SIMPLICIAL SPACES

09VI

Contents

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1. Introduction

09VJ This chapter develops some theory concerning simplicial topological spaces, simplicial ringed spaces, simplicial schemes, and simplicial algebraic spaces. The theory of simplicial spaces sometimes allows one to prove local to global principles which appear difficult to prove in other ways. Some example applications can be found in the papers [Fal03], [Kie72], and [Del74].

We assume throughout that the reader is familiar with the basic concepts and results of the chapter Simplicial Methods, see Simplicial, Section 1. In particular, we continue to write X and not $X_\bullet$ for a simplicial object.

2. Simplicial topological spaces

09VK A *simplicial space* is a simplicial object in the category of topological spaces where morphisms are continuous maps of topological spaces. (We will use "simplicial algebraic space" to refer to simplicial objects in the category of algebraic spaces.) We may picture a simplicial space X as follows

$$X_2 \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} X_1 \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} X_0$$

Here there are two morphisms $d^1_0, d^1_1 : X_1 \to X_0$ and a single morphism $s^0_0 : X_0 \to X_1$, etc. It is important to keep in mind that $d^n_i : X_n \to X_{n-1}$ should be thought of as a "projection forgetting the ith coordinate" and $s^n_j : X_n \to X_{n+1}$ as the diagonal map repeating the jth coordinate.

Let X be a simplicial space. We associate a site X_{Zar}[1] to X as follows.

(1) An object of X_{Zar} is an open U of X_n for some n,
(2) a morphism $U \to V$ of X_{Zar} is given by a $\varphi : [m] \to [n]$ where n, m are such that $U \subset X_n$, $V \subset X_m$ and φ is such that $X(\varphi)(U) \subset V$, and
(3) a covering $\{U_i \to U\}$ in X_{Zar} means that $U, U_i \subset X_n$ are open, the maps $U_i \to U$ are given by $\mathrm{id} : [n] \to [n]$, and $U = \bigcup U_i$.

Note that in particular, if $U \to V$ is a morphism of X_{Zar} given by φ, then $X(\varphi) : X_n \to X_m$ does in fact induce a continuous map $U \to V$ of topological spaces.
It is clear that the above is a special case of a construction that associates to any diagram of topological spaces a site. We formulate the obligatory lemma.

09VL **Lemma 2.1.** *Let X be a simplicial space. Then X_{Zar} as defined above is a site.*

Proof. Omitted. □

Let X be a simplicial space. Let $\mathcal{F}$ be a sheaf on X_{Zar}. It is clear from the definition of coverings, that the restriction of $\mathcal{F}$ to the opens of X_n defines a sheaf $\mathcal{F}_n$ on the topological space X_n. For every $\varphi : [m] \to [n]$ the restriction maps of $\mathcal{F}$ for pairs $U \subset X_n$, $V \subset X_m$ with $X(\varphi)(U) \subset V$, define an $X(\varphi)$-map $\mathcal{F}(\varphi) : \mathcal{F}_m \to \mathcal{F}_n$, see Sheaves, Definition 21.7. Moreover, given $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ we have

$$\mathcal{F}(\varphi) \circ \mathcal{F}(\psi) = \mathcal{F}(\varphi \circ \psi)$$

(LHS uses composition of f-maps, see Sheaves, Definition 21.9). Clearly, the converse is true as well: if we have a system $(\{\mathcal{F}_n\}_{n \geq 0}, \{\mathcal{F}(\varphi)\}_{\varphi \in \text{Arrows}(\Delta)})$ as above, satisfying the displayed equalities, then we obtain a sheaf on X_{Zar}.

[1] This notation is similar to the notation in Sites, Example 6.4 and Topologies, Definition 3.7.

09VM **Lemma 2.2.** *Let X be a simplicial space. There is an equivalence of categories between*

1. *$Sh(X_{Zar})$, and*
2. *category of systems $(\mathcal{F}_n, \mathcal{F}(\varphi))$ described above.*

Proof. See discussion above. □

09VN **Lemma 2.3.** *Let $f : Y \to X$ be a morphism of simplicial spaces. Then the functor $u : X_{Zar} \to Y_{Zar}$ which associates to the open $U \subset X_n$ the open $f_n^{-1}(U) \subset Y_n$ defines a morphism of sites $f_{Zar} : Y_{Zar} \to X_{Zar}$.*

Proof. It is clear that u is a continuous functor. Hence we obtain functors $f_{Zar,*} = u^s$ and $f_{Zar}^{-1} = u_s$, see Sites, Section 14. To see that we obtain a morphism of sites we have to show that u_s is exact. We will use Sites, Lemma 14.6 to see this. Let $V \subset Y_n$ be an open subset. The category $\mathcal{I}_V^u$ (see Sites, Section 5) consists of pairs (U, φ) where $\varphi : [m] \to [n]$ and $U \subset X_m$ open such that $Y(\varphi)(V) \subset f_m^{-1}(U)$. Moreover, a morphism $(U, \varphi) \to (U', \varphi')$ is given by a $\psi : [m'] \to [m]$ such that $X(\psi)(U) \subset U'$ and $\varphi \circ \psi = \varphi'$. It is our task to show that $\mathcal{I}_V^u$ is cofiltered.

We verify the conditions of Categories, Definition 20.1. Condition (1) holds because $(X_n, \mathrm{id}_{[n]})$ is an object. Let (U, φ) be an object. The condition $Y(\varphi)(V) \subset f_m^{-1}(U)$ is equivalent to $V \subset f_n^{-1}(X(\varphi)^{-1}(U))$. Hence we obtain a morphism $(X(\varphi)^{-1}(U), \mathrm{id}_{[n]}) \to (U, \varphi)$ given by setting $\psi = \varphi$. Moreover, given a pair of objects of the form $(U, \mathrm{id}_{[n]})$ and $(U', \mathrm{id}_{[n]})$ we see there exists an object, namely $(U \cap U', \mathrm{id}_{[n]})$, which maps to both of them. Thus condition (2) holds. To verify condition (3) suppose given two morphisms $a, a' : (U, \varphi) \to (U', \varphi')$ given by $\psi, \psi' : [m'] \to [m]$. Then precomposing with the morphism $(X(\varphi)^{-1}(U), \mathrm{id}_{[n]}) \to (U, \varphi)$ given by φ equalizes a, a' because $\varphi \circ \psi = \varphi' = \varphi \circ \psi'$. This finishes the proof. □

09VP **Lemma 2.4.** *Let $f : Y \to X$ be a morphism of simplicial spaces. In terms of the description of sheaves in Lemma 2.2 the morphism f_{Zar} of Lemma 2.3 can be described as follows.*

1. *If $\mathcal{G}$ is a sheaf on Y, then $(f_{Zar,*}\mathcal{G})_n = f_{n,*}\mathcal{G}_n$.*
2. *If $\mathcal{F}$ is a sheaf on X, then $(f_{Zar}^{-1}\mathcal{F})_n = f_n^{-1}\mathcal{F}_n$.*

Proof. The first part is immediate from the definitions. For the second part, note that in the proof of Lemma 2.3 we have shown that for a $V \subset Y_n$ open the category $(\mathcal{I}_V^u)^{opp}$ contains as a cofinal subcategory the category of opens $U \subset X_n$ with $f_n^{-1}(U) \supset V$ and morphisms given by inclusions. Hence we see that the restriction of $u_p\mathcal{F}$ to opens of Y_n is the presheaf $f_{n,p}\mathcal{F}_n$ as defined in Sheaves, Lemma 21.3. Since $f_{Zar}^{-1}\mathcal{F} = u_s\mathcal{F}$ is the sheafification of $u_p\mathcal{F}$ and since sheafification uses only coverings and since coverings in Y_{Zar} use only inclusions between opens on the same Y_n, the result follows from the fact that $f_n^{-1}\mathcal{F}_n$ is (correspondingly) the sheafification of $f_{n,p}\mathcal{F}_n$, see Sheaves, Section 21. □

Let X be a topological space. In Sites, Example 6.4 we denoted X_{Zar} the site consisting of opens of X with inclusions as morphisms and coverings given by open coverings. We identify the topos $Sh(X_{Zar})$ with the category of sheaves on X.

09W0 **Lemma 2.5.** *Let X be a simplicial space. The functor $X_{n,Zar} \to X_{Zar}$, $U \mapsto U$ is continuous and cocontinuous. The associated morphism of topoi $g_n : Sh(X_n) \to Sh(X_{Zar})$ satisfies*

(1) g_n^{-1} *associates to the sheaf* $\mathcal{F}$ *on* X *the sheaf* $\mathcal{F}_n$ *on* X_n,
(2) $g_n^{-1} : Sh(X_{Zar}) \to Sh(X_n)$ *has a left adjoint* $g_{n!}^{Sh}$,
(3) $g_{n!}^{Sh}$ *commutes with finite connected limits,*
(4) $g_n^{-1} : Ab(X_{Zar}) \to Ab(X_n)$ *has a left adjoint* $g_{n!}$, *and*
(5) $g_{n!}$ *is exact.*

Proof. Besides the properties of our functor mentioned in the statement, the category $X_{n,Zar}$ has fibre products and equalizers and the functor commutes with them (beware that X_{Zar} does not have all fibre products). Hence the lemma follows from the discussion in Sites, Sections 20 and 21 and Modules on Sites, Section 16. More precisely, Sites, Lemmas 21.1, 21.5, and 21.6 and Modules on Sites, Lemmas 16.2 and 16.3. □

09W1 **Lemma 2.6.** *Let* X *be a simplicial space. If* $\mathcal{I}$ *is an injective abelian sheaf on* X_{Zar}, *then* $\mathcal{I}_n$ *is an injective abelian sheaf on* X_n.

Proof. This follows from Homology, Lemma 29.1 and Lemma 2.5. □

09W2 **Lemma 2.7.** *Let* $f : Y \to X$ *be a morphism of simplicial spaces. Then*

$$\begin{array}{ccc} Sh(Y_n) & \xrightarrow{f_n} & Sh(X_n) \\ \downarrow & & \downarrow \\ Sh(Y_{Zar}) & \xrightarrow{f_{Zar}} & Sh(X_{Zar}) \end{array}$$

is a commutative diagram of topoi.

Proof. Direct from the description of pullback functors in Lemmas 2.4 and 2.5. □

09W4 **Lemma 2.8.** *Let* Y *be a simplicial space and let* $a : Y \to X$ *be an augmentation (Simplicial, Definition 20.1). Let* $a_n : Y_n \to X$ *be the corresponding morphisms of topological spaces. There is a canonical morphism of topoi*

$$a : Sh(Y_{Zar}) \to Sh(X)$$

with the following properties:

(1) $a^{-1}\mathcal{F}$ *is the sheaf restricting to* $a_n^{-1}\mathcal{F}$ *on* Y_n,
(2) $a_m \circ Y(\varphi) = a_n$ *for all* $\varphi : [m] \to [n]$,
(3) $a \circ g_n = a_n$ *as morphisms of topoi with* g_n *as in Lemma 2.5,*
(4) $a_*\mathcal{G}$ *for* $\mathcal{G} \in Sh(Y_{Zar})$ *is the equalizer of the two maps* $a_{0,*}\mathcal{G}_0 \to a_{1,*}\mathcal{G}_1$.

Proof. Part (2) holds for augmentations of simplicial objects in any category. Thus $Y(\varphi)^{-1}a_m^{-1}\mathcal{F} = a_n^{-1}\mathcal{F}$ which defines an $Y(\varphi)$-map from $a_m^{-1}\mathcal{F}$ to $a_n^{-1}\mathcal{F}$. Thus we can use (1) as the definition of $a^{-1}\mathcal{F}$ (using Lemma 2.2) and (4) as the definition of a_*. If this defines a morphism of topoi then part (3) follows because we'll have $g_n^{-1} \circ a^{-1} = a_n^{-1}$ by construction. To check a is a morphism of topoi we have to show that a^{-1} is left adjoint to a_* and we have to show that a^{-1} is exact. The last fact is immediate from the exactness of the functors a_n^{-1}.

Let $\mathcal{F}$ be an object of $Sh(X)$ and let $\mathcal{G}$ be an object of $Sh(Y_{Zar})$. Given $\beta : a^{-1}\mathcal{F} \to \mathcal{G}$ we can look at the components $\beta_n : a_n^{-1}\mathcal{F} \to \mathcal{G}_n$. These maps are adjoint to maps $\beta_n : \mathcal{F} \to a_{n,*}\mathcal{G}_n$. Compatibility with the simplicial structure shows that β_0 maps into $a_*\mathcal{G}$. Conversely, suppose given a map $\alpha : \mathcal{F} \to a_*\mathcal{G}$. For any n choose a $\varphi : [0] \to [n]$. Then we can look at the composition

$$\mathcal{F} \xrightarrow{\alpha} a_*\mathcal{G} \to a_{0,*}\mathcal{G}_0 \xrightarrow{\mathcal{G}(\varphi)} a_{n,*}\mathcal{G}_n$$

These are adjoint to maps $a_n^{-1}\mathcal{F} \to \mathcal{G}_n$ which define a morphism of sheaves $a^{-1}\mathcal{F} \to \mathcal{G}$. We omit the proof that the constructions given above define mutually inverse bijections

$$\mathrm{Mor}_{Sh(Y_{Zar})}(a^{-1}\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(X)}(\mathcal{F}, a_*\mathcal{G})$$

This finishes the proof. An interesting observation is here that this morphism of topoi does not correspond to any obvious geometric functor between the sites defining the topoi. □

09W5 **Lemma 2.9.** *Let X be a simplicial topological space. The complex of abelian presheaves on X_{Zar}*

$$\ldots \to \mathbf{Z}_{X_2} \to \mathbf{Z}_{X_1} \to \mathbf{Z}_{X_0}$$

with boundary $\sum (-1)^i d_i^n$ is a resolution of the constant presheaf $\mathbf{Z}$.

Proof. Let $U \subset X_m$ be an object of X_{Zar}. Then the value of the complex above on U is the complex of abelian groups

$$\ldots \to \mathbf{Z}[\mathrm{Mor}_\Delta([2], [m])] \to \mathbf{Z}[\mathrm{Mor}_\Delta([1], [m])] \to \mathbf{Z}[\mathrm{Mor}_\Delta([0], [m])]$$

In other words, this is the complex associated to the free abelian group on the simplicial set $\Delta[m]$, see Simplicial, Example 11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 26.7, and since "taking free abelian groups" is a functor, we see that the complex above is homotopy equivalent to the free abelian group on $\Delta[0]$ (Simplicial, Remark 26.4 and Lemma 27.2). This complex is acyclic in positive degrees and equal to $\mathbf{Z}$ in degree 0. □

09W6 **Lemma 2.10.** *Let X be a simplicial topological space. Let $\mathcal{F}$ be an abelian sheaf on X. There is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_1^{p,q} = H^q(X_p, \mathcal{F}_p)$$

converging to $H^{p+q}(X_{Zar}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Consider the double complex with terms

$$A^{p,q} = \mathcal{I}^q(X_p)$$

and first differential given by the alternating sum along the maps d_i^{p+1}-maps $\mathcal{I}_p^q \to \mathcal{I}_{p+1}^q$, see Lemma 2.2. Note that

$$A^{p,q} = \Gamma(X_p, \mathcal{I}_p^q) = \mathrm{Mor}_{PSh}(h_{X_p}, \mathcal{I}^q) = \mathrm{Mor}_{PAb}(\mathbf{Z}_{X_p}, \mathcal{I}^q)$$

Hence it follows from Lemma 2.9 and Cohomology on Sites, Lemma 10.1 that the rows of the double complex are exact in positive degrees and evaluate to $\Gamma(X_{Zar}, \mathcal{I}^q)$ in degree 0. On the other hand, since restriction is exact (Lemma 2.5) the map

$$\mathcal{F}_p \to \mathcal{I}_p^\bullet$$

is a resolution. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on X_p (Lemma 2.6). Hence the cohomology of the columns computes the groups $H^q(X_p, \mathcal{F}_p)$. We conclude by applying Homology, Lemmas 25.3 and 25.4. □

0D84 **Lemma 2.11.** *Let X be a simplicial space and let $a : X \to Y$ be an augmentation. Let $\mathcal{F}$ be an abelian sheaf on X_{Zar}. Then $R^n a_* \mathcal{F}$ is the sheaf associated to the presheaf*

$$V \longmapsto H^n((X \times_Y V)_{Zar}, \mathcal{F}|_{(X \times_Y V)_{Zar}})$$

Proof. This is the analogue of Cohomology, Lemma 7.3 or of Cohomology on Sites, Lemma 7.4 and we strongly encourage the reader to skip the proof. Choosing an injective resolution of $\mathcal{F}$ on X_{Zar} and using the definitions we see that it suffices to show: (1) the restriction of an injective abelian sheaf on X_{Zar} to $(X \times_Y V)_{Zar}$ is an injective abelian sheaf and (2) $a_*\mathcal{F}$ is equal to the rule

$$V \longmapsto H^0((X \times_Y V)_{Zar}, \mathcal{F}|_{(X \times_Y V)_{Zar}})$$

Part (2) follows from the following facts

(2a) $a_*\mathcal{F}$ is the equalizer of the two maps $a_{0,*}\mathcal{F}_0 \to a_{1,*}\mathcal{F}_1$ by Lemma 2.8,
(2b) $a_{0,*}\mathcal{F}_0(V) = H^0(a_0^{-1}(V), \mathcal{F}_0)$ and $a_{1,*}\mathcal{F}_1(V) = H^0(a_1^{-1}(V), \mathcal{F}_1)$,
(2c) $X_0 \times_Y V = a_0^{-1}(V)$ and $X_1 \times_Y V = a_1^{-1}(V)$,
(2d) $H^0((X \times_Y V)_{Zar}, \mathcal{F}|_{(X \times_Y V)_{Zar}})$ is the equalizer of the two maps $H^0(X_0 \times_Y V, \mathcal{F}_0) \to H^0(X_1 \times_Y V, \mathcal{F}_1)$ for example by Lemma 2.10.

Part (1) follows after one defines an exact left adjoint $j_! : Ab((X \times_Y V)_{Zar}) \to Ab(X_{Zar})$ (extension by zero) to restriction $Ab(X_{Zar}) \to Ab((X \times_Y V)_{Zar})$ and using Homology, Lemma 29.1. □

Let X be a topological space. Denote $X_\bullet$ the constant simplicial topological space with value X. By Lemma 2.2 a sheaf on $X_{\bullet,Zar}$ is the same thing as a cosimplicial object in the category of sheaves on X.

