By cohomology and base change (Lemma 25.4) the complex  $E = Rf_*\mathcal{O}_X$  is perfect and its formation commutes with arbitrary base change. In particular, for  $y \in Y$  we see that  $H^0(E \otimes^{\mathbf{L}} \kappa(y)) = H^0(X_y, \mathcal{O}_{X_y}) = \kappa(y)$ . Thus  $\beta_0(y) \leq 1$  for all  $y \in Y$  with notation as in Lemma 26.1. Apply Lemma 26.6

with  $a = 0$  and  $r = 1$ . We obtain a universal closed subscheme  $j : Z \rightarrow Y$  with  $H^0(Lj^*E)$  invertible characterized by the equivalence of (4)(a), (b), and (c) of the lemma. Since formation of  $E$  commutes with base change, we have

$$Lf^*E = R\mathrm{pr}_{1,*}\mathcal{O}_{X \times_Y X}$$

The morphism  $\mathrm{pr}_1 : X \times_Y X$  has a section namely the diagonal morphism  $\Delta$  for  $X$  over  $Y$ . We obtain maps

$$\mathcal{O}_X \longrightarrow R\mathrm{pr}_{1,*}\mathcal{O}_{X \times_Y X} \longrightarrow \mathcal{O}_X$$

in  $D(\mathcal{O}_X)$  whose composition is the identity. Thus  $R\mathrm{pr}_{1,*}\mathcal{O}_{X \times_Y X} = \mathcal{O}_X \oplus E'$  in  $D(\mathcal{O}_X)$ . Thus  $\mathcal{O}_X$  is a direct summand of  $H^0(Lf^*E)$  and we conclude that  $X \rightarrow Y$  factors through  $Z$  by the equivalence of (4)(c) and (4)(a) of the lemma cited above. Since  $\{X \rightarrow Y\}$  is an fppf covering, we have  $Z = Y$ . Thus  $f_*\mathcal{O}_X$  is an invertible  $\mathcal{O}_Y$ -module. We conclude  $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$  is an isomorphism because a ring map  $A \rightarrow B$  such that  $B$  is invertible as an  $A$ -module is an isomorphism. Since the assumptions are preserved under base change, we see that (a) is true.

Proof of (b). Above we have seen that for every  $y \in Y$  the map  $\mathcal{O}_Y \rightarrow H^0(E \otimes^{\mathbf{L}} \kappa(y))$  is surjective. Thus we may apply More on Algebra, Lemma 76.2 to see that in an open neighbourhood of  $y$  we have a decomposition  $Rf_*\mathcal{O}_X = \mathcal{O}_Y \oplus P$   $\square$

**0E0S Lemma 26.8.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Assume*

- (1)  *$f$  is proper, flat, and of finite presentation, and*
- (2) *the geometric fibres of  $f$  are reduced and connected.*

*Then  $f_*\mathcal{O}_X = \mathcal{O}_Y$  and this holds after any base change.*

**Proof.** By Lemma 26.7 it suffices to show that  $k = H^0(X_k, \mathcal{O}_{X_k})$  for all morphisms  $\mathrm{Spec}(k) \rightarrow Y$  where  $k$  is a field. This follows from Spaces over Fields, Lemma 14.3 and the fact that  $X_k$  is geometrically connected and geometrically reduced.  $\square$

## 27. Other applications

**0CRT** In this section we state and prove some results that can be deduced from the theory worked out above.

**0CRU Lemma 27.1.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $K$  be an object of  $D_{QCoh}(\mathcal{O}_X)$  such that the cohomology sheaves  $H^i(K)$  have countable sets of sections over affine schemes étale over  $X$ . Then for any quasi-compact and quasi-separated étale morphism  $U \rightarrow X$  and any perfect object  $E$  in  $D(\mathcal{O}_X)$  the sets*

$$H^i(U, K \otimes^{\mathbf{L}} E), \quad \mathrm{Ext}^i(E|_U, K|_U)$$

*are countable.*

**Proof.** Using Cohomology on Sites, Lemma 48.4 we see that it suffices to prove the result for the groups  $H^i(U, K \otimes^{\mathbf{L}} E)$ . We will use the induction principle to prove the lemma, see Lemma 9.3.

When  $U = \mathrm{Spec}(A)$  is affine the result follows from the case of schemes, see Derived Categories of Schemes, Lemma 33.2.