09W3 **Lemma 2.12.** *Let X be a topological space. Let $X_\bullet$ be the constant simplicial topological space with value X. The functor*

$$X_{\bullet,Zar} \longrightarrow X_{Zar}, \quad U \longmapsto U$$

is continuous and cocontinuous and defines a morphism of topoi $g : Sh(X_{\bullet,Zar}) \to Sh(X)$ as well as a left adjoint $g_!$ to g^{-1}. We have

(1) *g^{-1} associates to a sheaf on X the constant cosimplicial sheaf on X,*
(2) *$g_!$ associates to a sheaf $\mathcal{F}$ on $X_{\bullet,Zar}$ the sheaf $\mathcal{F}_0$, and*
(3) *g_* associates to a sheaf $\mathcal{F}$ on $X_{\bullet,Zar}$ the equalizer of the two maps $\mathcal{F}_0 \to \mathcal{F}_1$.*

Proof. The statements about the functor are straightforward to verify. The existence of g and $g_!$ follow from Sites, Lemmas 21.1 and 21.5. The description of g^{-1} is immediate from Sites, Lemma 21.5. The description of g_* and $g_!$ follows as the functors given are right and left adjoint to g^{-1}. □

3. Simplicial sites and topoi

09WB It seems natural to define a *simplicial site* as a simplicial object in the (big) category whose objects are sites and whose morphisms are morphisms of sites. See Sites, Definitions 6.2 and 14.1 with composition of morphisms as in Sites, Lemma 14.4. But here are some variants one might want to consider: (a) we could work with cocontinuous functors (see Sites, Sections 20 and 21) between sites instead, (b) we could work in a suitable 2-category of sites where one introduces the notion of a 2-morphism between morphisms of sites, (c) we could work in a 2-category constructed out of cocontinuous functors. Instead of picking one of these variants as a definition we will simply develop theory as needed.

Certainly a *simplicial topos* should probably be defined as a pseudo-functor from Δ^{opp} into the 2-category of topoi. See Categories, Definition 29.5 and Sites, Section 15 and 36. We will try to avoid working with such a beast if possible.

Case A. Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites. This means that for every morphism $\varphi : [m] \to [n]$ of Δ we have a morphism of sites $f_\varphi : \mathcal{C}_n \to \mathcal{C}_m$. This morphism is given by a continuous functor in the opposite direction which we will denote $u_\varphi : \mathcal{C}_m \to \mathcal{C}_n$.

09WC **Lemma 3.1.** *Let $\mathcal{C}$ be a simplicial object in the category of sites. With notation as above we construct a site $\mathcal{C}_{total}$ as follows.*

1. *An object of $\mathcal{C}_{total}$ is an object U of $\mathcal{C}_n$ for some n,*
2. *a morphism $(\varphi, f) : U \to V$ of $\mathcal{C}_{total}$ is given by a map $\varphi : [m] \to [n]$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$, $V \in \mathrm{Ob}(\mathcal{C}_m)$ and a morphism $f : U \to u_\varphi(V)$ of $\mathcal{C}_n$, and*
3. *a covering $\{(id, f_i) : U_i \to U\}$ in $\mathcal{C}_{total}$ is given by an n and a covering $\{f_i : U_i \to U\}$ of $\mathcal{C}_n$.*

Proof. Composition of $(\varphi, f) : U \to V$ with $(\psi, g) : V \to W$ is given by $(\varphi \circ \psi, u_\varphi(g) \circ f)$. This uses that $u_\varphi \circ u_\psi = u_{\varphi \circ \psi}$.

Let $\{(\mathrm{id}, f_i) : U_i \to U\}$ be a covering as in (3) and let $(\varphi, g) : W \to U$ be a morphism with $W \in \mathrm{Ob}(\mathcal{C}_m)$. We claim that

$$W \times_{(\varphi,g),U,(\mathrm{id},f_i)} U_i = W \times_{g,u_\varphi(U),u_\varphi(f_i)} u_\varphi(U_i)$$

in the category $\mathcal{C}_{total}$. This makes sense as by our definition of morphisms of sites, the required fibre products in $\mathcal{C}_m$ exist since u_φ transforms coverings into coverings. The same reasoning implies the claim (details omitted). Thus we see that the collection of coverings is stable under base change. The other axioms of a site are immediate. $\square$

Case B. Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors. This means that for every morphism $\varphi : [m] \to [n]$ of Δ we have a cocontinuous functor denoted $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$. The associated morphism of topoi is denoted $f_\varphi : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_m)$.

09WD **Lemma 3.2.** *Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors. With notation as above, assume the functors $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ have property P of Sites, Remark 20.5. Then we can construct a site $\mathcal{C}_{total}$ as follows.*

1. *An object of $\mathcal{C}_{total}$ is an object U of $\mathcal{C}_n$ for some n,*
2. *a morphism $(\varphi, f) : U \to V$ of $\mathcal{C}_{total}$ is given by a map $\varphi : [m] \to [n]$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$, $V \in \mathrm{Ob}(\mathcal{C}_m)$ and a morphism $f : u_\varphi(U) \to V$ of $\mathcal{C}_m$, and*
3. *a covering $\{(id, f_i) : U_i \to U\}$ in $\mathcal{C}_{total}$ is given by an n and a covering $\{f_i : U_i \to U\}$ of $\mathcal{C}_n$.*

Proof. Composition of $(\varphi, f) : U \to V$ with $(\psi, g) : V \to W$ is given by $(\varphi \circ \psi, g \circ u_\psi(f))$. This uses that $u_\psi \circ u_\varphi = u_{\varphi \circ \psi}$.

Let $\{(\mathrm{id}, f_i) : U_i \to U\}$ be a covering as in (3) and let $(\varphi, g) : W \to U$ be a morphism with $W \in \mathrm{Ob}(\mathcal{C}_m)$. We claim that

$$W \times_{(\varphi,g),U,(\mathrm{id},f_i)} U_i = W \times_{g,U,f_i} U_i$$

in the category $\mathcal{C}_{total}$ where the right hand side is the object of $\mathcal{C}_m$ defined in Sites, Remark 20.5 which exists by property P. Compatibility of this type of fibre product with compositions of functors implies the claim (details omitted). Since the family $\{W \times_{g,U,f_i} U_i \to W\}$ is a covering of $\mathcal{C}_m$ by property P we see that the collection of coverings is stable under base change. The other axioms of a site are immediate. $\square$

09WE **Situation 3.3.** Here we have one of the following two cases:

(A) $\mathcal{C}$ is a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites. For every morphism $\varphi : [m] \to [n]$ of Δ we have a morphism of sites $f_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ given by a continuous functor $u_\varphi : \mathcal{C}_m \to \mathcal{C}_n$.

(B) $\mathcal{C}$ is a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors having property P of Sites, Remark 20.5. For every morphism $\varphi : [m] \to [n]$ of Δ we have a cocontinuous functor $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ which induces a morphism of topoi $f_\varphi : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_m)$.

As usual we will denote f_φ^{-1} and $f_{\varphi,*}$ the pullback and pushforward. We let $\mathcal{C}_{total}$ denote the site defined in Lemma 3.1 (case A) or Lemma 3.2 (case B).

Let $\mathcal{C}$ be as in Situation 3.3. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}_{total}$. It is clear from the definition of coverings, that the restriction of $\mathcal{F}$ to the objects of $\mathcal{C}_n$ defines a sheaf $\mathcal{F}_n$ on the site $\mathcal{C}_n$. For every $\varphi : [m] \to [n]$ the restriction maps of $\mathcal{F}$ along the morphisms $(\varphi, f) : U \to V$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$ and $V \in \mathrm{Ob}(\mathcal{C}_m)$ define an element $\mathcal{F}(\varphi)$ of

$$\mathrm{Mor}_{Sh(\mathcal{C}_m)}(\mathcal{F}_m, f_{\varphi,*}\mathcal{F}_n) = \mathrm{Mor}_{Sh(\mathcal{C}_n)}(f_\varphi^{-1}\mathcal{F}_m, \mathcal{F}_n)$$

Moreover, given $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ the diagrams

$$\begin{tikzcd} \mathcal{F}_l \arrow[rr, "\mathcal{F}(\varphi\circ\psi)"] \arrow[dr, "\mathcal{F}(\psi)"'] & & f_{\varphi\circ\psi,*}\mathcal{F}_n \\ & f_{\psi,*}\mathcal{F}_m \arrow[ur, "f_{\psi,*}\mathcal{F}(\varphi)"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} f_{\varphi\circ\psi}^{-1}\mathcal{F}_l \arrow[rr, "\mathcal{F}(\varphi\circ\psi)"] \arrow[dr, "f_\varphi^{-1}\mathcal{F}(\psi)"'] & & \mathcal{F}_n \\ & f_\varphi^{-1}\mathcal{F}_m \arrow[ur, "\mathcal{F}(\varphi)"'] & \end{tikzcd}$$

commute. Clearly, the converse statement is true as well: if we have a system $(\{\mathcal{F}_n\}_{n\geq 0}, \{\mathcal{F}(\varphi)\}_{\varphi\in\mathrm{Arrows}(\Delta)})$ satisfying the commutativity constraints above, then we obtain a sheaf on $\mathcal{C}_{total}$.

09WF **Lemma 3.4.** *In Situation 3.3 there is an equivalence of categories between*

(1) *$Sh(\mathcal{C}_{total})$, and*
(2) *the category of systems $(\mathcal{F}_n, \mathcal{F}(\varphi))$ described above.*

In particular, the topos $Sh(\mathcal{C}_{total})$ only depends on the topoi $Sh(\mathcal{C}_n)$ and the morphisms of topoi f_φ.

Proof. See discussion above. $\square$

09WG **Lemma 3.5.** *In Situation 3.3 the functor $\mathcal{C}_n \to \mathcal{C}_{total}$, $U \mapsto U$ is continuous and cocontinuous. The associated morphism of topoi $g_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{total})$ satisfies*

(1) *g_n^{-1} associates to the sheaf $\mathcal{F}$ on $\mathcal{C}_{total}$ the sheaf $\mathcal{F}_n$ on $\mathcal{C}_n$,*

(2) *$g_n^{-1} : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{C}_n)$ has a left adjoint $g_{n!}^{Sh}$,*
(3) *for $\mathcal{G}$ in $Sh(\mathcal{C}_n)$ the restriction of $g_{n!}^{Sh}\mathcal{G}$ to $\mathcal{C}_m$ is $\coprod_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$,*
(4) *$g_{n!}^{Sh}$ commutes with finite connected limits,*
(5) *$g_n^{-1} : Ab(\mathcal{C}_{total}) \to Ab(\mathcal{C}_n)$ has a left adjoint $g_{n!}$,*
(6) *for $\mathcal{G}$ in $Ab(\mathcal{C}_n)$ the restriction of $g_{n!}\mathcal{G}$ to $\mathcal{C}_m$ is $\bigoplus_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$, and*
(7) *$g_{n!}$ is exact.*

Proof. Case A. If $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{C}_n$ then the image $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{C}_{total}$ by definition (Lemma 3.1). For a morphism $V \to U$ of $\mathcal{C}_n$, the fibre product $V \times_U U_i$ in $\mathcal{C}_n$ is also the fibre product in $\mathcal{C}_{total}$ (by the claim in the proof of Lemma 3.1). Therefore our functor is continuous. On the other hand, our functor defines a bijection between coverings of U in $\mathcal{C}_n$ and coverings of U in $\mathcal{C}_{total}$. Therefore it is certainly the case that our functor is cocontinuous.

Case B. If $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{C}_n$ then the image $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{C}_{total}$ by definition (Lemma 3.2). For a morphism $V \to U$ of $\mathcal{C}_n$, the fibre product $V \times_U U_i$ in $\mathcal{C}_n$ is also the fibre product in $\mathcal{C}_{total}$ (by the claim in the proof of Lemma 3.2). Therefore our functor is continuous. On the other hand, our functor defines a bijection between coverings of U in $\mathcal{C}_n$ and coverings of U in $\mathcal{C}_{total}$. Therefore it is certainly the case that our functor is cocontinuous.

At this point part (1) and the existence of $g_{n!}^{Sh}$ and $g_{n!}$ in cases A and B follows from Sites, Lemmas 21.1 and 21.5 and Modules on Sites, Lemma 16.2.

Proof of (3). Let $\mathcal{G}$ be a sheaf on $\mathcal{C}_n$. Consider the sheaf $\mathcal{H}$ on $\mathcal{C}_{total}$ whose degree m part is the sheaf

$$\mathcal{H}_m = \coprod\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$$

given in part (3) of the statement of the lemma. Given a map $\psi : [m] \to [m']$ the map $\mathcal{H}(\psi) : f_\psi^{-1}\mathcal{H}_m \to \mathcal{H}_{m'}$ is given on components by the identifications

$$f_\psi^{-1} f_\varphi^{-1}\mathcal{G} \to f_{\psi\circ\varphi}^{-1}\mathcal{G}$$

Observe that given a map $\alpha : \mathcal{H} \to \mathcal{F}$ of sheaves on $\mathcal{C}_{total}$ we obtain a map $\mathcal{G} \to \mathcal{F}_n$ corresponding to the restriction of α_n to the component $\mathcal{G}$ in $\mathcal{H}_n$. Conversely, given a map $\beta : \mathcal{G} \to \mathcal{F}_n$ of sheaves on $\mathcal{C}_n$ we can define $\alpha : \mathcal{H} \to \mathcal{F}$ by letting α_m be the map which on components

$$f_\varphi^{-1}\mathcal{G} \to \mathcal{F}_m$$

uses the maps adjoint to $\mathcal{F}(\varphi) \circ f_\varphi^{-1}\beta$. We omit the arguments showing these two constructions give mutually inverse maps

$$\mathrm{Mor}_{Sh(\mathcal{C}_n)}(\mathcal{G}, \mathcal{F}_n) = \mathrm{Mor}_{Sh(\mathcal{C}_{total})}(\mathcal{H}, \mathcal{F})$$

Thus $\mathcal{H} = g_{n!}^{Sh}\mathcal{G}$ as desired.

Proof of (4). If $\mathcal{G}$ is an abelian sheaf on $\mathcal{C}_n$, then we proceed in exactly the same ammner as above, except that we define $\mathcal{H}$ is the abelian sheaf on $\mathcal{C}_{total}$ whose degree m part is the sheaf

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$$

with transition maps defined exactly as above. The bijection

$$\mathrm{Mor}_{Ab(\mathcal{C}_n)}(\mathcal{G}, \mathcal{F}_n) = \mathrm{Mor}_{Ab(\mathcal{C}_{total})}(\mathcal{H}, \mathcal{F})$$

is proved exactly as above. Thus $\mathcal{H} = g_{n!}\mathcal{G}$ as desired.

The exactness properties of $g_{n!}^{Sh}$ and $g_{n!}$ follow from formulas given for these functors. $\square$

09WH **Lemma 3.6.** *In Situation 3.3. If $\mathcal{I}$ is injective in $Ab(\mathcal{C}_{total})$, then $\mathcal{I}_n$ is injective in $Ab(\mathcal{C}_n)$. If $\mathcal{I}^\bullet$ is a K-injective complex in $Ab(\mathcal{C}_{total})$, then $\mathcal{I}_n^\bullet$ is K-injective in $Ab(\mathcal{C}_n)$.*

Proof. The first statement follows from Homology, Lemma 29.1 and Lemma 3.5. The second statement from Derived Categories, Lemma 31.9 and Lemma 3.5. $\square$

4. Augmentations of simplicial sites

0D93 We continue in the fashion described in Section 3 working out the meaning of augmentations in cases A and B treated in that section.

0D6Z **Remark 4.1.** In Situation 3.3 an *augmentation a_0 towards a site $\mathcal{D}$* will mean

(A) $a_0 : \mathcal{C}_0 \to \mathcal{D}$ is a morphism of sites given by a continuous functor $u_0 : \mathcal{D} \to \mathcal{C}_0$ such that for all $\varphi, \psi : [0] \to [n]$ we have $u_\varphi \circ u_0 = u_\psi \circ u_0$.
(B) $a_0 : Sh(\mathcal{C}_0) \to Sh(\mathcal{D})$ is a morphism of topoi given by a cocontinuous functor $u_0 : \mathcal{C}_0 \to \mathcal{D}$ such that for all $\varphi, \psi : [0] \to [n]$ we have $u_0 \circ u_\varphi = u_0 \circ u_\psi$.

0D70 **Lemma 4.2.** *In Situation 3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 4.1. Then a_0 induces*

1. *a morphism of topoi $a_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{D})$ for all $n \geq 0$,*
2. *a morphism of topoi $a : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{D})$*

such that

1. *for all $\varphi : [m] \to [n]$ we have $a_m \circ f_\varphi = a_n$,*
2. *if $g_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{total})$ is as in Lemma 3.5, then $a \circ g_n = a_n$, and*
3. *$a_*\mathcal{F}$ for $\mathcal{F} \in Sh(\mathcal{C}_{total})$ is the equalizer of the two maps $a_{0,*}\mathcal{F}_0 \to a_{1,*}\mathcal{F}_1$.*

Proof. Case A. Let $u_n : \mathcal{D} \to \mathcal{C}_n$ be the common value of the functors $u_\varphi \circ u_0$ for $\varphi : [0] \to [n]$. Then u_n corresponds to a morphism of sites $a_n : \mathcal{C}_n \to \mathcal{D}$, see Sites, Lemma 14.4. The same lemma shows that for all $\varphi : [m] \to [n]$ we have $a_m \circ f_\varphi = a_n$.

Case B. Let $u_n : \mathcal{C}_n \to \mathcal{D}$ be the common value of the functors $u_0 \circ u_\varphi$ for $\varphi : [0] \to [n]$. Then u_n is cocontinuous and hence defines a morphism of topoi $a_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{D})$, see Sites, Lemma 21.2. The same lemma shows that for all $\varphi : [m] \to [n]$ we have $a_m \circ f_\varphi = a_n$.

Consider the functor $a^{-1} : Sh(\mathcal{D}) \to Sh(\mathcal{C}_{total})$ which to a sheaf of sets $\mathcal{G}$ associates the sheaf $\mathcal{F} = a^{-1}\mathcal{G}$ whose components are $a_n^{-1}\mathcal{G}$ and whose transition maps $\mathcal{F}(\varphi)$ are the identifications

$$f_\varphi^{-1}\mathcal{F}_m = f_\varphi^{-1}a_m^{-1}\mathcal{G} = a_n^{-1}\mathcal{G} = \mathcal{F}_n$$

for $\varphi : [m] \to [n]$, see the description of $Sh(\mathcal{C}_{total})$ in Lemma 3.4. Since the functors a_n^{-1} are exact, a^{-1} is an exact functor. Finally, for $a_* : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{D})$ we take the functor which to a sheaf $\mathcal{F}$ on $Sh(\mathcal{D})$ associates

$$a_*\mathcal{F} = \text{Equalizer}(a_{0,*}\mathcal{F}_0 \rightrightarrows a_{1,*}\mathcal{F}_1)$$

Here the two maps come from the two maps $\varphi : [0] \to [1]$ via

$$a_{0,*}\mathcal{F}_0 \to a_{0,*}f_{\varphi,*}f_\varphi^{-1}\mathcal{F}_0 \xrightarrow{\mathcal{F}(\varphi)} a_{0,*}f_{\varphi,*}\mathcal{F}_1 = a_{1,*}\mathcal{F}_1$$

where the first arrow comes from $1 \to f_{\varphi,*}f_\varphi^{-1}$. Let $\mathcal{G}_\bullet$ denote the constant coosimplicial sheaf with value $\mathcal{G}$ and let $a_{\bullet,*}\mathcal{F}$ denote the cosimplicial sheaf having $a_{n,*}\mathcal{F}_n$ in degree n. By the usual adjuntion for the morphisms of topoi a_n we see that a map $a^{-1}\mathcal{G} \to \mathcal{F}$ is the same thing as a map

$$\mathcal{G}_\bullet \longrightarrow a_{\bullet,*}\mathcal{F}$$

of cosimplicial sheaves. By the dual to Simplicial, Lemma 20.2 this is the same thing as a map $\mathcal{G} \to a_*\mathcal{F}$. Thus a^{-1} and a_* are adjoint functors and we obtain our morphism of topoi a[2]. The equalities $a \circ g_n = f_n$ follow immediately from the definitions. □

5. Morphisms of simplicial sites

0D94 We continue in the fashion described in Section 3 working out the meaning of morphisms of simplicial sites in cases A and B treated in that section.