To finish the proof it suffices to show: if  $(U \subset W, V \rightarrow W)$  is an elementary distinguished triangle and the result holds for  $U$ ,  $V$ , and  $U \times_W V$ , then the result

holds for  $W$ . This is an immediate consequence of the Mayer-Vietoris sequence, see Lemma 10.5.  $\square$

0CRV **Lemma 27.2.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Assume the sets of sections of  $\mathcal{O}_X$  over affines étale over  $X$  are countable. Let  $K$  be an object of  $D_{QCoh}(\mathcal{O}_X)$ . The following are equivalent*

- (1)  $K = \text{hocolim} E_n$  with  $E_n$  a perfect object of  $D(\mathcal{O}_X)$ , and
- (2) the cohomology sheaves  $H^i(K)$  have countable sets of sections over affines étale over  $X$ .

**Proof.** If (1) is true, then (2) is true because homotopy colimits commutes with taking cohomology sheaves (by Derived Categories, Lemma 33.8) and because a perfect complex is locally isomorphic to a finite complex of finite free  $\mathcal{O}_X$ -modules and therefore satisfies (2) by assumption on  $X$ .

Assume (2). Choose a K-injective complex  $\mathcal{K}^\bullet$  representing  $K$ . Choose a perfect generator  $E$  of  $D_{QCoh}(\mathcal{O}_X)$  and represent it by a K-injective complex  $\mathcal{I}^\bullet$ . According to Theorem 17.3 and its proof there is an equivalence of triangulated categories  $F : D_{QCoh}(\mathcal{O}_X) \rightarrow D(A, d)$  where  $(A, d)$  is the differential graded algebra

$$(A, d) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$$

which maps  $K$  to the differential graded module

$$M = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{K}^\bullet)$$

Note that  $H^i(A) = \text{Ext}^i(E, E)$  and  $H^i(M) = \text{Ext}^i(E, K)$ . Moreover, since  $F$  is an equivalence it and its quasi-inverse commute with homotopy colimits. Therefore, it suffices to write  $M$  as a homotopy colimit of compact objects of  $D(A, d)$ . By Differential Graded Algebra, Lemma 38.3 it suffices show that  $\text{Ext}^i(E, E)$  and  $\text{Ext}^i(E, K)$  are countable for each  $i$ . This follows from Lemma 27.1.  $\square$

0CRW **Lemma 27.3.** *Let  $A$  be a ring. Let  $f : U \rightarrow X$  be a flat morphism of algebraic spaces of finite presentation over  $A$ . Then*

- (1) *there exists an inverse system of perfect objects  $L_n$  of  $D(\mathcal{O}_X)$  such that*

$$R\Gamma(U, Lf^*K) = \text{hocolim } R\text{Hom}_X(L_n, K)$$

*in  $D(A)$  functorially in  $K$  in  $D_{QCoh}(\mathcal{O}_X)$ , and*

- (2) *there exists a system of perfect objects  $E_n$  of  $D(\mathcal{O}_X)$  such that*

$$R\Gamma(U, Lf^*K) = \text{hocolim } R\Gamma(X, E_n \otimes^{\mathbf{L}} K)$$

*in  $D(A)$  functorially in  $K$  in  $D_{QCoh}(\mathcal{O}_X)$ .*

**Proof.** By Lemma 20.1 we have

$$R\Gamma(U, Lf^*K) = R\Gamma(X, Rf_*\mathcal{O}_U \otimes^{\mathbf{L}} K)$$

functorially in  $K$ . Observe that  $R\Gamma(X, -)$  commutes with homotopy colimits because it commutes with direct sums by Lemma 6.2. Similarly,  $-\otimes^{\mathbf{L}} K$  commutes with derived colimits because  $-\otimes^{\mathbf{L}} K$  commutes with direct sums (because direct sums in  $D(\mathcal{O}_X)$  are given by direct sums of representing complexes). Hence to prove (2) it suffices to write  $Rf_*\mathcal{O}_U = \text{hocolim} E_n$  for a system of perfect objects  $E_n$  of  $D(\mathcal{O}_X)$ . Once this is done we obtain (1) by setting  $L_n = E_n^\vee$ , see Cohomology on Sites, Lemma 48.4.

Write  $A = \operatorname{colim} A_i$  with  $A_i$  of finite type over  $\mathbf{Z}$ . By Limits of Spaces, Lemma 7.1 we can find an  $i$  and morphisms  $U_i \rightarrow X_i \rightarrow \operatorname{Spec}(A_i)$  of finite presentation whose base change to  $\operatorname{Spec}(A)$  recovers  $U \rightarrow X \rightarrow \operatorname{Spec}(A)$ . After increasing  $i$  we may assume that  $f_i : U_i \rightarrow X_i$  is flat, see Limits of Spaces, Lemma 6.12. By Lemma 20.4 the derived pullback of  $Rf_{i,*}\mathcal{O}_{U_i}$  by  $g : X \rightarrow X_i$  is equal to  $Rf_*\mathcal{O}_U$ . Since  $Lg^*$  commutes with derived colimits, it suffices to prove what we want for  $f_i$ . Hence we may assume that  $U$  and  $X$  are of finite type over  $\mathbf{Z}$ .

Assume  $f : U \rightarrow X$  is a morphism of algebraic spaces of finite type over  $\mathbf{Z}$ . To finish the proof we will show that  $Rf_*\mathcal{O}_U$  is a homotopy colimit of perfect complexes. To see this we apply Lemma 27.2. Thus it suffices to show that  $R^if_*\mathcal{O}_U$  has countable sets of sections over affines étale over  $X$ . This follows from Lemma 27.1 applied to the structure sheaf.  $\square$

## 28. The resolution property

0GUR This section is the analogue of Derived Categories of Schemes, Section 36 for algebraic spaces; please read that section first. It is currently not known if a smooth proper algebraic space over a field always has the resolution property or if this is false. If you know the answer to this question, please email [stacks.project@gmail.com](mailto:stacks.project@gmail.com).

We can make the following definition although it scarcely makes sense to consider it for general algebraic spaces.

0GUS **Definition 28.1.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . We say  $X$  has the *resolution property* if every quasi-coherent  $\mathcal{O}_X$ -module of finite type is the quotient of a finite locally free  $\mathcal{O}_X$ -module.

If  $X$  is a quasi-compact and quasi-separated algebraic space, then it suffices to check every  $\mathcal{O}_X$ -module of finite presentation (automatically quasi-coherent) is the quotient of a finite locally free  $\mathcal{O}_X$ -module, see Limits of Spaces, Lemma 9.3. If  $X$  is a Noetherian algebraic space, then finite type quasi-coherent modules are exactly the coherent  $\mathcal{O}_X$ -modules, see Cohomology of Spaces, Lemma 12.2.

0GUT **Lemma 28.2.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Assume

- (1)  $Y$  is quasi-compact and quasi-separated and has the resolution property,
- (2) there exists an  $f$ -ample invertible module on  $X$  (Divisors on Spaces, Definition 14.1).

Then  $X$  has the resolution property.

**Proof.** Let  $\mathcal{F}$  be a finite type quasi-coherent  $\mathcal{O}_X$ -module. Let  $\mathcal{L}$  be an  $f$ -ample invertible module. Choose an affine scheme  $V$  and a surjective étale morphism  $V \rightarrow Y$ . Set  $U = V \times_Y X$ . Then  $\mathcal{L}|_U$  is ample on  $U$ . By Properties, Proposition 26.13 we know there exists finitely many maps  $s_i : \mathcal{L}^{\otimes n_i}|_U \rightarrow \mathcal{F}|_U$  which are jointly surjective. Consider the quasi-coherent  $\mathcal{O}_Y$ -modules

$$\mathcal{H}_n = f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

We may think of  $s_i$  as a section over  $V$  of the sheaf  $\mathcal{H}_{-n_i}$ . Suppose we can find finite locally free  $\mathcal{O}_Y$ -modules  $\mathcal{E}_i$  and maps  $\mathcal{E}_i \rightarrow \mathcal{H}_{-n_i}$  such that  $s_i$  is in the image. Then the corresponding maps

$$f^*\mathcal{E}_i \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n_i} \longrightarrow \mathcal{F}$$

are going to be jointly surjective and the lemma is proved. By Limits of Spaces, Lemma 9.2 for each  $i$  we can find a finite type quasi-coherent submodule  $\mathcal{H}'_i \subset \mathcal{H}_{-n_i}$  which contains the section  $s_i$  over  $V$ . Thus the resolution property of  $Y$  produces surjections  $\mathcal{E}_i \rightarrow \mathcal{H}'_i$  and we conclude.  $\square$

0GUU **Lemma 28.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be an affine or quasi-affine morphism of algebraic spaces over  $S$  with  $Y$  quasi-compact and quasi-separated. If  $Y$  has the resolution property, so does  $X$ .*

**Proof.** By Divisors on Spaces, Lemma 14.7 this is a special case of Lemma 28.2.  $\square$

Here is a case where one can prove the resolution property goes down.