0D95 **Remark 5.1.** Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 3.3. A *morphism h between simplicial sites* will mean

(A) Morphisms of sites $h_n : \mathcal{C}_n \to \mathcal{C}'_n$ such that $f'_\varphi \circ h_n = h_m \circ f_\varphi$ as morphisms of sites for all $\varphi : [m] \to [n]$.

(B) Cocontinuous functors $v_n : \mathcal{C}_n \to \mathcal{C}'_n$ inducing morphisms of topoi $h_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}'_n)$ such that $u'_\varphi \circ v_n = v_m \circ u_\varphi$ as functors for all $\varphi : [m] \to [n]$.

In both cases we have $f'_\varphi \circ h_n = h_m \circ f_\varphi$ as morphisms of topoi, see Sites, Lemma 21.2 for case B and Sites, Definition 14.5 for case A.

0D96 **Lemma 5.2.** *Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 3.3. Let h be a morphism between simplicial sites as in Remark 5.1. Then we obtain a morphism of topoi*

$$h_{total} : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{C}'_{total})$$

and commutative diagrams

$$\begin{tikzcd} Sh(\mathcal{C}_n) \arrow[r, "h_n"'] \arrow[d, "g_n"'] & Sh(\mathcal{C}'_n) \arrow[d, "g'_n"] \\ Sh(\mathcal{C}_{total}) \arrow[r, "h_{total}"] & Sh(\mathcal{C}'_{total}) \end{tikzcd}$$

Moreover, we have $(g'_n)^{-1} \circ h_{total,*} = h_{n,*} \circ g_n^{-1}$.

Proof. Case A. Say h_n corresponds to the continuous functor $v_n : \mathcal{C}'_n \to \mathcal{C}_n$. Then we can define a functor $v_{total} : \mathcal{C}'_{total} \to \mathcal{C}_{total}$ by using v_n in degree n. This is clearly a continuous functor (see definition of coverings in Lemma 3.1). Let $h_{total}^{-1} = v_{total,s} : Sh(\mathcal{C}'_{total}) \to Sh(\mathcal{C}_{total})$ and $h_{total,*} = v^s_{total} = v^p_{total} : Sh(\mathcal{C}_{total}) \to Sh(\mathcal{C}'_{total})$ be the adjoint pair of functors constructed and studied in Sites, Sections 13 and 14. To see that h_{total} is a morphism of topoi we still have to verify that h_{total}^{-1} is exact. We first observe that $(g'_n)^{-1} \circ h_{total,*} = h_{n,*} \circ g_n^{-1}$; this is immediate by computing sections over an object U of $\mathcal{C}'_n$. Thus, if we think of a sheaf $\mathcal{F}$ on $\mathcal{C}_{total}$ as a system $(\mathcal{F}_n, \mathcal{F}(\varphi))$ as in Lemma 3.4, then $h_{total,*}\mathcal{F}$ corresponds to the system $(h_{n,*}\mathcal{F}_n, h_{n,*}\mathcal{F}(\varphi))$. Clearly, the functor $(\mathcal{F}'_n, \mathcal{F}'(\varphi)) \to (h_n^{-1}\mathcal{F}'_n, h_n^{-1}\mathcal{F}'(\varphi))$

[2]In case B the morphism a corresponds to the cocontinuous functor $\mathcal{C}_{total} \to \mathcal{D}$ sending U in $\mathcal{C}_n$ to $u_n(U)$.

is its left adjoint. By uniqueness of adjoints, we conclude that h_{total}^{-1} is given by this rule on systems. In particular, h_{total}^{-1} is exact (by the description of sheaves on $\mathcal{C}_{total}$ given in the lemma and the exactness of the functors h_n^{-1}) and we have our morphism of topoi. Finally, we obtain $g_n^{-1} \circ h_{total}^{-1} = h_n^{-1} \circ (g'_n)^{-1}$ as well, which proves that the displayed diagram of the lemma commutes.

Case B. Here we have a functor $v_{total} : \mathcal{C}_{total} \to \mathcal{C}'_{total}$ by using v_n in degree n. This is clearly a cocontinuous functor (see definition of coverings in Lemma 3.2). Let h_{total} be the morphism of topoi associated to v_{total}. The commutativity of the displayed diagram of the lemma follows immediately from Sites, Lemma 21.2. Taking left adjoints the final equality of the lemma becomes

$$h_{total}^{-1} \circ (g'_n)^{Sh}_! = g^{Sh}_{n!} \circ h_n^{-1}$$

This follows immediately from the explicit description of the functors $(g'_n)^{Sh}_!$ and $g^{Sh}_{n!}$ in Lemma 3.5, the fact that $h_n^{-1} \circ (f'_\varphi)^{-1} = f_\varphi^{-1} \circ h_m^{-1}$ for $\varphi : [m] \to [n]$, and the fact that we already know h_{total}^{-1} commutes with restrictions to the degree n parts of the simplicial sites. □

0D97 **Lemma 5.3.** *With notation and hypotheses as in Lemma 5.2. For $K \in D(\mathcal{C}_{total})$ we have $(g'_n)^{-1} Rh_{total,*}K = Rh_{n,*}g_n^{-1}K$.*

Proof. Let $\mathcal{I}^\bullet$ be a K-injective complex on $\mathcal{C}_{total}$ representing K. Then $g_n^{-1}K$ is represented by $g_n^{-1}\mathcal{I}^\bullet = \mathcal{I}_n^\bullet$ which is K-injective by Lemma 3.6. We have $(g'_n)^{-1}h_{total,*}\mathcal{I}^\bullet = h_{n,*}g_n^{-1}\mathcal{I}_n^\bullet$ by Lemma 5.2 which gives the desired equality. □

0D98 **Remark 5.4.** Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 3.3. Let a_0, resp. a'_0 be an augmentation towards a site $\mathcal{D}$, resp. $\mathcal{D}'$ as in Remark 4.1. Let h be a morphism between simplicial sites as in Remark 5.1. We say a morphism of topoi $h_{-1} : Sh(\mathcal{D}) \to Sh(\mathcal{D}')$ is *compatible with h, a_0, a'_0* if

(A) h_{-1} comes from a morphism of sites $h_{-1} : \mathcal{D} \to \mathcal{D}'$ such that $a'_0 \circ h_0 = h_{-1} \circ a_0$ as morphisms of sites.
(B) h_{-1} comes from a cocontinuous functor $v_{-1} : \mathcal{D} \to \mathcal{D}'$ such that $u'_0 \circ v_0 = v_{-1} \circ u_0$ as functors.

In both cases we have $a'_0 \circ h_0 = h_{-1} \circ a_0$ as morphisms of topoi, see Sites, Lemma 21.2 for case B and Sites, Definition 14.5 for case A.

0D99 **Lemma 5.5.** *Let $\mathcal{C}_n, f_\varphi, u_\varphi, \mathcal{D}, a_0$, $\mathcal{C}'_n, f'_\varphi, u'_\varphi, \mathcal{D}', a'_0$, and h_n, $n \geq -1$ be as in Remark 5.4. Then we obtain a commutative diagram*

$$\begin{tikzcd} Sh(\mathcal{C}_{total}) \arrow[r, "h_{total}"'] \arrow[d, "a"'] & Sh(\mathcal{C}'_{total}) \arrow[d, "a'"] \\ Sh(\mathcal{D}) \arrow[r, "h_{-1}"] & Sh(\mathcal{D}') \end{tikzcd}$$

Proof. The morphism h is defined in Lemma 5.2. The morphisms a and a' are defined in Lemma 4.2. Thus the only thing is to prove the commutativity of the diagram. To do this, we prove that $a^{-1} \circ h_{-1}^{-1} = h_{total}^{-1} \circ (a')^{-1}$. By the commutative diagrams of Lemma 5.2 and the description of $Sh(\mathcal{C}_{total})$ and $Sh(\mathcal{C}'_{total})$ in terms of components in Lemma 3.4, it suffices to show that

$$\begin{tikzcd} Sh(\mathcal{C}_n) \arrow[r, "h_n"'] \arrow[d, "a_n"'] & Sh(\mathcal{C}'_n) \arrow[d, "a'_n"] \\ Sh(\mathcal{D}) \arrow[r, "h_{-1}"] & Sh(\mathcal{D}') \end{tikzcd}$$

commutes for all n. This follows from the case for $n = 0$ (which is an assumption in Remark 5.4) and for $n > 0$ we pick $\varphi : [0] \to [n]$ and then the required commutativity follows from the case $n = 0$ and the relations $a_n = a_0 \circ f_\varphi$ and $a'_n = a'_0 \circ f'_\varphi$ as well as the commutation relations $f'_\varphi \circ h_n = h_0 \circ f_\varphi$. □

6. Ringed simplicial sites

0D71 Let us endow our simplicial topos with a sheaf of rings.

0D72 **Lemma 6.1.** *In Situation 3.3. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. There is a canonical morphism of ringed topoi $g_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{total}), \mathcal{O})$ agreeing with the morphism g_n of Lemma 3.5 on underlying topoi. The functor $g_n^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_n)$ has a left adjoint $g_{n!}$. For $\mathcal{G}$ in $Mod(\mathcal{O}_n)$-modules the restriction of $g_{n!}\mathcal{G}$ to $\mathcal{C}_m$ is*

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^*\mathcal{G}$$

where $f_\varphi : (Sh(\mathcal{C}_m), \mathcal{O}_m) \to (Sh(\mathcal{C}_n), \mathcal{O}_n)$ is the morphism of ringed topoi agreeing with the previously defined f_φ on topoi and using the map $\mathcal{O}(\varphi) : f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ on sheaves of rings.

Proof. By Lemma 3.5 we have $g_n^{-1}\mathcal{O} = \mathcal{O}_n$ and hence we obtain our morphism of ringed topoi. By Modules on Sites, Lemma 41.1 we obtain the adjoint $g_{n!}$. To prove the formula for $g_{n!}$ we first define a sheaf of $\mathcal{O}$-modules $\mathcal{H}$ on $\mathcal{C}_{total}$ with degree m component the $\mathcal{O}_m$-module

$$\mathcal{H}_m = \bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^*\mathcal{G}$$

Given a map $\psi : [m] \to [m']$ the map $\mathcal{H}(\psi) : f_\psi^{-1}\mathcal{H}_m \to \mathcal{H}_{m'}$ is given on components by

$$f_\psi^{-1} f_\varphi^*\mathcal{G} \to f_\psi^* f_\varphi^*\mathcal{G} \to f_{\psi\circ\varphi}^*\mathcal{G}$$

Since this map $f_\psi^{-1}\mathcal{H}_m \to \mathcal{H}_{m'}$ is $\mathcal{O}(\psi) : f_\psi^{-1}\mathcal{O}_m \to \mathcal{O}_{m'}$-semi-linear, this indeed does define an $\mathcal{O}$-module (use Lemma 3.4). Then one proves directly that

$$\mathrm{Mor}_{\mathcal{O}_n}(\mathcal{G}, \mathcal{F}_n) = \mathrm{Mor}_{\mathcal{O}}(\mathcal{H}, \mathcal{F})$$

proceeding as in the proof of Lemma 3.5. Thus $\mathcal{H} = g_{n!}\mathcal{G}$ as desired. □

0D73 **Lemma 6.2.** *In Situation 3.3. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. If $\mathcal{I}$ is injective in $Mod(\mathcal{O})$, then $\mathcal{I}_n$ is a totally acyclic sheaf on $\mathcal{C}_n$.*

Proof. This follows from Cohomology on Sites, Lemma 37.4 applied to the inclusion functor $\mathcal{C}_n \to \mathcal{C}_{total}$ and its properties proven in Lemma 3.5. □

0D74 **Lemma 6.3.** *With assumptions as in Lemma 6.1 the functor $g_{n!} : Mod(\mathcal{O}_n) \to Mod(\mathcal{O})$ is exact if the maps $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ are flat for all $\varphi : [n] \to [m]$.*

Proof. Recall that $g_{n!}\mathcal{G}$ is the $\mathcal{O}$-module whose degree m part is the $\mathcal{O}_m$-module

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^*\mathcal{G}$$

Here the morphism of ringed topoi $f_\varphi : (Sh(\mathcal{C}_m), \mathcal{O}_m) \to (Sh(\mathcal{C}_n), \mathcal{O}_n)$ uses the map $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ of the statement of the lemma. If these maps are flat, then f_φ^* is exact (Modules on Sites, Lemma 31.2). By definition of the site $\mathcal{C}_{total}$ we see that these functors have the desired exactness properties and we conclude. ☐

0D75 **Lemma 6.4.** *In Situation 3.3. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ such that $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ is flat for all $\varphi : [n] \to [m]$. If $\mathcal{I}$ is injective in $Mod(\mathcal{O})$, then $\mathcal{I}_n$ is injective in $Mod(\mathcal{O}_n)$.*

Proof. This follows from Homology, Lemma 29.1 and Lemma 6.3. ☐

7. Morphisms of ringed simplicial sites

0DGY We continue the discussion of Section 5.

0DGZ **Remark 7.1.** Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 3.3. Let $\mathcal{O}$ and $\mathcal{O}'$ be a sheaf of rings on $\mathcal{C}_{total}$ and $\mathcal{C}'_{total}$. We will say that $(h, h^\sharp)$ is a *morphism between ringed simplicial sites* if h is a morphism between simplicial sites as in Remark 5.1 and $h^\sharp : h_{total}^{-1}\mathcal{O}' \to \mathcal{O}$ or equivalently $h^\sharp : \mathcal{O}' \to h_{total,*}\mathcal{O}$ is a homomorphism of sheaves of rings.

0DH0 **Lemma 7.2.** *Let $\mathcal{C}_n, f_\varphi, u_\varphi$ and $\mathcal{C}'_n, f'_\varphi, u'_\varphi$ be as in Situation 3.3. Let $\mathcal{O}$ and $\mathcal{O}'$ be a sheaf of rings on $\mathcal{C}_{total}$ and $\mathcal{C}'_{total}$. Let $(h, h^\sharp)$ be a morphism between simplicial sites as in Remark 7.1. Then we obtain a morphism of ringed topoi*

$$h_{total} : (Sh(\mathcal{C}_{total}), \mathcal{O}) \to (Sh(\mathcal{C}'_{total}), \mathcal{O}')$$

and commutative diagrams

$$\begin{tikzcd}
(Sh(\mathcal{C}_n), \mathcal{O}_n) \arrow[r, "h_n"'] \arrow[d, "g_n"'] & (Sh(\mathcal{C}'_n), \mathcal{O}'_n) \arrow[d, "g'_n"] \\
(Sh(\mathcal{C}_{total}), \mathcal{O}) \arrow[r, "h_{total}"] & (Sh(\mathcal{C}'_{total}), \mathcal{O}')
\end{tikzcd}$$

of ringed topoi where g_n and g'_n are as in Lemma 6.1. Moreover, we have $(g'_n)^* \circ h_{total,*} = h_{n,*} \circ g_n^*$ as functor $Mod(\mathcal{O}) \to Mod(\mathcal{O}'_n)$.

Proof. Follows from Lemma 5.2 and 6.1 by keeping track of the sheaves of rings. A small point is that in order to define h_n as a morphism of ringed topoi we set $h_n^\sharp = g_n^{-1}h^\sharp : g_n^{-1}h_{total}^{-1}\mathcal{O}' \to g_n^{-1}\mathcal{O}$ which makes sense because $g_n^{-1}h_{total}^{-1}\mathcal{O}' = h_n^{-1}(g'_n)^{-1}\mathcal{O}' = h_n^{-1}\mathcal{O}'_n$ and $g_n^{-1}\mathcal{O} = \mathcal{O}_n$. Note that $g_n^*\mathcal{F} = g_n^{-1}\mathcal{F}$ for a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ and similarly for g'_n and this helps explain why $(g'_n)^* \circ h_{total,*} = h_{n,*} \circ g_n^*$ follows from the corresponding statement of Lemma 5.2. ☐

0DH1 **Lemma 7.3.** *With notation and hypotheses as in Lemma 7.2. For $K \in D(\mathcal{O})$ we have $(g'_n)^*Rh_{total,*}K = Rh_{n,*}g_n^*K$.*

Proof. Recall that $g_n^* = g_n^{-1}$ because $g_n^{-1}\mathcal{O} = \mathcal{O}_n$ by the construction in Lemma 6.1. In particular g_n^* is exact and Lg_n^* is given by applying g_n^* to any representative complex of modules. Similarly for g'_n. There is a canonical base change map

$(g'_n)^*Rh_{total,*}K \to Rh_{n,*}g_n^*K$, see Cohomology on Sites, Remark 19.3. By Cohomology on Sites, Lemma 20.7 the image of this in $D(\mathcal{C}'_n)$ is the map $(g'_n)^{-1}Rh_{total,*}K_{ab} \to Rh_{n,*}g_n^{-1}K_{ab}$ where K_{ab} is the image of K in $D(\mathcal{C}_{total})$. This we proved to be an isomorphism in Lemma 5.3 and the result follows. □

8. Cohomology on simplicial sites

0D76 Let $\mathcal{C}$ be as in Situation 3.3. In statement of the following lemmas we will let $g_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{total})$ be the morphism of topoi of Lemma 3.5. If $\varphi : [m] \to [n]$ is a morphism of Δ, then the diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{C}_n) \arrow[rr, "f_\varphi"'] \arrow[dr, "g_n"'] & & Sh(\mathcal{C}_m) \arrow[dl, "g_m"] \\ & Sh(\mathcal{C}_{total}) & \end{tikzcd}$$

is not commutative, but there is a 2-morphism $g_n \to g_m \circ f_\varphi$ coming from the maps $\mathcal{F}(\varphi) : f_\varphi^{-1}\mathcal{F}_m \to \mathcal{F}_n$. See Sites, Section 36.