0GUV **Lemma 28.4.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a surjective finite locally free morphism of algebraic spaces over  $S$ . If  $X$  has the resolution property, so does  $Y$ .*

**Proof.** The condition means that  $f$  is affine and that  $f_*\mathcal{O}_X$  is a finite locally free  $\mathcal{O}_Y$ -module of positive rank. Let  $\mathcal{G}$  be a quasi-coherent  $\mathcal{O}_Y$ -module of finite type. By assumption there exists a surjection  $\mathcal{E} \rightarrow f^*\mathcal{G}$  for some finite locally free  $\mathcal{O}_X$ -module  $\mathcal{E}$ . Since  $f_*$  is exact (Cohomology of Spaces, Section 4) we get a surjection

$$f_*\mathcal{E} \longrightarrow f_*f^*\mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X$$

Taking duals we get a surjection

$$f_*\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y) \longrightarrow \mathcal{G}$$

Since  $f_*\mathcal{E}$  is finite locally free, we conclude.  $\square$

For more on the resolution property of algebraic spaces, please see More on Morphisms of Spaces, Section 56.

## 29. Detecting Boundedness

0GFE In this section, we show that compact generators of  $D_{Qcoh}$  of a quasi-compact, quasi-separated scheme, as constructed in Section 15, have a special property. We recommend reading that section first as it is very similar to this one.

0GFF **Lemma 29.1.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $P \in D_{perf}(\mathcal{O}_X)$  and  $E \in D_{Qcoh}(\mathcal{O}_X)$ . Let  $a \in \mathbf{Z}$ . The following are equivalent*

- (1)  $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], E) = 0$  for  $i \gg 0$ , and
- (2)  $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{\geq a}E) = 0$  for  $i \gg 0$ .

**Proof.** Using the triangle  $\tau_{<a}E \rightarrow E \rightarrow \tau_{\geq a}E \rightarrow$  we see that the equivalence follows if we can show

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{<a}E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(P, (\tau_{<a}E)[i]) = 0$$

for  $i \gg 0$ . As  $P$  is perfect this is true by Lemma 17.2.  $\square$

0GFG **Lemma 29.2.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $P \in D_{perf}(\mathcal{O}_X)$  and  $E \in D_{Qcoh}(\mathcal{O}_X)$ . Let  $a \in \mathbf{Z}$ . The following are equivalent*

- (1)  $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], E) = 0$  for  $i \ll 0$ , and
- (2)  $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{\leq a}E) = 0$  for  $i \ll 0$ .

**Proof.** Using the triangle  $\tau_{\leq a}E \rightarrow E \rightarrow \tau_{>a}E \rightarrow$  we see that the equivalence follows if we can show

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{>a}E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(P, (\tau_{>a}E)[i]) = 0$$

for  $i \ll 0$ . As  $P$  is perfect this is true by Lemma 17.2.  $\square$

0GFH **Proposition 29.3.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $G \in D_{\mathrm{perf}}(\mathcal{O}_X)$  be a perfect complex which generates  $D_{Q\mathrm{Coh}}(\mathcal{O}_X)$ . Let  $E \in D_{Q\mathrm{Coh}}(\mathcal{O}_X)$ . The following are equivalent*

- (1)  $E \in D_{Q\mathrm{Coh}}^-(\mathcal{O}_X)$ ,
- (2)  $\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = 0$  for  $i \gg 0$ ,
- (3)  $\mathrm{Ext}_X^i(G, E) = 0$  for  $i \gg 0$ ,
- (4)  $R\mathrm{Hom}_X(G, E)$  is in  $D^-(\mathbf{Z})$ ,
- (5)  $H^i(X, G^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} E) = 0$  for  $i \gg 0$ ,
- (6)  $R\Gamma(X, G^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} E)$  is in  $D^-(\mathbf{Z})$ ,
- (7) for every perfect object  $P$  of  $D(\mathcal{O}_X)$ 
  - (a) the assertions (2), (3), (4) hold with  $G$  replaced by  $P$ , and
  - (b)  $H^i(X, P \otimes_{\mathcal{O}_X}^{\mathbf{L}} E) = 0$  for  $i \gg 0$ ,
  - (c)  $R\Gamma(X, P \otimes_{\mathcal{O}_X}^{\mathbf{L}} E)$  is in  $D^-(\mathbf{Z})$ .

**Proof.** Assume (1). Since  $\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(G, E[i])$  we see that this is zero for  $i \gg 0$  by Lemma 17.2. This proves that (1) implies (2).

Parts (2), (3), (4) are equivalent by the discussion in Cohomology on Sites, Section 36. Part (5) and (6) are equivalent as  $H^i(X, -) = H^i(R\Gamma(X, -))$  by definition. The equivalent conditions (2), (3), (4) are equivalent to the equivalent conditions (5), (6) by Cohomology on Sites, Lemma 48.4 and the fact that  $(G[-i])^\vee = G^\vee[i]$ .

It is clear that (7) implies (2). Conversely, let us prove that the equivalent conditions (2) – (6) imply (7). Recall that  $G$  is a classical generator for  $D_{\mathrm{perf}}(\mathcal{O}_X)$  by Remark 16.2. For  $P \in D_{\mathrm{perf}}(\mathcal{O}_X)$  let  $T(P)$  be the assertion that  $R\mathrm{Hom}_X(P, E)$  is in  $D^-(\mathbf{Z})$ . Clearly,  $T$  is inherited by direct sums, satisfies the 2-out-of-three property for distinguished triangles, is inherited by direct summands, and is preserved by shifts. Hence by Derived Categories, Remark 36.7 we see that (4) implies  $T$  holds on all of  $D_{\mathrm{perf}}(\mathcal{O}_X)$ . The same argument works for all other properties, except that for property (7)(b) and (7)(c) we also use that  $P \mapsto P^\vee$  is a self equivalence of  $D_{\mathrm{perf}}(\mathcal{O}_X)$ . Small detail omitted.

We will prove the equivalent conditions (2) – (7) imply (1) using the induction principle of Lemma 9.3.

First, we prove (2) – (7)  $\Rightarrow$  (1) if  $X$  is affine. This follows from the case of schemes, see Derived Categories of Schemes, Proposition 40.5.

Now assume  $(U \subset X, j : V \rightarrow X)$  is an elementary distinguished square of quasi-compact and quasi-separated algebraic spaces over  $S$  and assume the implication (2) – (7)  $\Rightarrow$  (1) is known for  $U$ ,  $V$ , and  $U \times_X V$ . To finish the proof we have to show the implication (2) – (7)  $\Rightarrow$  (1) for  $X$ . Suppose  $E \in D_{Q\mathrm{Coh}}(\mathcal{O}_X)$  satisfies (2) – (7). By Lemma 15.3 and Theorem 15.4 there exists a perfect complex  $Q$  on  $X$  such that  $Q|_U$  generates  $D_{Q\mathrm{Coh}}(\mathcal{O}_U)$ .

Say  $V = \text{Spec}(A)$ . Let  $Z \subset V$  be the reduced closed subscheme which is the inverse image of  $X \setminus U$  and maps isomorphically to it (see Definition 9.1). This is a retro-compact closed subset of  $V$ . Choose  $f_1, \dots, f_r \in A$  such that  $Z = V(f_1, \dots, f_r)$ . Let  $K \in D(\mathcal{O}_V)$  be the perfect object corresponding to the Koszul complex on  $f_1, \dots, f_r$  over  $A$ . Note that since  $K$  is supported on  $Z$ , the pushforward  $K' = Rj_*K$  is a perfect object of  $D(\mathcal{O}_X)$  whose restriction to  $V$  is  $K$  (see Lemmas 14.3 and 10.7). By assumption, we know  $R\text{Hom}_{\mathcal{O}_X}(Q, E)$  and  $R\text{Hom}_{\mathcal{O}_X}(K', E)$  are bounded above.

By Lemma 10.7 we have  $K' = j_!K$  and hence

$$\text{Hom}_{D(\mathcal{O}_X)}(K'[-i], E) = \text{Hom}_{D(\mathcal{O}_V)}(K[-i], E|_V) = 0$$

for  $i \gg 0$ . Therefore, we may apply Derived Categories of Schemes, Lemma 40.1 to  $E|_V$  to obtain an integer  $a$  such that  $\tau_{\geq a}(E|_V) = \tau_{\geq a}R(U \times_X V \rightarrow V)_*(E|_{U \times_X V})$ . Then  $\tau_{\geq a}E = \tau_{\geq a}R(U \rightarrow X)_*(E|_U)$  (check that the canonical map is an isomorphism after restricting to  $U$  and to  $V$ ). Hence using Lemma 29.1 twice we see that

$$\text{Hom}_{D(\mathcal{O}_U)}(Q|_U[-i], E|_U) = \text{Hom}_{D(\mathcal{O}_X)}(Q[-i], R(U \rightarrow X)_*(E|_U)) = 0$$

for  $i \gg 0$ . Since the Proposition holds for  $U$  and the generator  $Q|_U$ , we have  $E|_U \in D_{QCoh}^-(\mathcal{O}_U)$ . But then since the functor  $R(U \rightarrow X)_*$  preserves  $D_{QCoh}^-$  (by Lemma 6.1), we get  $\tau_{\geq a}E \in D_{QCoh}^-(\mathcal{O}_X)$ . Thus  $E \in D_{QCoh}^-(\mathcal{O}_X)$ .  $\square$