09WI **Lemma 8.1.** *In Situation 3.3 and with notation as above there is a complex*

$$\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}$$

of abelian sheaves on $\mathcal{C}_{total}$ which forms a resolution of the constant sheaf with value $\mathbf{Z}$ on $\mathcal{C}_{total}$.

Proof. We will use the description of the functors $g_{n!}$ in Lemma 3.5 without further mention. As maps of the complex we take $\sum(-1)^i d_i^n$ where $d_i^n : g_{n!}\mathbf{Z} \to g_{n-1!}\mathbf{Z}$ is the adjoint to the map $\mathbf{Z} \to \bigoplus_{[n-1]\to[n]} \mathbf{Z} = g_n^{-1}g_{n-1!}\mathbf{Z}$ corresponding to the factor labeled with $\delta_i^n : [n-1] \to [n]$. Then g_m^{-1} applied to the complex gives the complex

$$\ldots \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([2],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([1],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([0],[m])} \mathbf{Z}$$

on $\mathcal{C}_m$. In other words, this is the complex associated to the free abelian sheaf on the simplicial set $\Delta[m]$, see Simplicial, Example 11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 26.7, and since "taking free abelian sheaf on" is a functor, we see that the complex above is homotopy equivalent to the free abelian sheaf on $\Delta[0]$ (Simplicial, Remark 26.4 and Lemma 27.2). This complex is acyclic in positive degrees and equal to $\mathbf{Z}$ in degree 0. □

0D77 **Lemma 8.2.** *In Situation 3.3. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{total}$ there is a canonical complex*

$$0 \to \Gamma(\mathcal{C}_{total}, \mathcal{F}) \to \Gamma(\mathcal{C}_0, \mathcal{F}_0) \to \Gamma(\mathcal{C}_1, \mathcal{F}_1) \to \Gamma(\mathcal{C}_2, \mathcal{F}_2) \to \ldots$$

which is exact in degrees $-1, 0$ and exact everywhere if $\mathcal{F}$ is injective.

Proof. Observe that $\mathrm{Hom}(\mathbf{Z}, \mathcal{F}) = \Gamma(\mathcal{C}_{total}, \mathcal{F})$ and $\mathrm{Hom}(g_{n!}\mathbf{Z}, \mathcal{F}) = \Gamma(\mathcal{C}_n, \mathcal{F}_n)$. Hence this lemma is an immediate consequence of Lemma 8.1 and the fact that $\mathrm{Hom}(-, \mathcal{F})$ is exact if $\mathcal{F}$ is injective. □

09WJ **Lemma 8.3.** *In Situation 3.3. For K in $D^+(\mathcal{C}_{total})$ there is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_1^{p,q} = H^q(\mathcal{C}_p, K_p), \quad d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

converging to $H^{p+q}(\mathcal{C}_{total}, K)$. This spectral sequence is functorial in K.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives representing K. Consider the double complex with terms

$$A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q)$$

where the horizontal arrows come from Lemma 8.2 and the vertical arrows from the differentials of the complex $\mathcal{I}^\bullet$. The rows of the double complex are exact in positive degrees and evaluate to $\Gamma(\mathcal{C}_{total}, \mathcal{I}^q)$ in degree 0. On the other hand, since restriction to $\mathcal{C}_p$ is exact (Lemma 3.5) the complex $\mathcal{I}_p^\bullet$ represents K_p in $D(\mathcal{C}_p)$. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on $\mathcal{C}_p$ (Lemma 3.6). Hence the cohomology of the columns computes the groups $H^q(\mathcal{C}_p, K_p)$. We conclude by applying Homology, Lemmas 25.3 and 25.4. □

0H0V **Remark 8.4.** Assumptions and notation as in Lemma 8.3 except we do not require K in $D(\mathcal{C}_{total})$ to be bounded below. We claim there is a natural spectral sequence in this case also. Namely, suppose that $\mathcal{I}^\bullet$ is a K-injective complex of sheaves on $\mathcal{C}_{total}$ with injective terms representing K. We have

$$\begin{aligned}
R\Gamma(\mathcal{C}_{total}, K) &= R\operatorname{Hom}(\mathbf{Z}, K) \\
&= R\operatorname{Hom}(\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}, K) \\
&= \Gamma(\mathcal{C}_{total}, \mathcal{H}om^\bullet(\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}, \mathcal{I}^\bullet)) \\
&= \operatorname{Tot}_\pi(A^{\bullet,\bullet})
\end{aligned}$$

where $A^{\bullet,\bullet}$ is the double complex with terms $A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q)$ and Tot_π denotes the product totalization of this double complex. Namely, the first equality holds in any site. The second equality holds by Lemma 8.1. The third equality holds because $\mathcal{I}^\bullet$ is K-injective, see Cohomology on Sites, Sections 34 and 35. The final equality holds by the construction of $\mathcal{H}om^\bullet$ and the fact that $\operatorname{Hom}(g_{p!}\mathbf{Z}, \mathcal{I}^q) = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q)$. Then we get our spectral sequence by viewing $\operatorname{Tot}_\pi(A^{\bullet,\bullet})$ as a filtered complex with $F^i\operatorname{Tot}_\pi^n(A^{\bullet,\bullet}) = \prod_{p+q=n,\ p\geq i} A^{p,q}$. The spectral sequence we obtain behaves like the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ in Homology, Section 25 (where the case of the direct sum totalization is discussed) except for regularity, boundedness, convergence, and abutment issues. In particular we obtain $E_1^{p,q} = H^q(\mathcal{C}_p, K_p)$ as in Lemma 8.3.

0H0W **Lemma 8.5.** *In Situation 3.3. Let K be an object of $D(\mathcal{C}_{total})$.*

(1) *If $H^{-p}(\mathcal{C}_p, K_p) = 0$ for all $p \geq 0$, then $H^0(\mathcal{C}_{total}, K) = 0$.*
(2) *If $R\Gamma(\mathcal{C}_p, K_p) = 0$ for all $p \geq 0$, then $R\Gamma(\mathcal{C}_{total}, K) = 0$.*

Proof. With notation as in Remark 8.4 we see that $R\Gamma(\mathcal{C}_{total}, K)$ is represented by $\operatorname{Tot}_\pi(A^{\bullet,\bullet})$. The assumption in (1) tells us that $H^{-p}(A^{p,\bullet}) = 0$. Thus the vanishing in (1) follows from More on Algebra, Lemma 103.1. Part (2) follows from part (1) and taking shifts. □

0DBZ **Lemma 8.6.** *Let $\mathcal{C}$ be as in Situation 3.3. Let $U \in \operatorname{Ob}(\mathcal{C}_n)$. Let $\mathcal{F} \in Ab(\mathcal{C}_{total})$. Then $H^p(U, \mathcal{F}) = H^p(U, g_n^{-1}\mathcal{F})$ where on the left hand side U is viewed as an object of $\mathcal{C}_{total}$.*

Proof. Observe that "U viewed as object of $\mathcal{C}_{total}$" is explained by the construction of $\mathcal{C}_{total}$ in Lemma 3.1 in case (A) and Lemma 3.2 in case (B). The equality then follows from Lemma 3.6 and the definition of cohomology. □

9. Cohomology and augmentations of simplicial sites

0D9A Consider a simplicial site $\mathcal{C}$ as in Situation 3.3. Let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 4.1. By Lemma 4.2 we obtain a morphism of topoi

$$a : Sh(\mathcal{C}_{total}) \longrightarrow Sh(\mathcal{D})$$

and morphisms of topoi $g_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{total})$ as in Lemma 3.5. The compositions $a \circ g_n$ are denoted $a_n : Sh(\mathcal{C}_n) \to Sh(\mathcal{D})$. Furthermore, the simplicial structure gives morphisms of topoi $f_\varphi : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_m)$ such that $a_n \circ f_\varphi = a_m$ for all $\varphi : [m] \to [n]$.

0D78 **Lemma 9.1.** *In Situation 3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 4.1. For any abelian sheaf $\mathcal{G}$ on $\mathcal{D}$ there is an exact complex*

$$\ldots \to g_{2!}(a_2^{-1}\mathcal{G}) \to g_{1!}(a_1^{-1}\mathcal{G}) \to g_{0!}(a_0^{-1}\mathcal{G}) \to a^{-1}\mathcal{G} \to 0$$

of abelian sheaves on $\mathcal{C}_{total}$.

Proof. We encourage the reader to read the proof of Lemma 8.1 first. We will use Lemma 4.2 and the description of the functors $g_{n!}$ in Lemma 3.5 without further mention. In particular $g_{n!}(a_n^{-1}\mathcal{G})$ is the sheaf on $\mathcal{C}_{total}$ whose restriction to $\mathcal{C}_m$ is the sheaf

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}a_n^{-1}\mathcal{G} = \bigoplus\nolimits_{\varphi:[n]\to[m]} a_m^{-1}\mathcal{G}$$

As maps of the complex we take $\sum(-1)^i d_i^n$ where $d_i^n : g_{n!}(a_n^{-1}\mathcal{G}) \to g_{n-1!}(a_{n-1}^{-1}\mathcal{G})$ is the adjoint to the map $a_n^{-1}\mathcal{G} \to \bigoplus_{[n-1]\to[n]} a_n^{-1}\mathcal{G} = g_n^{-1}g_{n-1!}(a_{n-1}^{-1}\mathcal{G})$ corresponding to the factor labeled with $\delta_i^n : [n-1] \to [n]$. The map $g_{0!}(a_0^{-1}\mathcal{G}) \to a^{-1}\mathcal{G}$ is adjoint to the identity map of $a_0^{-1}\mathcal{G}$. Then g_m^{-1} applied to the chain complex in degrees $\ldots, 2, 1, 0$ gives the complex

$$\ldots \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([2],[m])} a_m^{-1}\mathcal{G} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([1],[m])} a_m^{-1}\mathcal{G} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([0],[m])} a_m^{-1}\mathcal{G}$$

on $\mathcal{C}_m$. This is equal to $a_m^{-1}\mathcal{G}$ tensored over the constant sheaf $\mathbf{Z}$ with the complex

$$\ldots \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([2],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([1],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([0],[m])} \mathbf{Z}$$

discussed in the proof of Lemma 8.1. There we have seen that this complex is homotopy equivalent to $\mathbf{Z}$ placed in degree 0 which finishes the proof. □

0D79 **Lemma 9.2.** *In Situation 3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 4.1. For an abelian sheaf $\mathcal{F}$ on $\mathcal{C}_{total}$ there is a canonical complex*

$$0 \to a_*\mathcal{F} \to a_{0,*}\mathcal{F}_0 \to a_{1,*}\mathcal{F}_1 \to a_{2,*}\mathcal{F}_2 \to \ldots$$

on $\mathcal{D}$ which is exact in degrees $-1, 0$ and exact everywhere if $\mathcal{F}$ is injective.

Proof. To construct the complex, by the Yoneda lemma, it suffices for any abelian sheaf $\mathcal{G}$ on $\mathcal{D}$ to construct a complex

$$0 \to \mathrm{Hom}(\mathcal{G}, a_*\mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, a_{0,*}\mathcal{F}_0) \to \mathrm{Hom}(\mathcal{G}, a_{1,*}\mathcal{F}_1) \to \ldots$$

functorially in $\mathcal{G}$. To do this apply $\mathrm{Hom}(-, \mathcal{F})$ to the exact complex of Lemma 9.1 and use adjointness of pullback and pushforward. The exactness properties in degrees $-1, 0$ follow from the construction as $\mathrm{Hom}(-, \mathcal{F})$ is left exact. If $\mathcal{F}$ is an injective abelian sheaf, then the complex is exact because $\mathrm{Hom}(-, \mathcal{F})$ is exact. □

0D7A **Lemma 9.3.** *In Situation 3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 4.1. For any K in $D^+(\mathcal{C}_{total})$ there is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_1^{p,q} = R^q a_{p,*}K_p, \quad d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

converging to $R^{p+q}a_*K$. This spectral sequence is functorial in K.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives representing K. Consider the double complex with terms

$$A^{p,q} = a_{p,*}\mathcal{I}_p^q$$

where the horizontal arrows come from Lemma 9.2 and the vertical arrows from the differentials of the complex $\mathcal{I}^\bullet$. The rows of the double complex are exact in positive degrees and evaluate to $a_*\mathcal{I}^q$ in degree 0. On the other hand, since restriction to $\mathcal{C}_p$ is exact (Lemma 3.5) the complex $\mathcal{I}_p^\bullet$ represents K_p in $D(\mathcal{C}_p)$. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on $\mathcal{C}_p$ (Lemma 3.6). Hence the cohomology of the columns computes $R^q a_{p,*}K_p$. We conclude by applying Homology, Lemmas 25.3 and 25.4. $\square$

10. Cohomology on ringed simplicial sites

0D7B This section is the analogue of Section 8 for sheaves of modules.

In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. In statement of the following lemmas we will let $g_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{total}), \mathcal{O})$ be the morphism of ringed topoi of Lemma 6.1. If $\varphi : [m] \to [n]$ is a morphism of Δ, then the diagram of ringed topoi

$$\begin{tikzcd} (Sh(\mathcal{C}_n), \mathcal{O}_n) \arrow[rr, "f_\varphi"'] \arrow[dr, "g_n"'] & & (Sh(\mathcal{C}_m), \mathcal{O}_m) \arrow[dl, "g_m"] \\ & (Sh(\mathcal{C}_{total}), \mathcal{O}) & \end{tikzcd}$$

is not commutative, but there is a 2-morphism $g_n \to g_m \circ f_\varphi$ coming from the maps $\mathcal{F}(\varphi) : f_\varphi^{-1}\mathcal{F}_m \to \mathcal{F}_n$. See Sites, Section 36.

0D9B **Lemma 10.1.** *In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. There is a complex*

$$\ldots \to g_{2!}\mathcal{O}_2 \to g_{1!}\mathcal{O}_1 \to g_{0!}\mathcal{O}_0$$

of $\mathcal{O}$-modules which forms a resolution of $\mathcal{O}$. Here $g_{n!}$ is as in Lemma 6.1.

Proof. We will use the description of $g_{n!}$ given in Lemma 3.5. As maps of the complex we take $\sum (-1)^i d_i^n$ where $d_i^n : g_{n!}\mathcal{O}_n \to g_{n-1!}\mathcal{O}_{n-1}$ is the adjoint to the map $\mathcal{O}_n \to \bigoplus_{[n-1]\to[n]} \mathcal{O}_n = g_n^* g_{n-1!}\mathcal{O}_{n-1}$ corresponding to the factor labeled with $\delta_i^n : [n-1] \to [n]$. Then g_m^{-1} applied to the complex gives the complex

$$\ldots \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([2],[m])]} \mathcal{O}_m \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([1],[m])]} \mathcal{O}_m \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([0],[m])]} \mathcal{O}_m$$

on $\mathcal{C}_m$. In other words, this is the complex associated to the free $\mathcal{O}_m$-module on the simplicial set $\Delta[m]$, see Simplicial, Example 11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 26.7, and since "taking free abelian sheaf on" is a functor, we see that the complex above is homotopy equivalent to the free abelian sheaf on $\Delta[0]$ (Simplicial, Remark 26.4 and Lemma 27.2). This complex is acyclic in positive degrees and equal to $\mathcal{O}_m$ in degree 0. $\square$

0D9C **Lemma 10.2.** *In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. There is a canonical complex*

$$0 \to \Gamma(\mathcal{C}_{total}, \mathcal{F}) \to \Gamma(\mathcal{C}_0, \mathcal{F}_0) \to \Gamma(\mathcal{C}_1, \mathcal{F}_1) \to \Gamma(\mathcal{C}_2, \mathcal{F}_2) \to \ldots$$

which is exact in degrees $-1, 0$ and exact everywhere if $\mathcal{F}$ is an injective $\mathcal{O}$-module.

Proof. Observe that $\mathrm{Hom}(\mathcal{O}, \mathcal{F}) = \Gamma(\mathcal{C}_{total}, \mathcal{F})$ and $\mathrm{Hom}(g_{n!}\mathcal{O}_n, \mathcal{F}) = \Gamma(\mathcal{C}_n, \mathcal{F}_n)$. Hence this lemma is an immediate consequence of Lemma 10.1 and the fact that $\mathrm{Hom}(-, \mathcal{F})$ is exact if $\mathcal{F}$ is injective. □

0D7E **Lemma 10.3.** *In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings. For K in $D^+(\mathcal{O})$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_1^{p,q} = H^q(\mathcal{C}_p, K_p), \quad d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

converging to $H^{p+q}(\mathcal{C}_{total}, K)$. This spectral sequence is functorial in K.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injective $\mathcal{O}$-modules representing K. Consider the double complex with terms

$$A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q)$$

where the horizontal arrows come from Lemma 10.2 and the vertical arrows from the differentials of the complex $\mathcal{I}^\bullet$. Observe that $\Gamma(\mathcal{D}, -) = \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{O}_\mathcal{D}, -)$ on $Mod(\mathcal{O}_\mathcal{D})$. Hence the lemma says rows of the double complex are exact in positive degrees and evaluate to $\Gamma(\mathcal{C}_{total}, \mathcal{I}^q)$ in degree 0. Thus the total complex associated to the double complex computes $R\Gamma(\mathcal{C}_{total}, K)$ by Homology, Lemma 25.4. On the other hand, since restriction to $\mathcal{C}_p$ is exact (Lemma 3.5) the complex $\mathcal{I}_p^\bullet$ represents K_p in $D(\mathcal{C}_p)$. The sheaves $\mathcal{I}_p^q$ are totally acyclic on $\mathcal{C}_p$ (Lemma 6.2). Hence the cohomology of the columns computes the groups $H^q(\mathcal{C}_p, K_p)$ by Leray's acyclicity lemma (Derived Categories, Lemma 16.7) and Cohomology on Sites, Lemma 14.3. We conclude by applying Homology, Lemma 25.3. □

0DH2 **Lemma 10.4.** *In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings. Let $U \in \mathrm{Ob}(\mathcal{C}_n)$. Let $\mathcal{F} \in Mod(\mathcal{O})$. Then $H^p(U, \mathcal{F}) = H^p(U, g_n^*\mathcal{F})$ where on the left hand side U is viewed as an object of $\mathcal{C}_{total}$.*

Proof. Observe that "U viewed as object of $\mathcal{C}_{total}$" is explained by the construction of $\mathcal{C}_{total}$ in Lemma 3.1 in case (A) and Lemma 3.2 in case (B). In both cases the functor $\mathcal{C}_n \to \mathcal{C}$ is continuous and cocontinuous, see Lemma 3.5, and $g_n^{-1}\mathcal{O} = \mathcal{O}_n$ by definition. Hence the result is a special case of Cohomology on Sites, Lemma 37.5. □

11. Cohomology and augmentations of ringed simplicial sites

0D9D This section is the analogue of Section 9 for sheaves of modules.