0GFI **Proposition 29.4.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $G \in D_{perf}(\mathcal{O}_X)$  be a perfect complex which generates  $D_{QCoh}(\mathcal{O}_X)$ . Let  $E \in D_{QCoh}(\mathcal{O}_X)$ . The following are equivalent*

- (1)  $E \in D_{QCoh}^+(\mathcal{O}_X)$ ,
- (2)  $\text{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = 0$  for  $i \ll 0$ ,
- (3)  $\text{Ext}_X^i(G, E) = 0$  for  $i \ll 0$ ,
- (4)  $R\text{Hom}_X(G, E)$  is in  $D^+(\mathbf{Z})$ ,
- (5)  $H^i(X, G^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} E) = 0$  for  $i \ll 0$ ,
- (6)  $R\Gamma(X, G^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} E)$  is in  $D^+(\mathbf{Z})$ ,
- (7) for every perfect object  $P$  of  $D(\mathcal{O}_X)$ 
  - (a) the assertions (2), (3), (4) hold with  $G$  replaced by  $P$ , and
  - (b)  $H^i(X, P \otimes_{\mathcal{O}_X}^{\mathbf{L}} E) = 0$  for  $i \ll 0$ ,
  - (c)  $R\Gamma(X, P \otimes_{\mathcal{O}_X}^{\mathbf{L}} E)$  is in  $D^+(\mathbf{Z})$ .

**Proof.** Assume (1). Since  $\text{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = \text{Hom}_{D(\mathcal{O}_X)}(G, E[i])$  we see that this is zero for  $i \ll 0$  by Lemma 17.2. This proves that (1) implies (2).

Parts (2), (3), (4) are equivalent by the discussion in Cohomology on Sites, Section 36. Part (5) and (6) are equivalent as  $H^i(X, -) = H^i(R\Gamma(X, -))$  by definition. The equivalent conditions (2), (3), (4) are equivalent to the equivalent conditions (5), (6) by Cohomology on Sites, Lemma 48.4 and the fact that  $(G[-i])^\vee = G^\vee[i]$ .

It is clear that (7) implies (2). Conversely, let us prove that the equivalent conditions (2) – (6) imply (7). Recall that  $G$  is a classical generator for  $D_{perf}(\mathcal{O}_X)$  by Remark 16.2. For  $P \in D_{perf}(\mathcal{O}_X)$  let  $T(P)$  be the assertion that  $R\text{Hom}_X(P, E)$  is in  $D^+(\mathbf{Z})$ . Clearly,  $T$  is inherited by direct sums, satisfies the 2-out-of-three property for distinguished triangles, is inherited by direct summands, and is preserved by shifts. Hence by Derived Categories, Remark 36.7 we see that (4) implies  $T$  holds on all of  $D_{perf}(\mathcal{O}_X)$ . The same argument works for all other properties, except

that for property (7)(b) and (7)(c) we also use that  $P \mapsto P^\vee$  is a self equivalence of  $D_{perf}(\mathcal{O}_X)$ . Small detail omitted.

We will prove the equivalent conditions (2) – (7) imply (1) using the induction principle of Lemma 9.3.

First, we prove (2) – (7)  $\Rightarrow$  (1) if  $X$  is affine. This follows from the case of schemes, see Derived Categories of Schemes, Proposition 40.6.

Now assume  $(U \subset X, j : V \rightarrow X)$  is an elementary distinguished square of quasi-compact and quasi-separated algebraic spaces over  $S$  and assume the implication (2) – (7)  $\Rightarrow$  (1) is known for  $U$ ,  $V$ , and  $U \times_X V$ . To finish the proof we have to show the implication (2) – (7)  $\Rightarrow$  (1) for  $X$ . Suppose  $E \in D_{Qcoh}(\mathcal{O}_X)$  satisfies (2) – (7). By Lemma 15.3 and Theorem 15.4 there exists a perfect complex  $Q$  on  $X$  such that  $Q|_U$  generates  $D_{Qcoh}(\mathcal{O}_U)$ .

Say  $V = \text{Spec}(A)$ . Let  $Z \subset V$  be the reduced closed subscheme which is the inverse image of  $X \setminus U$  and maps isomorphically to it (see Definition 9.1). This is a retro-compact closed subset of  $V$ . Choose  $f_1, \dots, f_r \in A$  such that  $Z = V(f_1, \dots, f_r)$ . Let  $K \in D(\mathcal{O}_V)$  be the perfect object corresponding to the Koszul complex on  $f_1, \dots, f_r$  over  $A$ . Note that since  $K$  is supported on  $Z$ , the pushforward  $K' = Rj_* K$  is a perfect object of  $D(\mathcal{O}_X)$  whose restriction to  $V$  is  $K$  (see Lemmas 14.3 and 10.7). By assumption, we know  $R\text{Hom}_{\mathcal{O}_X}(Q, E)$  and  $R\text{Hom}_{\mathcal{O}_X}(K', E)$  are bounded below.

By Lemma 10.7 we have  $K' = j_! K$  and hence

$$\text{Hom}_{D(\mathcal{O}_X)}(K'[-i], E) = \text{Hom}_{D(\mathcal{O}_V)}(K[-i], E|_V) = 0$$

for  $i \ll 0$ . Therefore, we may apply Derived Categories of Schemes, Lemma 40.2 to  $E|_V$  to obtain an integer  $a$  such that  $\tau_{\leq a}(E|_V) = \tau_{\leq a}R(U \times_X V \rightarrow V)_*(E|_{U \times_X V})$ . Then  $\tau_{\leq a}E = \tau_{\leq a}R(U \rightarrow X)_*(E|_U)$  (check that the canonical map is an isomorphism after restricting to  $U$  and to  $V$ ). Hence using Lemma 29.2 twice we see that

$$\text{Hom}_{D(\mathcal{O}_U)}(Q|_U[-i], E|_U) = \text{Hom}_{D(\mathcal{O}_X)}(Q[-i], R(U \rightarrow X)_*(E|_U)) = 0$$

for  $i \ll 0$ . Since the Proposition holds for  $U$  and the generator  $Q|_U$ , we have  $E|_U \in D_{Qcoh}^+(\mathcal{O}_U)$ . But then since the functor  $R(U \rightarrow X)_*$  preserves  $D_{Qcoh}^+$  (by Lemma 6.1), we get  $\tau_{\leq a}E \in D_{Qcoh}^+(\mathcal{O}_X)$ . Thus  $E \in D_{Qcoh}^+(\mathcal{O}_X)$ .  $\square$

### 30. Quasi-coherent objects in the derived category

0H05 Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Recall that  $X_{affine, \acute{e}tale}$  denotes the category of affine objects of  $X_{\acute{e}tale}$  with topology given by standard étale coverings, see Properties of Spaces, Definition 18.5. We remind the reader that the topos of  $X_{affine, \acute{e}tale}$  is the small étale topos of  $X$ , see Properties of Spaces, Lemma 18.6. The site  $X_{\acute{e}tale}$  comes with a structure sheaf  $\mathcal{O}_X$  whose restriction to  $X_{affine, \acute{e}tale}$  we also denote  $\mathcal{O}_X$ . Then there is an equivalence of ringed topoi

$$(Sh(X_{affine, \acute{e}tale}), \mathcal{O}_X) \longrightarrow (Sh(X_{\acute{e}tale}), \mathcal{O}_X)$$

See Descent on Spaces, Equation (5.0.1) and the discussion in Descent on Spaces, Section 5.

In this section we denote  $X_{\text{affine}}$  the underlying category of  $X_{\text{affine},\text{étale}}$  endowed with the chaotic topology, i.e., such that sheaves agree with presheaves. In particular, the structure sheaf  $\mathcal{O}_X$  becomes a sheaf on  $X_{\text{affine}}$  as well. We obtain a morphisms of ringed sites

$$\epsilon : (X_{\text{affine},\text{étale}}, \mathcal{O}_X) \longrightarrow (X_{\text{affine}}, \mathcal{O}_X)$$

as in Cohomology on Sites, Section 27. In this section we will identify  $D_{QCoh}(\mathcal{O}_X)$  with the category  $QC(X_{\text{affine}}, \mathcal{O}_X)$  introduced in Cohomology on Sites, Section 43.