Consider a simplicial site $\mathcal{C}$ as in Situation 3.3. Let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 4.1. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings on $\mathcal{D}$. Suppose we are given a morphism

$$a^\sharp : \mathcal{O}_\mathcal{D} \longrightarrow a_*\mathcal{O}$$

where a is as in Lemma 4.2. Consequently, we obtain a morphism of ringed topoi

$$a : (Sh(\mathcal{C}_{total}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$$

We will think of $g_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{total}), \mathcal{O})$ as a morphism of ringed topoi as in Lemma 6.1, then taking the composition $a_n = a \circ g_n$ (Lemma 4.2) as morphisms of ringed topoi we obtain

$$a_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$$

Using the transition maps $f_\varphi^{-1}\mathcal{O}_m \to \mathcal{O}_n$ we obtain morphisms of ringed topoi

$$f_\varphi : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_m), \mathcal{O}_m)$$

such that $a_m \circ f_\varphi = a_n$ as morphisms of ringed topoi for all $\varphi : [m] \to [n]$.

0DH3 **Lemma 11.1.** *With notation as above. The morphism $a : (Sh(\mathcal{C}_{total}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ is flat if and only if $a_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ is flat for $n \geq 0$.*

Proof. Since $g_n : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{total}), \mathcal{O})$ is flat, we see that if a is flat, then $a_n = a \circ g_n$ is flat as a composition. Conversely, suppose that a_n is flat for all n. We have to check that $\mathcal{O}$ is flat as a sheaf of $a^{-1}\mathcal{O}_\mathcal{D}$-modules. Let $\mathcal{F} \to \mathcal{G}$ be an injective map of $a^{-1}\mathcal{O}_\mathcal{D}$-modules. We have to show that

$$\mathcal{F} \otimes_{a^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O} \to \mathcal{G} \otimes_{a^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}$$

is injective. We can check this on $\mathcal{C}_n$, i.e., after applying g_n^{-1}. Since $g_n^* = g_n^{-1}$ because $g_n^{-1}\mathcal{O} = \mathcal{O}_n$ we obtain

$$g_n^{-1}\mathcal{F} \otimes_{g_n^{-1}a^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}_n \to g_n^{-1}\mathcal{G} \otimes_{g_n^{-1}a^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}_n$$

which is injective because $g_n^{-1}a^{-1}\mathcal{O}_\mathcal{D} = a_n^{-1}\mathcal{O}_\mathcal{D}$ and we assume a_n was flat. $\square$

0D7C **Lemma 11.2.** *With notation as above. For a $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ there is an exact complex*

$$\ldots \to g_{2!}(a_2^*\mathcal{G}) \to g_{1!}(a_1^*\mathcal{G}) \to g_{0!}(a_0^*\mathcal{G}) \to a^*\mathcal{G} \to 0$$

of sheaves of $\mathcal{O}$-modules on $\mathcal{C}_{total}$. Here $g_{n!}$ is as in Lemma 6.1.

Proof. Observe that $a^*\mathcal{G}$ is the $\mathcal{O}$-module on $\mathcal{C}_{total}$ whose restriction to $\mathcal{C}_m$ is the $\mathcal{O}_m$-module $a_m^*\mathcal{G}$. The description of the functors $g_{n!}$ on modules in Lemma 6.1 shows that $g_{n!}(a_n^*\mathcal{G})$ is the $\mathcal{O}$-module on $\mathcal{C}_{total}$ whose restriction to $\mathcal{C}_m$ is the $\mathcal{O}_m$-module

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^* a_n^*\mathcal{G} = \bigoplus\nolimits_{\varphi:[n]\to[m]} a_m^*\mathcal{G}$$

The rest of the proof is exactly the same as the proof of Lemma 9.1, replacing $a_m^{-1}\mathcal{G}$ by $a_m^*\mathcal{G}$. $\square$

0D7D **Lemma 11.3.** *With notation as above. For an $\mathcal{O}$-module $\mathcal{F}$ on $\mathcal{C}_{total}$ there is a canonical complex*

$$0 \to a_*\mathcal{F} \to a_{0,*}\mathcal{F}_0 \to a_{1,*}\mathcal{F}_1 \to a_{2,*}\mathcal{F}_2 \to \ldots$$

of $\mathcal{O}_\mathcal{D}$-modules which is exact in degrees $-1, 0$. If $\mathcal{F}$ is an injective $\mathcal{O}$-module, then the complex is exact in all degrees and remains exact on applying the functor $\mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, -)$ for any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$.

Proof. To construct the complex, by the Yoneda lemma, it suffices for any $\mathcal{O}_\mathcal{D}$-modules $\mathcal{G}$ on $\mathcal{D}$ to construct a complex

$$0 \to \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, a_*\mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, a_{0,*}\mathcal{F}_0) \to \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, a_{1,*}\mathcal{F}_1) \to \ldots$$

functorially in $\mathcal{G}$. To do this apply $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ to the exact complex of Lemma 11.2 and use adjointness of pullback and pushforward. The exactness properties in degrees $-1, 0$ follow from the construction as $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ is left exact. If $\mathcal{F}$ is an injective $\mathcal{O}$-module, then the complex is exact because $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ is exact. □

0D7F **Lemma 11.4.** *With notation as above for any K in $D^+(\mathcal{O})$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ in $Mod(\mathcal{O}_\mathcal{D})$ with*

$$E_1^{p,q} = R^q a_{p,*} K_p$$

converging to $R^{p+q}a_*K$. This spectral sequence is functorial in K.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injective $\mathcal{O}$-modules representing K. Consider the double complex with terms

$$A^{p,q} = a_{p,*}\mathcal{I}_p^q$$

where the horizontal arrows come from Lemma 11.3 and the vertical arrows from the differentials of the complex $\mathcal{I}^\bullet$. The lemma says rows of the double complex are exact in positive degrees and evaluate to $a_*\mathcal{I}^q$ in degree 0. Thus the total complex associated to the double complex computes Ra_*K by Homology, Lemma 25.4. On the other hand, since restriction to $\mathcal{C}_p$ is exact (Lemma 3.5) the complex $\mathcal{I}_p^\bullet$ represents K_p in $D(\mathcal{C}_p)$. The sheaves $\mathcal{I}_p^q$ are totally acyclic on $\mathcal{C}_p$ (Lemma 6.2). Hence the cohomology of the columns are the sheaves $R^q a_{p,*}K_p$ by Leray's acyclicity lemma (Derived Categories, Lemma 16.7) and Cohomology on Sites, Lemma 14.3. We conclude by applying Homology, Lemma 25.3. □

12. Cartesian sheaves and modules

0D7G Here is the definition.

07TF **Definition 12.1.** In Situation 3.3.

(1) A sheaf $\mathcal{F}$ of sets or of abelian groups on $\mathcal{C}_{total}$ is *cartesian* if the maps $\mathcal{F}(\varphi) : f_\varphi^{-1}\mathcal{F}_m \to \mathcal{F}_n$ are isomorphisms for all $\varphi : [m] \to [n]$.
(2) If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}_{total}$, then a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules is *cartesian* if the maps $f_\varphi^*\mathcal{F}_m \to \mathcal{F}_n$ are isomorphisms for all $\varphi : [m] \to [n]$.
(3) An object K of $D(\mathcal{C}_{total})$ is *cartesian* if the maps $f_\varphi^{-1}K_m \to K_n$ are isomorphisms for all $\varphi : [m] \to [n]$.
(4) If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}_{total}$, then an object K of $D(\mathcal{O})$ is *cartesian* if the maps $Lf_\varphi^*K_m \to K_n$ are isomorphisms for all $\varphi : [m] \to [n]$.

Of course there is a general notion of a cartesian section of a fibred category and the above are merely examples of this. The property on pullbacks needs only be checked for the degeneracies.

07TG **Lemma 12.2.** *In Situation 3.3.*

(1) *A sheaf $\mathcal{F}$ of sets or abelian groups is cartesian if and only if the maps $(f_{\delta_j^n})^{-1}\mathcal{F}_{n-1} \to \mathcal{F}_n$ are isomorphisms.*
(2) *An object K of $D(\mathcal{C}_{total})$ is cartesian if and only if the maps $(f_{\delta_j^n})^{-1}K_{n-1} \to K_n$ are isomorphisms.*
(3) *If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}_{total}$ a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules is cartesian if and only if the maps $(f_{\delta_j^n})^*\mathcal{F}_{n-1} \to \mathcal{F}_n$ are isomorphisms.*
(4) *If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}_{total}$ an object K of $D(\mathcal{O})$ is cartesian if and only if the maps $L(f_{\delta_j^n})^*K_{n-1} \to K_n$ are isomorphisms.*

(5) *Add more here.*

Proof. In each case the key is that the pullback functors compose to pullback functor; for part (4) see Cohomology on Sites, Lemma 18.3. We show how the argument works in case (1) and omit the proof in the other cases. The category Δ is generated by the morphisms the morphisms δ^n_j and σ^n_j, see Simplicial, Lemma 2.2. Hence we only need to check the maps $(f_{\delta^n_j})^{-1}\mathcal{F}_{n-1} \to \mathcal{F}_n$ and $(f_{\sigma^n_j})^{-1}\mathcal{F}_{n+1} \to \mathcal{F}_n$ are isomorphisms, see Simplicial, Lemma 3.2 for notation. Since $\sigma^n_j \circ \delta^{n+1}_j = \text{id}_{[n]}$ the composition

$$\mathcal{F}_n = (f_{\sigma^n_j})^{-1}(f_{\delta^{n+1}_j})^{-1}\mathcal{F}_n \to (f_{\sigma^n_j})^{-1}\mathcal{F}_{n+1} \to \mathcal{F}_n$$

is the identity. Thus the result for δ^{n+1}_j implies the result for σ^n_j. $\square$

0D7H **Lemma 12.3.** *In Situation 3.3 let a_0 be an augmentation towards a site $\mathcal{D}$ as in Remark 4.1.*

(1) *The pullback $a^{-1}\mathcal{G}$ of a sheaf of sets or abelian groups on $\mathcal{D}$ is cartesian.*
(2) *The pullback $a^{-1}K$ of an object K of $D(\mathcal{D})$ is cartesian.*

Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ and $\mathcal{O}_\mathcal{D}$ a sheaf of rings on $\mathcal{D}$ and $a^\sharp : \mathcal{O}_\mathcal{D} \to a_*\mathcal{O}$ a morphism as in Section 11.

(3) *The pullback $a^*\mathcal{F}$ of a sheaf of $\mathcal{O}_\mathcal{D}$-modules is cartesian.*
(4) *The derived pullback La^*K of an object K of $D(\mathcal{O}_\mathcal{D})$ is cartesian.*

Proof. This follows immediately from the identities $a_m \circ f_\varphi = a_n$ for all $\varphi : [m] \to [n]$. See Lemma 4.2 and the discussion in Section 11. $\square$

0D7I **Lemma 12.4.** *In Situation 3.3. The category of cartesian sheaves of sets (resp. abelian groups) is equivalent to the category of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a sheaf of sets (resp. abelian groups) on $\mathcal{C}_0$ and*

$$\alpha : (f_{\delta^1_1})^{-1}\mathcal{F} \longrightarrow (f_{\delta^1_0})^{-1}\mathcal{F}$$

is an isomorphism of sheaves of sets (resp. abelian groups) on $\mathcal{C}_1$ such that $(f_{\delta^2_1})^{-1}\alpha = (f_{\delta^2_0})^{-1}\alpha \circ (f_{\delta^2_2})^{-1}\alpha$ as maps of sheaves on $\mathcal{C}_2$.

Proof. We abbreviate $d^n_j = f_{\delta^n_j} : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{n-1})$. The condition on α in the statement of the lemma makes sense because

$$d^1_1 \circ d^2_2 = d^1_1 \circ d^2_1, \quad d^1_1 \circ d^2_0 = d^1_0 \circ d^2_2, \quad d^1_0 \circ d^2_0 = d^1_0 \circ d^2_1$$

as morphisms of topoi $Sh(\mathcal{C}_2) \to Sh(\mathcal{C}_0)$, see Simplicial, Remark 3.3. Hence we can picture these maps as follows

$$\begin{tikzcd}
 & (d^2_0)^{-1}(d^1_1)^{-1}\mathcal{F} \arrow[r, "(d^2_0)^{-1}\alpha"'] \arrow[dl, equal] & (d^2_0)^{-1}(d^1_0)^{-1}\mathcal{F} \arrow[dr, equal] & \\
(d^2_2)^{-1}(d^1_0)^{-1}\mathcal{F} & & & (d^2_1)^{-1}(d^1_0)^{-1}\mathcal{F} \\
 & (d^2_2)^{-1}(d^1_1)^{-1}\mathcal{F} \arrow[ul, "(d^2_2)^{-1}\alpha"] \arrow[r, equal] & (d^2_1)^{-1}(d^1_1)^{-1}\mathcal{F} \arrow[ur, "(d^2_1)^{-1}\alpha"'] &
\end{tikzcd}$$

and the condition signifies the diagram is commutative. It is clear that given a cartesian sheaf $\mathcal{G}$ of sets (resp. abelian groups) on $\mathcal{C}_{total}$ we can set $\mathcal{F} = \mathcal{G}_0$ and α equal to the composition

$$(d^1_1)^{-1}\mathcal{G}_0 \to \mathcal{G}_1 \leftarrow (d^0_1)^{-1}\mathcal{G}_0$$

where the arrows are invertible as $\mathcal{G}$ is cartesian. To prove this functor is an equivalence we construct a quasi-inverse. The construction of the quasi-inverse is analogous to the construction discussed in Descent, Section 3 from which we borrow the notation $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and $\tau^n_{ij} : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Namely, given a pair $(\mathcal{F}, \alpha)$ as in the lemma we set $\mathcal{G}_n = (f_{\tau^n_n})^{-1}\mathcal{F}$. Given $\varphi : [n] \to [m]$ we define $\mathcal{G}(\varphi) : (f_\varphi)^{-1}\mathcal{G}_n \to \mathcal{G}_m$ using

$$\begin{tikzcd}
(f_\varphi)^{-1}\mathcal{G}_n \arrow[r, equal] & (f_\varphi)^{-1}(f_{\tau^n_n})^{-1}\mathcal{F} \arrow[r, equal] & (f_{\tau^m_{\varphi(n)}})^{-1}\mathcal{F} \arrow[r, equal] & (f_{\tau^m_{\varphi(n)m}})^{-1}(d^1_1)^{-1}\mathcal{F} \arrow[d, "(f_{\tau^m_{\varphi(n)m}})^{-1}\alpha"] \\
& \mathcal{G}_m \arrow[r, equal] & (f_{\tau^m_m})^{-1}\mathcal{F} \arrow[r, equal] & (f_{\tau^m_{\varphi(n)m}})^{-1}(d^1_0)^{-1}\mathcal{F}
\end{tikzcd}$$

We omit the verification that the commutativity of the displayed diagram above implies the maps compose correctly and hence give rise to a sheaf on $\mathcal{C}_{total}$, see Lemma 3.4. We also omit the verification that the two functors are quasi-inverse to each other. $\square$

07TH **Lemma 12.5.** *In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. The category of cartesian $\mathcal{O}$-modules is equivalent to the category of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a $\mathcal{O}_0$-module and*

$$\alpha : (f_{\delta^1_1})^*\mathcal{F} \longrightarrow (f_{\delta^1_0})^*\mathcal{F}$$

is an isomorphism of $\mathcal{O}_1$-modules such that $(f_{\delta^2_1})^*\alpha = (f_{\delta^2_0})^*\alpha \circ (f_{\delta^2_2})^*\alpha$ as $\mathcal{O}_2$-module maps.

Proof. The proof is identical to the proof of Lemma 12.4 with pullback of sheaves of abelian groups replaced by pullback of modules. $\square$

0D7J **Lemma 12.6.** *In Situation 3.3.*

(1) *The full subcategory of cartesian abelian sheaves forms a weak Serre subcategory of $Ab(\mathcal{C}_{total})$. Colimits of systems of cartesian abelian sheaves are cartesian.*
(2) *Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ such that the morphisms*

$$f_{\delta^n_j} : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{n-1}), \mathcal{O}_{n-1})$$

are flat. The full subcategory of cartesian $\mathcal{O}$-modules forms a weak Serre subcategory of $Mod(\mathcal{O})$. Colimits of systems of cartesian $\mathcal{O}$-modules are cartesian.

Proof. To see we obtain a weak Serre subcategory in (1) we check the conditions listed in Homology, Lemma 10.3. First, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a map between cartesian abelian sheaves, then $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are cartesian too because the restriction functors $Sh(\mathcal{C}_{total}) \to Sh(\mathcal{C}_n)$ and the functors f_φ^{-1} are exact. Similarly, if

$$0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{G} \to 0$$

is a short exact sequence of abelian sheaves on $\mathcal{C}_{total}$ with $\mathcal{F}$ and $\mathcal{G}$ cartesian, then it follows that $\mathcal{H}$ is cartesian from the 5-lemma. To see the property of colimits, use that colimits commute with pullback as pullback is a left adjoint. In the case of modules we argue in the same manner, using the exactness of flat pullback (Modules on Sites, Lemma 31.2) and the fact that it suffices to check the condition for $f_{\delta_j^n}$, see Lemma 12.2. $\square$

0D7K **Remark 12.7** (Warning)**.** Lemma 12.6 notwithstanding, it can happen that the category of cartesian $\mathcal{O}$-modules is abelian without being a Serre subcategory of $Mod(\mathcal{O})$. Namely, suppose that we only know that $f_{\delta_1^1}$ and $f_{\delta_0^1}$ are flat. Then it follows easily from Lemma 12.5 that the category of cartesian $\mathcal{O}$-modules is abelian. But if $f_{\delta_0^2}$ is not flat (for example), there is no reason for the inclusion functor from the category of cartesian $\mathcal{O}$-modules to all $\mathcal{O}$-modules to be exact.

0D7L **Lemma 12.8.** *In Situation 3.3.*

(1) *An object K of $D(\mathcal{C}_{total})$ is cartesian if and only if $H^q(K)$ is a cartesian abelian sheaf for all q.*
(2) *Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ such that the morphisms $f_{\delta_j^n} : (Sh(\mathcal{C}_n), \mathcal{O}_n) \to (Sh(\mathcal{C}_{n-1}), \mathcal{O}_{n-1})$ are flat. Then an object K of $D(\mathcal{O})$ is cartesian if and only if $H^q(K)$ is a cartesian $\mathcal{O}$-module for all q.*

Proof. Part (1) is true because the pullback functors $(f_\varphi)^{-1}$ are exact. Part (2) follows from the characterization in Lemma 12.2 and the fact that $L(f_{\delta_j^n})^* = (f_{\delta_j^n})^*$ by flatness. $\square$

0D9E **Lemma 12.9.** *In Situation 3.3.*

(1) *An object K of $D(\mathcal{C}_{total})$ is cartesian if and only the canonical map*
$$g_{n!}K_n \longrightarrow g_{n!}\mathbf{Z} \otimes_{\mathbf{Z}}^{\mathbf{L}} K$$
is an isomorphism for all n.
(2) *Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$ such that the morphisms $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ are flat for all $\varphi : [n] \to [m]$. Then an object K of $D(\mathcal{O})$ is cartesian if and only if the canonical map*
$$g_{n!}K_n \longrightarrow g_{n!}\mathcal{O}_n \otimes_{\mathcal{O}}^{\mathbf{L}} K$$
is an isomorphism for all n.

Proof. Proof of (1). Since $g_{n!}$ is exact, it induces a functor on derived categories adjoint to g_n^{-1}. The map is the adjoint of the map $K_n \to (g_n^{-1}g_{n!}\mathbf{Z}) \otimes_{\mathbf{Z}}^{\mathbf{L}} K_n$ corresponding to $\mathbf{Z} \to g_n^{-1}g_{n!}\mathbf{Z}$ which in turn is adjoint to $\text{id} : g_{n!}\mathbf{Z} \to g_{n!}\mathbf{Z}$. Using the description of $g_{n!}$ given in Lemma 3.5 we see that the restriction to $\mathcal{C}_m$ of this map is
$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}K_n \longrightarrow \bigoplus\nolimits_{\varphi:[n]\to[m]} K_m$$
Thus the statement is clear.

Proof of (2). Since $g_{n!}$ is exact (Lemma 6.3), it induces a functor on derived categories adjoint to g_n^* (also exact). The map is the adjoint of the map $K_n \to (g_n^*g_{n!}\mathcal{O}_n) \otimes_{\mathcal{O}_n}^{\mathbf{L}} K_n$ corresponding to $\mathcal{O}_n \to g_n^*g_{n!}\mathcal{O}_n$ which in turn is adjoint to $\text{id} : g_{n!}\mathcal{O}_n \to g_{n!}\mathcal{O}_n$. Using the description of $g_{n!}$ given in Lemma 6.1 we see that the restriction to $\mathcal{C}_m$ of this map is
$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^*K_n \longrightarrow \bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^*\mathcal{O}_n \otimes_{\mathcal{O}_m} K_m = \bigoplus\nolimits_{\varphi:[n]\to[m]} K_m$$
Thus the statement is clear. $\square$

0D7M **Lemma 12.10.** *In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Then $\mathcal{F}$ is quasi-coherent in the sense of Modules on Sites, Definition 23.1 if and only if $\mathcal{F}$ is cartesian and $\mathcal{F}_n$ is a quasi-coherent $\mathcal{O}_n$-module for all n.*

Proof. Assume $\mathcal{F}$ is quasi-coherent. Since pullbacks of quasi-coherent modules are quasi-coherent (Modules on Sites, Lemma 23.4) we see that $\mathcal{F}_n$ is a quasi-coherent $\mathcal{O}_n$-module for all n. To show that $\mathcal{F}$ is cartesian, let U be an object of $\mathcal{C}_n$ for some n. Let us view U as an object of $\mathcal{C}_{total}$. Because $\mathcal{F}$ is quasi-coherent there exists a covering $\{U_i \to U\}$ and for each i a presentation

$$\bigoplus\nolimits_{j \in J_i} \mathcal{O}_{\mathcal{C}_{total}/U_i} \to \bigoplus\nolimits_{k \in K_i} \mathcal{O}_{\mathcal{C}_{total}/U_i} \to \mathcal{F}|_{\mathcal{C}_{total}/U_i} \to 0$$

Observe that $\{U_i \to U\}$ is a covering of $\mathcal{C}_n$ by the construction of the site $\mathcal{C}_{total}$. Next, let V be an object of $\mathcal{C}_m$ for some m and let $V \to U$ be a morphism of $\mathcal{C}_{total}$ lying over $\varphi : [n] \to [m]$. The fibre products $V_i = V \times_U U_i$ exist and we get an induced covering $\{V_i \to V\}$ in $\mathcal{C}_m$. Restricting the presentation above to the sites $\mathcal{C}_n/U_i$ and $\mathcal{C}_m/V_i$ we obtain presentations

$$\bigoplus\nolimits_{j \in J_i} \mathcal{O}_{\mathcal{C}_m/U_i} \to \bigoplus\nolimits_{k \in K_i} \mathcal{O}_{\mathcal{C}_m/U_i} \to \mathcal{F}_n|_{\mathcal{C}_n/U_i} \to 0$$

and

$$\bigoplus\nolimits_{j \in J_i} \mathcal{O}_{\mathcal{C}_m/V_i} \to \bigoplus\nolimits_{k \in K_i} \mathcal{O}_{\mathcal{C}_m/V_i} \to \mathcal{F}_m|_{\mathcal{C}_m/V_i} \to 0$$

These presentations are compatible with the map $\mathcal{F}(\varphi) : f_\varphi^*\mathcal{F}_n \to \mathcal{F}_m$ (as this map is defined using the restriction maps of $\mathcal{F}$ along morphisms of $\mathcal{C}_{total}$ lying over φ). We conclude that $\mathcal{F}(\varphi)|_{\mathcal{C}_m/V_i}$ is an isomorphism. As $\{V_i \to V\}$ is a covering we conclude $\mathcal{F}(\varphi)|_{\mathcal{C}_m/V}$ is an isomorphism. Since V and U were arbitrary this proves that $\mathcal{F}$ is cartesian. (In case A use Sites, Lemma 14.10.)

Conversely, assume $\mathcal{F}_n$ is quasi-coherent for all n and that $\mathcal{F}$ is cartesian. Then for any n and object U of $\mathcal{C}_n$ we can choose a covering $\{U_i \to U\}$ of $\mathcal{C}_n$ and for each i a presentation

$$\bigoplus\nolimits_{j \in J_i} \mathcal{O}_{\mathcal{C}_m/U_i} \to \bigoplus\nolimits_{k \in K_i} \mathcal{O}_{\mathcal{C}_m/U_i} \to \mathcal{F}_n|_{\mathcal{C}_n/U_i} \to 0$$

Pulling back to $\mathcal{C}_{total}/U_i$ we obtain complexes

$$\bigoplus\nolimits_{j \in J_i} \mathcal{O}_{\mathcal{C}_{total}/U_i} \to \bigoplus\nolimits_{k \in K_i} \mathcal{O}_{\mathcal{C}_{total}/U_i} \to \mathcal{F}|_{\mathcal{C}_{total}/U_i} \to 0$$

of modules on $\mathcal{C}_{total}/U_i$. Then the property that $\mathcal{F}$ is cartesian implies that this is exact. We omit the details. □

13. Simplicial systems of the derived category

0D9F In this section we are going to prove a special case of [BBD82, Proposition 3.2.9] in the setting of derived categories of abelian sheaves. The case of modules is discussed in Section 14.

0D9G **Definition 13.1.** In Situation 3.3. A *simplicial system of the derived category* consists of the following data

(1) for every n an object K_n of $D(\mathcal{C}_n)$,
(2) for every $\varphi : [m] \to [n]$ a map $K_\varphi : f_\varphi^{-1}K_m \to K_n$ in $D(\mathcal{C}_n)$

subject to the condition that

$$K_{\varphi \circ \psi} = K_\varphi \circ f_\varphi^{-1}K_\psi : f_{\varphi \circ \psi}^{-1}K_l = f_\varphi^{-1}f_\psi^{-1}K_l \longrightarrow K_n$$

for any morphisms $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ of Δ. We say the simplicial system is *cartesian* if the maps K_φ are isomorphisms for all φ. Given two simplicial systems of the derived category there is an obvious notion of a *morphism of simplicial systems of the derived category.*

We have given this notion a ridiculously long name intentionally. The goal is to show that a simplicial system of the derived category comes from an object of $D(\mathcal{C}_{total})$ under certain hypotheses.

0D9H **Lemma 13.2.** *In Situation 3.3. If $K \in D(\mathcal{C}_{total})$ is an object, then $(K_n, K(\varphi))$ is a simplicial system of the derived category. If K is cartesian, so is the system.*

Proof. This is obvious. $\square$

0GME **Lemma 13.3.** *In Situation 3.3 suppose given $K_0 \in D(\mathcal{C}_0)$ and an isomorphism*

$$\alpha : f_{\delta^1_1}^{-1}K_0 \longrightarrow f_{\delta^1_0}^{-1}K_0$$

satisfying the cocycle condition. Set $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and set $K_n = f_{\tau^n_n}^{-1}K_0$. Then the K_n form a cartesian simplicial system of the derived category.

Proof. Please compare with Lemma 12.4 and its proof (also to see the cocycle condition spelled out). The construction is analogous to the construction discussed in Descent, Section 3 from which we borrow the notation $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and $\tau^n_{ij} : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Given $\varphi : [n] \to [m]$ we define $K_\varphi : f_\varphi^{-1}K_n \to K_m$ using

$$\begin{tikzcd} f_\varphi^{-1}K_n \arrow[r, equal] & f_\varphi^{-1}f_{\tau^n_n}^{-1}K_0 \arrow[r, equal] & f_{\tau^m_{\varphi(n)}}^{-1}K_0 \arrow[r, equal] & f_{\tau^m_{\varphi(n)m}}^{-1}f_{\delta^1_1}^{-1}K_0 \arrow[d, "f_{\tau^m_{\varphi(n)m}}^{-1}\alpha"'] \\ & K_m \arrow[r, equal] & f_{\tau^m_m}^{-1}K_0 \arrow[r, equal] & f_{\tau^m_{\varphi(n)m}}^{-1}f_{\delta^1_0}^{-1}K_0 \end{tikzcd}$$

We omit the verification that the cocycle condition implies the maps compose correctly (in their respective derived categories) and hence give rise to a simplicial system in the derived category. $\square$

0D9I **Lemma 13.4.** *In Situation 3.3. Let K be an object of $D(\mathcal{C}_{total})$. Set*

$$X_n = (g_{n!}\mathbf{Z}) \otimes^{\mathbf{L}}_{\mathbf{Z}} K \quad \text{and} \quad Y_n = (g_{n!}\mathbf{Z} \to \ldots \to g_{0!}\mathbf{Z})[-n] \otimes^{\mathbf{L}}_{\mathbf{Z}} K$$

as objects of $D(\mathcal{C}_{total})$ where the maps are as in Lemma 8.1. With the evident canonical maps $Y_n \to X_n$ and $Y_0 \to Y_1[1] \to Y_2[2] \to \ldots$ we have

1. *the distinguished triangles $Y_n \to X_n \to Y_{n-1} \to Y_n[1]$ define a Postnikov system (Derived Categories, Definition 41.1) for $\ldots \to X_2 \to X_1 \to X_0$,*
2. *$K = hocolim Y_n[n]$ in $D(\mathcal{C}_{total})$.*

Proof. First, if $K = \mathbf{Z}$, then this is the construction of Derived Categories, Example 41.2 applied to the complex

$$\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}$$

in $Ab(\mathcal{C}_{total})$ combined with the fact that this complex represents $K = \mathbf{Z}$ in $D(\mathcal{C}_{total})$ by Lemma 8.1. The general case follows from this, the fact that the exact functor

$-\otimes^{\mathbf{L}}_{\mathbf{Z}} K$ sends Postnikov systems to Postnikov systems, and that $-\otimes^{\mathbf{L}}_{\mathbf{Z}} K$ commutes with homotopy colimits. □

0D9J **Lemma 13.5.** *In Situation 3.3. If $K, K' \in D(\mathcal{C}_{total})$. Assume*

1. *K is cartesian,*
2. *$\text{Hom}(K_i[i], K'_i) = 0$ for $i > 0$, and*
3. *$\text{Hom}(K_i[i+1], K'_i) = 0$ for $i \geq 0$.*

Then any map $K \to K'$ which induces the zero map $K_0 \to K'_0$ is zero.

Proof. Consider the objects X_n and the Postnikov system Y_n associated to K in Lemma 13.4. As $K = \text{hocolim} Y_n[n]$ the map $K \to K'$ induces a compatible family of morphisms $Y_n[n] \to K'$. By (1) and Lemma 12.9 we have $X_n = g_{n!}K_n$. Since $Y_0 = X_0$ we find that $K_0 \to K'_0$ being zero implies $Y_0 \to K'$ is zero. Suppose we've shown that the map $Y_n[n] \to K'$ is zero for some $n \geq 0$. From the distinguished triangle

$$Y_n[n] \to Y_{n+1}[n+1] \to X_{n+1}[n+1] \to Y_n[n+1]$$

we get an exact sequence

$$\text{Hom}(X_{n+1}[n+1], K') \to \text{Hom}(Y_{n+1}[n+1], K') \to \text{Hom}(Y_n[n], K')$$

As $X_{n+1}[n+1] = g_{n+1!}K_{n+1}[n+1]$ the first group is equal to

$$\text{Hom}(K_{n+1}[n+1], K'_{n+1})$$

which is zero by assumption (2). By induction we conclude all the maps $Y_n[n] \to K'$ are zero. Consider the defining distinguished triangle

$$\bigoplus Y_n[n] \to \bigoplus Y_n[n] \to K \to (\bigoplus Y_n[n])[1]$$

for the homotopy colimit. Arguing as above, we find that it suffices to show that

$$\text{Hom}((\bigoplus Y_n[n])[1], K') = \prod \text{Hom}(Y_n[n+1], K')$$

is zero for all $n \geq 0$. To see this, arguing as above, it suffices to show that

$$\text{Hom}(K_n[n+1], K'_n) = 0$$

for all $n \geq 0$ which follows from condition (3). □

0D9K **Lemma 13.6.** *In Situation 3.3. If $K, K' \in D(\mathcal{C}_{total})$. Assume*

1. *K is cartesian,*
2. *$\text{Hom}(K_i[i-1], K'_i) = 0$ for $i > 1$.*

Then any map $\{K_n \to K'_n\}$ between the associated simplicial systems of K and K' comes from a map $K \to K'$ in $D(\mathcal{C}_{total})$.

Proof. Let $\{K_n \to K'_n\}_{n \geq 0}$ be a morphism of simplicial systems of the derived category. Consider the objects X_n and Postnikov system Y_n associated to K of Lemma 13.4. By (1) and Lemma 12.9 we have $X_n = g_{n!}K_n$. In particular, the map $K_0 \to K'_0$ induces a morphism $X_0 \to K'$. Since $\{K_n \to K'_n\}$ is a morphism of systems, a computation (omitted) shows that the composition

$$X_1 \to X_0 \to K'$$

is zero. As $Y_0 = X_0$ and as Y_1 fits into a distinguished triangle

$$Y_1 \to X_1 \to Y_0 \to Y_1[1]$$

we conclude that there exists a morphism $Y_1[1] \to K'$ whose composition with $X_0 = Y_0 \to Y_1[1]$ is the morphism $X_0 \to K'$ given above. Suppose given a map $Y_n[n] \to K'$ for $n \geq 1$. From the distinguished triangle

$$X_{n+1}[n] \to Y_n[n] \to Y_{n+1}[n+1] \to X_{n+1}[n+1]$$

we get an exact sequence

$$\mathrm{Hom}(Y_{n+1}[n+1], K') \to \mathrm{Hom}(Y_n[n], K') \to \mathrm{Hom}(X_{n+1}[n], K')$$

As $X_{n+1}[n] = g_{n+1!}K_{n+1}[n]$ the last group is equal to

$$\mathrm{Hom}(K_{n+1}[n], K'_{n+1})$$

which is zero by assumption (2). By induction we get a system of maps $Y_n[n] \to K'$ compatible with transition maps and reducing to the given map on Y_0. This produces a map

$$\gamma : K = \mathrm{hocolim} Y_n[n] \longrightarrow K'$$

This map in any case has the property that the diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[rd] & K \arrow[d, "\gamma"] \\ & K' \end{tikzcd}$$

is commutative. Restricting to $\mathcal{C}_0$ we deduce that the map $\gamma_0 : K_0 \to K'_0$ is the same as the first map $K_0 \to K'_0$ of the morphism of simplicial systems. Since K is cartesian, this easily gives that $\{\gamma_n\}$ is the map of simplicial systems we started out with. $\square$

0D9L **Lemma 13.7.** *In Situation 3.3. Let (K_n, K_φ) be a simplicial system of the derived category. Assume*

(1) *(K_n, K_φ) is cartesian,*
(2) *$\mathrm{Hom}(K_i[t], K_i) = 0$ for $i \geq 0$ and $t > 0$.*

Then there exists a cartesian object K of $D(\mathcal{C}_{total})$ whose associated simplicial system is isomorphic to (K_n, K_φ).

Proof. Set $X_n = g_{n!}K_n$ in $D(\mathcal{C}_{total})$. For each $n \geq 1$ we have

$$\mathrm{Hom}(X_n, X_{n-1}) = \mathrm{Hom}(K_n, g_n^{-1}g_{n-1!}K_{n-1}) = \bigoplus\nolimits_{\varphi:[n-1]\to[n]} \mathrm{Hom}(K_n, f_\varphi^{-1}K_{n-1})$$

Thus we get a map $X_n \to X_{n-1}$ corresponding to the alternating sum of the maps $K_\varphi^{-1} : K_n \to f_\varphi^{-1}K_{n-1}$ where φ runs over $\delta_0^n, \ldots, \delta_n^n$. We can do this because K_φ is invertible by assumption (1). Please observe the similarity with the definition of the maps in the proof of Lemma 8.1. We obtain a complex

$$\ldots \to X_2 \to X_1 \to X_0$$

in $D(\mathcal{C}_{total})$. We omit the computation which shows that the compositions are zero. By Derived Categories, Lemma 41.6 if we have

$$\mathrm{Hom}(X_i[i-j-2], X_j) = 0 \text{ for } i > j+2$$

then we can extend this complex to a Postnikov system. The group is equal to

$$\mathrm{Hom}(K_i[i-j-2], g_i^{-1}g_{j!}K_j)$$

Again using that (K_n, K_φ) is cartesian we see that $g_i^{-1}g_{j!}K_j$ is isomorphic to a finite direct sum of copies of K_i. Hence the group vanishes by assumption (2).