0H06 **Lemma 30.1.** *In the situation above there are canonical exact equivalences between the following triangulated categories*

- (1)  $D_{QCoh}(\mathcal{O}_X)$ ,
- (2)  $D_{QCoh}(X_{\text{affine},\text{étale}}, \mathcal{O}_X)$ ,
- (3)  $D_{QCoh}(X_{\text{affine}}, \mathcal{O}_X)$ , and
- (4)  $QC(X_{\text{affine}}, \mathcal{O}_X)$ .

**Proof.** If  $U \rightarrow V \rightarrow X$  are étale morphisms with  $U$  and  $V$  affine, then the ring map  $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$  is flat. Hence the equivalence between (3) and (4) is a special case of Cohomology on Sites, Lemma 43.11 (the proof also clarifies the statement).

The discussion preceding the lemma shows that we have an equivalence of ringed topoi  $(Sh(X_{\text{affine},\text{étale}}, \mathcal{O}_X) \rightarrow (Sh(X_{\text{étale}}), \mathcal{O}_X)$  and hence an equivalence between abelian categories of modules. Since the notion of quasi-coherent modules is intrinsic (Modules on Sites, Lemma 23.2) we see that this equivalence preserves the subcategories of quasi-coherent modules. Thus we get a canonical exact equivalence between the triangulated categories in (1) and (2).

To get an exact equivalence between the triangulated categories in (2) and (3) we will apply Cohomology on Sites, Lemma 29.1 to the morphism  $\epsilon : (X_{\text{affine},\text{étale}}, \mathcal{O}_X) \rightarrow (X_{\text{affine}}, \mathcal{O}_X)$  above. We take  $\mathcal{B} = \text{Ob}(X_{\text{affine}})$  and we take  $\mathcal{A} \subset PMod(X_{\text{affine}}, \mathcal{O})$  to be the full subcategory of those presheaves  $\mathcal{F}$  such that  $\mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \rightarrow \mathcal{F}(U)$  is an isomorphism. Observe that by Descent on Spaces, Lemma 5.1 objects of  $\mathcal{A}$  are exactly those sheaves in the étale topology which are quasi-coherent modules on  $(X_{\text{affine},\text{étale}}, \mathcal{O}_X)$ . On the other hand, by Modules on Sites, Lemma 24.2, the objects of  $\mathcal{A}$  are exactly the quasi-coherent modules on  $(X_{\text{affine}}, \mathcal{O}_X)$ , i.e., in the chaotic topology. Thus if we show that Cohomology on Sites, Lemma 29.1 applies, then we do indeed get the canonical equivalence between the categories of (2) and (3) using  $\epsilon^*$  and  $R\epsilon_*$ .

We have to verify 4 conditions:

- (1) Every object of  $\mathcal{A}$  is a sheaf for the étale topology. This we have seen above.
- (2)  $\mathcal{A}$  is a weak Serre subcategory of  $Mod(X_{\text{affine},\text{étale}}, \mathcal{O}_X)$ . Above we have seen that  $\mathcal{A} = QCoh(X_{\text{affine},\text{étale}}, \mathcal{O}_X)$  and we have seen above that these, via the equivalence  $Mod(X_{\text{affine},\text{étale}}, \mathcal{O}) = Mod(X_{\text{étale}}, \mathcal{O}_X)$ , correspond to the quasi-coherent modules on  $X$ . Thus the result by Properties of Spaces, Lemma 29.7 and Homology, Lemma 10.3.
- (3) Every object of  $X_{\text{affine}}$  has a covering in the chaotic topology whose members are elements of  $\mathcal{B}$ . This holds because  $\mathcal{B}$  contains all objects.
- (4) For every object  $U$  of  $X_{\text{affine}}$  and  $\mathcal{F}$  in  $\mathcal{A}$  we have  $H_{Zar}^p(U, \mathcal{F}) = 0$  for  $p > 0$ . This holds by the vanishing of cohomology of quasi-coherent modules on

affines, see discussion in Cohomology of Spaces, Section 3 and Cohomology of Schemes, Lemma 2.2.

This finishes the proof.  $\square$

0H07 **Remark 30.2.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . We will later show that also  $QC((\text{Aff}/X), \mathcal{O})$  is canonically equivalent to  $D_{QCoh}(\mathcal{O}_X)$ . See Sheaves on Stacks, Proposition 26.4.

### 31. Other chapters

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#### Schemes

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|                                       |                                      |
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