Let the Postnikov system be given by $Y_0 = X_0$ and distinguished sequences $Y_n \to X_n \to Y_{n-1} \to Y_n[1]$ for $n \geq 1$. We set

$$K = \text{hocolim} Y_n[n]$$

To finish the proof we have to show that $g_m^{-1}K$ is isomorphic to K_m for all m compatible with the maps K_φ. Observe that

$$g_m^{-1}K = \text{hocolim} g_m^{-1}Y_n[n]$$

and that $g_m^{-1}Y_n[n]$ is a Postnikov system for $g_m^{-1}X_n$. Consider the isomorphisms

$$g_m^{-1}X_n = \bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}K_n \xrightarrow{\bigoplus K_\varphi} \bigoplus\nolimits_{\varphi:[n]\to[m]} K_m$$

These maps define an isomorphism of complexes

$$\begin{tikzcd} \ldots \arrow[r] & g_m^{-1}X_2 \arrow[r] \arrow[d] & g_m^{-1}X_1 \arrow[r] \arrow[d] & g_m^{-1}X_0 \arrow[d] \\ \ldots \arrow[r] & \bigoplus_{\varphi:[2]\to[m]} K_m \arrow[r] & \bigoplus_{\varphi:[1]\to[m]} K_m \arrow[r] & \bigoplus_{\varphi:[0]\to[m]} K_m \end{tikzcd}$$

in $D(\mathcal{C}_m)$ where the arrows in the bottom row are as in the proof of Lemma 8.1. The squares commute by our choice of the arrows of the complex $\ldots \to X_2 \to X_1 \to X_0$; we omit the computation. The bottom row complex has a postnikov tower given by

$$Y'_{m,n} = \left(\bigoplus\nolimits_{\varphi:[n]\to[m]} \mathbf{Z} \to \ldots \to \bigoplus\nolimits_{\varphi:[0]\to[m]} \mathbf{Z}\right)[-n] \otimes_{\mathbf{Z}}^{\mathbf{L}} K_m$$

and $\text{hocolim}Y'_{m,n} = K_m$ (please compare with the proof of Lemma 13.4 and Derived Categories, Example 41.2). Applying the second part of Derived Categories, Lemma 41.6 the vertical maps in the big diagram extend to an isomorphism of Postnikov systems provided we have

$$\text{Hom}(g_m^{-1}X_i[i-j-1], \bigoplus\nolimits_{\varphi:[j]\to[m]} K_m) = 0 \text{ for } i > j+1$$

The is true if $\text{Hom}(K_m[i-j-1], K_m) = 0$ for $i > j+1$ which holds by assumption (2). Choose an isomorphism given by $\gamma_{m,n} : g_m^{-1}Y_n \to Y'_{m,n}$ of Postnikov systems in $D(\mathcal{C}_m)$. By uniqueness of homotopy colimits, we can find an isomorphism

$$g_m^{-1}K = \text{hocolim} g_m^{-1}Y_n[n] \xrightarrow{\gamma_m} \text{hocolim}Y'_{m,n} = K_m$$

compatible with $\gamma_{m,n}$.

We still have to prove that the maps γ_m fit into commutative diagrams

$$\begin{tikzcd} f_\varphi^{-1}g_m^{-1}K \arrow[r, "K(\varphi)"'] \arrow[d, "f_\varphi^{-1}\gamma_m"'] & g_n^{-1}K \arrow[d, "\gamma_n"] \\ f_\varphi^{-1}K_m \arrow[r, "K_\varphi"] & K_n \end{tikzcd}$$

for every $\varphi : [m] \to [n]$. Consider the diagram

$$\begin{tikzcd}
f_\varphi^{-1}(\bigoplus_{\psi:[0]\to[m]} f_\psi^{-1}K_0) \arrow[r, equal] \arrow[d, "f_\varphi^{-1}(\bigoplus K_\psi)"'] & f_\varphi^{-1}g_m^{-1}X_0 \arrow[r, "X_0(\varphi)"'] \arrow[d] & g_n^{-1}X_0 \arrow[r, equal] \arrow[d] & \bigoplus_{\chi:[0]\to[n]} f_\chi^{-1}K_0 \arrow[d, "\bigoplus K_\chi"] \\
f_\varphi^{-1}(\bigoplus_{\psi:[0]\to[m]} K_m) \arrow[d, equal] & f_\varphi^{-1}g_m^{-1}K \arrow[r, "K(\varphi)"'] \arrow[d, "f_\varphi^{-1}\gamma_m"'] & g_n^{-1}K \arrow[d, "\gamma_n"] & \bigoplus_{\chi:[0]\to[n]} K_n \arrow[d, equal] \\
f_\varphi^{-1}Y'_{0,m} \arrow[r] & f_\varphi^{-1}K_m \arrow[r, "K_\varphi"] & K_n & Y'_{0,n} \arrow[l]
\end{tikzcd}$$

The top middle square is commutative as $X_0 \to K$ is a morphism of simplicial objects. The left, resp. the right rectangles are commutative as γ_m, resp. γ_n is compatible with $\gamma_{0,m}$, resp. $\gamma_{0,n}$ which are the arrows $\bigoplus K_\psi$ and $\bigoplus K_\chi$ in the diagram. Going around the outer rectangle of the diagram is commutative as (K_n, K_φ) is a simplical system and the map $X_0(\varphi)$ is given by the obvious identifications $f_\varphi^{-1}f_\psi^{-1}K_0 = f_{\varphi\circ\psi}^{-1}K_0$. Note that the arrow $\bigoplus_\psi K_m \to Y'_{0,m} \to K_m$ induces an isomorphism on any of the direct summands (because of our explicit construction of the Postnikov systems $Y'_{i,j}$ above). Hence, if we take a direct summand of the upper left and corner, then this maps isomorphically to $f_\varphi^{-1}g_m^{-1}K$ as γ_m is an isomorphism. Working out what the above says, but looking only at this direct summand we conclude the lower middle square commutes as we well. This concludes the proof. □

14. Simplicial systems of the derived category: modules

0D9M In this section we are going to prove a special case of [BBD82, Proposition 3.2.9] in the setting of derived categories of $\mathcal{O}$-modules. The (slightly) easier case of abelian sheaves is discussed in Section 13.

0D9N **Definition 14.1.** In Situation 3.3. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. A *simplicial system of the derived category of modules* consists of the following data

(1) for every n an object K_n of $D(\mathcal{O}_n)$,
(2) for every $\varphi : [m] \to [n]$ a map $K_\varphi : Lf_\varphi^* K_m \to K_n$ in $D(\mathcal{O}_n)$

subject to the condition that

$$K_{\varphi\circ\psi} = K_\varphi \circ Lf_\varphi^* K_\psi : Lf_{\varphi\circ\psi}^* K_l = Lf_\varphi^* Lf_\psi^* K_l \longrightarrow K_n$$

for any morphisms $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ of Δ. We say the simplicial system is *cartesian* if the maps K_φ are isomorphisms for all φ. Given two simplicial systems of the derived category there is an obvious notion of a *morphism of simplicial systems of the derived category of modules*.

We have given this notion a ridiculously long name intentionally. The goal is to show that a simplicial system of the derived category of modules comes from an object of $D(\mathcal{O})$ under certain hypotheses.

0D9P **Lemma 14.2.** *In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. If $K \in D(\mathcal{O})$ is an object, then $(K_n, K(\varphi))$ is a simplicial system of the derived category of modules. If K is cartesian, so is the system.*

Proof. This is immediate from the definitions. □

0GMF **Lemma 14.3.** *In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Suppose given $K_0 \in D(\mathcal{O}_0)$ and an isomorphism*

$$\alpha : L(f_{\delta^1_1})^*K_0 \longrightarrow L(f_{\delta^1_0})^*K_0$$

satisfying the cocycle condition. Set $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and set $K_n = Lf^*_{\tau^n_n}K_0$. The objects K_n form the members of a cartesian simplicial system of the derived category of modules.

Proof. Please compare with Lemmas 13.3 and 12.4 and its proof (also to see the cocycle condition spelled out). The construction is analogous to the construction discussed in Descent, Section 3 from which we borrow the notation $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and $\tau^n_{ij} : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Given $\varphi : [n] \to [m]$ we define $K_\varphi : L(f_\varphi)^*K_n \to K_m$ using

$$\begin{tikzcd}
L(f_\varphi)^*K_n \arrow[r, equal] & L(f_\varphi)^*L(f_{\tau^n_n})^*K_0 \arrow[r, equal] & L(f_{\tau^m_{\varphi(n)}})^*K_0 \arrow[r, equal] & L(f_{\tau^m_{\varphi(n)m}})^*L(f_{\delta^1_1})^*K_0 \arrow[d, "L(f_{\tau^m_{\varphi(n)m}})^*\alpha"'] \\
& K_m \arrow[r, equal] & L(f_{\tau^m_m})^*K_0 \arrow[r, equal] & L(f_{\tau^m_{\varphi(n)m}})^*L(f_{\delta^1_0})^*K_0
\end{tikzcd}$$

We omit the verification that the cocycle condition implies the maps compose correctly (in their respective derived categories) and hence give rise to a simplicial systems of the derived category of modules. $\square$

0D9Q **Lemma 14.4.** *In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Let K be an object of $D(\mathcal{C}_{total})$. Set*

$$X_n = (g_{n!}\mathcal{O}_n) \otimes^{\mathbf{L}}_{\mathcal{O}} K \quad \textit{and} \quad Y_n = (g_{n!}\mathcal{O}_n \to \ldots \to g_{0!}\mathcal{O}_0)[-n] \otimes^{\mathbf{L}}_{\mathcal{O}} K$$

as objects of $D(\mathcal{O})$ where the maps are as in Lemma 8.1. With the evident canonical maps $Y_n \to X_n$ and $Y_0 \to Y_1[1] \to Y_2[2] \to \ldots$ we have

(1) *the distinguished triangles $Y_n \to X_n \to Y_{n-1} \to Y_n[1]$ define a Postnikov system (Derived Categories, Definition 41.1) for $\ldots \to X_2 \to X_1 \to X_0$,*
(2) *$K = hocolim Y_n[n]$ in $D(\mathcal{O})$.*

Proof. First, if $K = \mathcal{O}$, then this is the construction of Derived Categories, Example 41.2 applied to the complex

$$\ldots \to g_{2!}\mathcal{O}_2 \to g_{1!}\mathcal{O}_1 \to g_{0!}\mathcal{O}_0$$

in $Ab(\mathcal{C}_{total})$ combined with the fact that this complex represents $K = \mathcal{O}$ in $D(\mathcal{C}_{total})$ by Lemma 10.1. The general case follows from this, the fact that the exact functor $- \otimes^{\mathbf{L}}_{\mathcal{O}} K$ sends Postnikov systems to Postnikov systems, and that $- \otimes^{\mathbf{L}}_{\mathcal{O}} K$ commutes with homotopy colimits. $\square$

0D9R **Lemma 14.5.** *In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. If $K, K' \in D(\mathcal{O})$. Assume*

(1) *$f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ is flat for $\varphi : [m] \to [n]$,*
(2) *K is cartesian,*
(3) *$\mathrm{Hom}(K_i[i], K'_i) = 0$ for $i > 0$, and*
(4) *$\mathrm{Hom}(K_i[i+1], K'_i) = 0$ for $i \geq 0$.*

Then any map $K \to K'$ which induces the zero map $K_0 \to K'_0$ is zero.

Proof. The proof is exactly the same as the proof of Lemma 13.5 except using Lemma 14.4 instead of Lemma 13.4. ☐

0D9S **Lemma 14.6.** *In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. If $K, K' \in D(\mathcal{O})$. Assume*

1. $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ *is flat for* $\varphi : [m] \to [n]$,
2. K *is cartesian,*
3. $\mathrm{Hom}(K_i[i-1], K'_i) = 0$ *for* $i > 1$.

Then any map $\{K_n \to K'_n\}$ between the associated simplicial systems of K and K' comes from a map $K \to K'$ in $D(\mathcal{O})$.

Proof. The proof is exactly the same as the proof of Lemma 13.6 except using Lemma 14.4 instead of Lemma 13.4. ☐

0D9T **Lemma 14.7.** *In Situation 3.3 let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}_{total}$. Let (K_n, K_φ) be a simplicial system of the derived category of modules. Assume*

1. $f_\varphi^{-1}\mathcal{O}_n \to \mathcal{O}_m$ *is flat for* $\varphi : [m] \to [n]$,
2. (K_n, K_φ) *is cartesian,*
3. $\mathrm{Hom}(K_i[t], K_i) = 0$ *for* $i \geq 0$ *and* $t > 0$.

Then there exists a cartesian object K of $D(\mathcal{O})$ whose associated simplicial system is isomorphic to (K_n, K_φ).

Proof. The proof is exactly the same as the proof of Lemma 13.7 with the following changes

1. use $g_n^* = Lg_n^*$ everywhere instead of g_n^{-1},
2. use $f_\varphi^* = Lf_\varphi^*$ everywhere instead of f_φ^{-1},
3. refer to Lemma 10.1 instead of Lemma 8.1,
4. in the construction of $Y'_{m,n}$ use $\mathcal{O}_m$ instead of $\mathbf{Z}$,
5. compare with the proof of Lemma 14.4 rather than the proof of Lemma 13.4.

This ends the proof. ☐

15. The site associated to a semi-representable object

09WK Let $\mathcal{C}$ be a site. Recall that a *semi-representable object* of $\mathcal{C}$ is simply a family $\{U_i\}_{i\in I}$ of objects of $\mathcal{C}$. A *morphism* $\{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ *of semi-representable objects* is given by a map $\alpha : I \to J$ and for every $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ of $\mathcal{C}$. The category of semi-representable objects of $\mathcal{C}$ is denoted $\mathrm{SR}(\mathcal{C})$. See Hypercoverings, Definition 2.1 and the enclosing section for more information.

For a semi-representable object $K = \{U_i\}_{i\in I}$ of $\mathcal{C}$ we let

$$\mathcal{C}/K = \coprod\nolimits_{i\in I} \mathcal{C}/U_i$$

be the disjoint union of the localizations of $\mathcal{C}$ at U_i. There is a natural structure of a site on this category, with coverings inherited from the localizations $\mathcal{C}/U_i$. The site $\mathcal{C}/K$ is called the *localization of $\mathcal{C}$ at K*. Observe that a sheaf on $\mathcal{C}/K$ is the same thing as a family of sheaves $\mathcal{F}_i$ on $\mathcal{C}/U_i$, i.e.,

$$Sh(\mathcal{C}/K) = \prod\nolimits_{i\in I} Sh(\mathcal{C}/U_i)$$

This is occasionally useful to understand what is going on.

Let $\mathcal{C}$ be a site. Let $K = \{U_i\}_{i \in I}$ be an object of $\text{SR}(\mathcal{C})$. There is a continuous and cocontinuous localization functor $j : \mathcal{C}/K \to \mathcal{C}$ which is the product of the localization functors $j_i : \mathcal{C}/V_i \to \mathcal{C}$. We obtain functors $j_!$, j^{-1}, j_* exactly as in Sites, Section 25. In terms of the product decomposition $Sh(\mathcal{C}/K) = \prod_{i \in I} Sh(\mathcal{C}/U_i)$ we have

$$\begin{matrix} j_! & : & (\mathcal{F}_i)_{i \in I} & \longmapsto & \coprod j_{i,!}\mathcal{F}_i \\ j^{-1} & : & \mathcal{G} & \longmapsto & (j_i^{-1}\mathcal{G})_{i \in I} \\ j_* & : & (\mathcal{F}_i)_{i \in I} & \longmapsto & \prod j_{i,*}\mathcal{F}_i \end{matrix}$$

as the reader easily verifies.

Let $f : K \to L$ be a morphism of $\text{SR}(\mathcal{C})$. Then we obtain a continuous and cocontinuous functor

$$v : \mathcal{C}/K \longrightarrow \mathcal{C}/L$$

by applying the construction of Sites, Lemma 25.8 to the components. More precisely, suppose $f = (\alpha, f_i)$ where $K = \{U_i\}_{i \in I}$, $L = \{V_j\}_{j \in J}$, $\alpha : I \to J$, and $f_i : U_i \to V_{\alpha(i)}$. Then the functor v maps the component $\mathcal{C}/U_i$ into the component $\mathcal{C}/V_{\alpha(i)}$ via the construction of the aforementioned lemma. In particular we obtain a morphism

$$f : Sh(\mathcal{C}/K) \to Sh(\mathcal{C}/L)$$

of topoi. In terms of the product decompositions $Sh(\mathcal{C}/K) = \prod_{i \in I} Sh(\mathcal{C}/U_i)$ and $Sh(\mathcal{C}/L) = \prod_{j \in J} Sh(\mathcal{C}/V_j)$ the reader verifies that

$$\begin{matrix} f_! & : & (\mathcal{F}_i)_{i \in I} & \longmapsto & (\coprod_{i \in I, \alpha(i) = j} f_{i,!}\mathcal{F}_i)_{j \in J} \\ f^{-1} & : & (\mathcal{G}_j)_{j \in J} & \longmapsto & (f_i^{-1}\mathcal{G}_{\alpha(i)})_{i \in I} \\ f_* & : & (\mathcal{F}_i)_{i \in I} & \longmapsto & (\prod_{i \in I, \alpha(i) = j} f_{i,*}\mathcal{F}_i)_{j \in J} \end{matrix}$$

where $f_i : Sh(\mathcal{C}/U_i) \to Sh(\mathcal{C}/V_{\alpha(i)})$ is the morphism associated to the localization functor $\mathcal{C}/U_i \to \mathcal{C}/V_{\alpha(i)}$ corresponding to $f_i : U_i \to V_{\alpha(i)}$.

0D85 **Lemma 15.1.** *Let $\mathcal{C}$ be a site.*

1. *For K in $SR(\mathcal{C})$ the functor $j : \mathcal{C}/K \to \mathcal{C}$ is continuous, cocontinuous, and has property P of Sites, Remark 20.5.*
2. *For $f : K \to L$ in $SR(\mathcal{C})$ the functor $v : \mathcal{C}/K \to \mathcal{C}/L$ (see above) is continuous, cocontinuous, and has property P of Sites, Remark 20.5.*

Proof. Proof of (2). In the notation of the discussion preceding the lemma, the localization functors $\mathcal{C}/U_i \to \mathcal{C}/V_{\alpha(i)}$ are continuous and cocontinuous by Sites, Section 25 and satisfy P by Sites, Remark 25.11. It is formal to deduce v is continuous and cocontinuous and has P. We omit the details. We also omit the proof of (1). $\square$

0D86 **Lemma 15.2.** *Let $\mathcal{C}$ be a site and K in $SR(\mathcal{C})$. For $\mathcal{F}$ in $Sh(\mathcal{C})$ we have*

$$j_*j^{-1}\mathcal{F} = \mathcal{H}om(F(K)^\#, \mathcal{F})$$

where F is as in Hypercoverings, Definition 2.2.

Proof. Say $K = \{U_i\}_{i \in I}$. Using the description of the functors j^{-1} and j_* given above we see that

$$j_*j^{-1}\mathcal{F} = \prod\nolimits_{i \in I} j_{i,*}(\mathcal{F}|_{\mathcal{C}/U_i}) = \prod\nolimits_{i \in I} \mathcal{H}om(h_{U_i}^\#, \mathcal{F})$$

The second equality by Sites, Lemma 26.3. Since $F(K) = \coprod h_{U_i}$ in $PSh(\mathcal{C}$, we have $F(K)^\# = \coprod h_{U_i}^\#$ in $Sh(\mathcal{C})$ and since $\mathcal{H}om(-, \mathcal{F})$ turns coproducts into products (immediate from the construction in Sites, Section 26), we conclude. □

0D87 **Lemma 15.3.** *Let $\mathcal{C}$ be a site.*

1. *For K in $SR(\mathcal{C})$ the functor $j_!$ gives an equivalence $Sh(\mathcal{C}/K) \to Sh(\mathcal{C})/F(K)^\#$ where F is as in Hypercoverings, Definition 2.2.*
2. *The functor $j^{-1} : Sh(\mathcal{C}) \to Sh(\mathcal{C}/K)$ corresponds via the identification of (1) with $\mathcal{F} \mapsto (\mathcal{F} \times F(K)^\# \to F(K)^\#)$.*
3. *For $f : K \to L$ in $SR(\mathcal{C})$ the functor f^{-1} corresponds via the identifications of (1) to the functor $Sh(\mathcal{C})/F(L)^\# \to Sh(\mathcal{C})/F(K)^\#$, $(\mathcal{G} \to F(L)^\#) \mapsto (\mathcal{G} \times_{F(L)^\#} F(K)^\# \to F(K)^\#)$.*

Proof. Observe that if $K = \{U_i\}_{i \in I}$ then the category $Sh(\mathcal{C}/K)$ decomposes as the product of the categories $Sh(\mathcal{C}/U_i)$. Observe that $F(K)^\# = \coprod_{i \in I} h_{U_i}^\#$ (coproduct in sheaves). Hence $Sh(\mathcal{C})/F(K)^\#$ is the product of the categories $Sh(\mathcal{C})/h_{U_i}^\#$. Thus (1) and (2) follow from the corresponding statements for each i, see Sites, Lemmas 25.4 and 25.7. Similarly, if $L = \{V_j\}_{j \in J}$ and f is given by $\alpha : I \to J$ and $f_i : U_i \to V_{\alpha(i)}$, then we can apply Sites, Lemma 25.9 to each of the re-localization morphisms $\mathcal{C}/U_i \to \mathcal{C}/V_{\alpha(i)}$ to get (3). □

0D88 **Lemma 15.4.** *Let $\mathcal{C}$ be a site. For K in $SR(\mathcal{C})$ the functor j^{-1} sends injective abelian sheaves to injective abelian sheaves. Similarly, the functor j^{-1} sends K-injective complexes of abelian sheaves to K-injective complexes of abelian sheaves.*

Proof. The first statement is the natural generalization of Cohomology on Sites, Lemma 7.1 to semi-representable objects. In fact, it follows from this lemma by the product decomposition of $Sh(\mathcal{C}/K)$ and the description of the functor j^{-1} given above. The second statement is the natural generalization of Cohomology on Sites, Lemma 20.1 and follows from it by the product decomposition of the topos.

Alternative: since j induces a localization of topoi by Lemma 15.3 part (1) it also follows immediately from Cohomology on Sites, Lemmas 7.1 and 20.1 by enlarging the site; compare with the proof of Cohomology on Sites, Lemma 13.3 in the case of injective sheaves. □

0D89 **Remark 15.5** (Variant for over an object)**.** Let $\mathcal{C}$ be a site. Let $X \in \mathrm{Ob}(\mathcal{C})$. The category $\mathrm{SR}(\mathcal{C}, X)$ of *semi-representable objects over X* is defined by the formula $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$. See Hypercoverings, Definition 2.1. Thus we may apply the above discussion to the site $\mathcal{C}/X$. Briefly, the constructions above give

1. a site $\mathcal{C}/K$ for K in $\mathrm{SR}(\mathcal{C}, X)$,
2. a decomposition $Sh(\mathcal{C}/K) = \prod Sh(\mathcal{C}/U_i)$ if $K = \{U_i/X\}$,
3. a localization functor $j : \mathcal{C}/K \to \mathcal{C}/X$,
4. a morphism $f : Sh(\mathcal{C}/K) \to Sh(\mathcal{C}/L)$ for $f : K \to L$ in $\mathrm{SR}(\mathcal{C}, X)$.

All results of this section hold in this situation by replacing $\mathcal{C}$ everywhere by $\mathcal{C}/X$.

0D9U **Remark 15.6** (Ringed variant)**.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings on $\mathcal{C}$. In this case, for any semi-representable object K of $\mathcal{C}$ the site $\mathcal{C}/K$ is a ringed site with sheaf of rings $\mathcal{O}_K = j^{-1}\mathcal{O}_\mathcal{C}$. The constructions above give

1. a ringed site $(\mathcal{C}/K, \mathcal{O}_K)$ for K in $\mathrm{SR}(\mathcal{C})$,
2. a decomposition $Mod(\mathcal{O}_K) = \prod Mod(\mathcal{O}_{U_i})$ if $K = \{U_i\}$,

(3) a localization morphism $j : (Sh(\mathcal{C}/K), \mathcal{O}_K) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ of ringed topoi,
(4) a morphism $f : (Sh(\mathcal{C}/K), \mathcal{O}_K) \to (Sh(\mathcal{C}/L), \mathcal{O}_L)$ of ringed topoi for $f : K \to L$ in $\text{SR}(\mathcal{C})$.

Many of the results above hold in this setting. For example, the functor j^* has an exact left adjoint

$$j_! : Mod(\mathcal{O}_K) \to Mod(\mathcal{O}_\mathcal{C}),$$

which in terms of the product decomposition given in (2) sends $(\mathcal{F}_i)_{i \in I}$ to $\bigoplus j_{i,!}\mathcal{F}_i$. Similarly, given $f : K \to L$ as above, the functor f^* has an exact left adjoint $f_! : Mod(\mathcal{O}_K) \to Mod(\mathcal{O}_L)$. Thus the functors j^* and f^* are exact, i.e., j and f are flat morphisms of ringed topoi (also follows from the equalities $\mathcal{O}_K = j^{-1}\mathcal{O}_\mathcal{C}$ and $\mathcal{O}_K = f^{-1}\mathcal{O}_L$).

0D9V **Remark 15.7** (Ringed variant over an object)**.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings on $\mathcal{C}$. Let $X \in \text{Ob}(\mathcal{C})$ and denote $\mathcal{O}_X = \mathcal{O}_\mathcal{C}|_{\mathcal{C}/U}$. Then we can combine the constructions given in Remarks 15.5 and 15.6 to get

(1) a ringed site $(\mathcal{C}/K, \mathcal{O}_K)$ for K in $\text{SR}(\mathcal{C}, X)$,
(2) a decomposition $Mod(\mathcal{O}_K) = \prod Mod(\mathcal{O}_{U_i})$ if $K = \{U_i\}$,
(3) a localization morphism $j : (Sh(\mathcal{C}/K), \mathcal{O}_K) \to (Sh(\mathcal{C}/X), \mathcal{O}_X)$ of ringed topoi,
(4) a morphism $f : (Sh(\mathcal{C}/K), \mathcal{O}_K) \to (Sh(\mathcal{C}/L), \mathcal{O}_L)$ of ringed topoi for $f : K \to L$ in $\text{SR}(\mathcal{C}, X)$.

Of course all of the results mentioned in Remark 15.6 hold in this setting as well.

16. The site associate to a simplicial semi-representable object

0D8A Let $\mathcal{C}$ be a site. Let K be a simplicial object of $\text{SR}(\mathcal{C})$. As usual, set $K_n = K([n])$ and denote $K(\varphi) : K_n \to K_m$ the morphism associated to $\varphi : [m] \to [n]$. By the construction in Section 15 we obtain a simplicial object $n \mapsto \mathcal{C}/K_n$ in the category whose objects are sites and whose morphisms are cocontinuous functors. In other words, we get a gadget as in Case B of Section 3. The functors satisfy property P by Lemma 15.1. Hence we may apply Lemma 3.2 to obtain a site $(\mathcal{C}/K)_{total}$.

We can describe the site $(\mathcal{C}/K)_{total}$ explicitly as follows. Say $K_n = \{U_{n,i}\}_{i \in I_n}$. For $\varphi : [m] \to [n]$ the morphism $K(\varphi) : K_n \to K_m$ is given by a map $\alpha(\varphi) : I_n \to I_m$ and morphisms $f_{\varphi,i} : U_{n,i} \to U_{m,\alpha(\varphi)(i)}$ for $i \in I_n$. Then we have

(1) an object of $(\mathcal{C}/K)_{total}$ corresponds to an object $(U/U_{n,i})$ of $\mathcal{C}/U_{n,i}$ for some n and some $i \in I_n$,
(2) a morphism between $U/U_{n,i}$ and $V/U_{m,j}$ is a pair (φ, f) where $\varphi : [m] \to [n]$, $j = \alpha(\varphi)(i)$, and $f : U \to V$ is a morphism of $\mathcal{C}$ such that

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d] & V \arrow[d] \\ U_{n,i} \arrow[r, "f_{\varphi,i}"] & U_{m,j} \end{tikzcd}$$

is commutative, and
(3) coverings of the object $U/U_{n,i}$ are constructed by starting with a covering $\{f_j : U_j \to U\}$ in $\mathcal{C}$ and letting $\{(\text{id}, f_j) : U_j/U_{n,i} \to U/U_{n,i}\}$ be a covering in $(\mathcal{C}/K)_{total}$.

All of our general theory developed for simplicial sites applies to $(\mathcal{C}/K)_{total}$. Observe that the obvious forgetful functor

$$j_{total} : (\mathcal{C}/K)_{total} \longrightarrow \mathcal{C}$$

is continuous and cocontinuous. It turns out that the associated morphism of topoi comes from an (obvious) augmentation.

0D8B **Lemma 16.1.** *Let $\mathcal{C}$ be a site. Let K be a simplicial object of $SR(\mathcal{C})$. The localization functor $j_0 : \mathcal{C}/K_0 \to \mathcal{C}$ defines an augmentation $a_0 : Sh(\mathcal{C}/K_0) \to Sh(\mathcal{C})$, as in case (B) of Remark 4.1. The corresponding morphisms of topoi*

$$a_n : Sh(\mathcal{C}/K_n) \longrightarrow Sh(\mathcal{C}), \quad a : Sh((\mathcal{C}/K)_{total}) \longrightarrow Sh(\mathcal{C})$$

of Lemma 4.2 are equal to the morphisms of topoi associated to the continuous and cocontinuous localization functors $j_n : \mathcal{C}/K_n \to \mathcal{C}$ and $j_{total} : (\mathcal{C}/K)_{total} \to \mathcal{C}$.

Proof. This is immediate from working through the definitions. See in particular the footnote in the proof of Lemma 4.2 for the relationship between a and j_{total}. □

09WM **Lemma 16.2.** *With assumption and notation as in Lemma 16.1 we have the following properties:*

1. *there is a functor $a_!^{Sh} : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C})$ left adjoint to $a^{-1} : Sh(\mathcal{C}) \to Sh((\mathcal{C}/K)_{total})$,*
2. *there is a functor $a_! : Ab((\mathcal{C}/K)_{total}) \to Ab(\mathcal{C})$ left adjoint to $a^{-1} : Ab(\mathcal{C}) \to Ab((\mathcal{C}/K)_{total})$,*
3. *the functor a^{-1} associates to $\mathcal{F}$ in $Sh(\mathcal{C})$ the sheaf on $(\mathcal{C}/K)_{total}$ which in degree n is equal to $a_n^{-1}\mathcal{F}$,*
4. *the functor a_* associates to $\mathcal{G}$ in $Ab((\mathcal{C}/K)_{total})$ the equalizer of the two maps $j_{0,*}\mathcal{G}_0 \to j_{1,*}\mathcal{G}_1$,*

Proof. Parts (3) and (4) hold for any augmentation of a simplicial site, see Lemma 4.2. Parts (1) and (2) follow as j_{total} is continuous and cocontinuous. The functor $a_!^{Sh}$ is constructed in Sites, Lemma 21.5 and the functor $a_!$ is constructed in Modules on Sites, Lemma 16.2. □

0DC0 **Lemma 16.3.** *Let $\mathcal{C}$ be a site. Let K be a simplicial object of $SR(\mathcal{C})$. Let $U/U_{n,i}$ be an object of $\mathcal{C}/K_n$. Let $\mathcal{F} \in Ab((\mathcal{C}/K)_{total})$. Then*

$$H^p(U, \mathcal{F}) = H^p(U, \mathcal{F}_{n,i})$$

where

1. *on the left hand side U is viewed as an object of $\mathcal{C}_{total}$, and*
2. *on the right hand side $\mathcal{F}_{n,i}$ is the ith component of the sheaf $\mathcal{F}_n$ on $\mathcal{C}/K_n$ in the decomposition $Sh(\mathcal{C}/K_n) = \prod Sh(\mathcal{C}/U_{n,i})$ of Section 15.*

Proof. This follows immediately from Lemma 8.6 and the product decompositions of Section 15. □

0D8C **Remark 16.4** (Variant for over an object)**.** Let $\mathcal{C}$ be a site. Let $X \in \mathrm{Ob}(\mathcal{C})$. Recall that we have a category $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$ of semi-representable objects over X, see Remark 15.5. We may apply the above discussion to the site $\mathcal{C}/X$. Briefly, the constructions above give

1. a site $(\mathcal{C}/K)_{total}$ for a simplicial K object of $\mathrm{SR}(\mathcal{C}, X)$,
2. a localization functor $j_{total} : (\mathcal{C}/K)_{total} \to \mathcal{C}/X$,

(3) localization functors $j_n : \mathcal{C}/K_n \to \mathcal{C}/X$,
(4) a morphism of topoi $a : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C}/X)$,
(5) morphisms of topoi $a_n : Sh(\mathcal{C}/K_n) \to Sh(\mathcal{C}/X)$,
(6) a functor $a_!^{Sh} : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C}/X)$ left adjoint to a^{-1}, and
(7) a functor $a_! : Ab((\mathcal{C}/K)_{total}) \to Ab(\mathcal{C}/X)$ left adjoint to a^{-1}.

All of the results of this section hold in this setting. To prove this one replaces the site $\mathcal{C}$ everywhere by $\mathcal{C}/X$.

0D9W **Remark 16.5** (Ringed variant)**.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Given a simplicial semi-representable object K of $\mathcal{C}$ we set $\mathcal{O} = a^{-1}\mathcal{O}_\mathcal{C}$, where a is as in Lemmas 16.1 and 16.2. The constructions above, keeping track of the sheaves of rings as in Remark 15.6, give

(1) a ringed site $((\mathcal{C}/K)_{total}, \mathcal{O})$ for a simplicial K object of $\text{SR}(\mathcal{C})$,
(2) a morphism of ringed topoi $a : (Sh((\mathcal{C}/K)_{total}), \mathcal{O}) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$,
(3) morphisms of ringed topoi $a_n : (Sh(\mathcal{C}/K_n), \mathcal{O}_n) \to (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$,
(4) a functor $a_! : Mod(\mathcal{O}) \to Mod(\mathcal{O}_\mathcal{C})$ left adjoint to a^*.

The functor $a_!$ exists (but in general is not exact) because $a^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}$ and we can replace the use of Modules on Sites, Lemma 16.2 in the proof of Lemma 16.2 by Modules on Sites, Lemma 41.1. As discussed in Remark 15.6 there are exact functors $a_{n!} : Mod(\mathcal{O}_n) \to Mod(\mathcal{O}_\mathcal{C})$ left adjoint to a_n^*. Consequently, the morphisms a and a_n are flat. Remark 15.6 implies the morphism of ringed topoi $f_\varphi : (Sh(\mathcal{C}/K_n), \mathcal{O}_n) \to (Sh(\mathcal{C}/K_m), \mathcal{O}_m)$ for $\varphi : [m] \to [n]$ is flat and there exists an exact functor $f_{\varphi!} : Mod(\mathcal{O}_n) \to Mod(\mathcal{O}_m)$ left adjoint to f_φ^*. This in turn implies that for the flat morphism of ringed topoi $g_n : (Sh(\mathcal{C}/K_n), \mathcal{O}_n) \to (Sh((\mathcal{C}/K)_{total}), \mathcal{O})$ the functor $g_{n!} : Mod(\mathcal{O}_n) \to Mod(\mathcal{O})$ left adjoint to g_n^* is exact, see Lemma 6.3.

0D9X **Remark 16.6** (Ringed variant over an object)**.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let $X \in \text{Ob}(\mathcal{C})$ and denote $\mathcal{O}_X = \mathcal{O}_\mathcal{C}|_{\mathcal{C}/X}$. Then we can combine the constructions given in Remarks 16.4 and 16.5 to get

(1) a ringed site $((\mathcal{C}/K)_{total}, \mathcal{O})$ for a simplicial K object of $\text{SR}(\mathcal{C}, X)$,
(2) a morphism of ringed topoi $a : (Sh((\mathcal{C}/K)_{total}), \mathcal{O}) \to (Sh(\mathcal{C}/X), \mathcal{O}_X)$,
(3) morphisms of ringed topoi $a_n : (Sh(\mathcal{C}/K_n), \mathcal{O}_n) \to (Sh(\mathcal{C}/X), \mathcal{O}_X)$,
(4) a functor $a_! : Mod(\mathcal{O}) \to Mod(\mathcal{O}_X)$ left adjoint to a^*.

Of course, all the results mentioned in Remark 16.5 hold in this setting as well.

17. Cohomological descent for hypercoverings

0D8D Let $\mathcal{C}$ be a site. In this section we assume $\mathcal{C}$ has equalizers and fibre products. We let K be a hypercovering as defined in Hypercoverings, Definition 6.1. We will study the augmentation

$$a : Sh((\mathcal{C}/K)_{total}) \longrightarrow Sh(\mathcal{C})$$

of Section 16.

0D8E **Lemma 17.1.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Then*

(1) *$a^{-1} : Sh(\mathcal{C}) \to Sh((\mathcal{C}/K)_{total})$ is fully faithful with essential image the cartesian sheaves of sets,*
(2) *$a^{-1} : Ab(\mathcal{C}) \to Ab((\mathcal{C}/K)_{total})$ is fully faithful with essential image the cartesian sheaves of abelian groups.*

In both cases a_* provides the quasi-inverse functor.

Proof. The case of abelian sheaves follows immediately from the case of sheaves of sets as the functor a^{-1} commutes with products. In the rest of the proof we work with sheaves of sets. Observe that $a^{-1}\mathcal{F}$ is cartesian for $\mathcal{F}$ in $Sh(\mathcal{C})$ by Lemma 12.3. It suffices to show that the adjunction map $\mathcal{F} \to a_*a^{-1}\mathcal{F}$ is an isomorphism $\mathcal{F}$ in $Sh(\mathcal{C})$ and that for a cartesian sheaf $\mathcal{G}$ on $(\mathcal{C}/K)_{total}$ the adjunction map $a^{-1}a_*\mathcal{G} \to \mathcal{G}$ is an isomorphism.

Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Recall that $a_*a^{-1}\mathcal{F}$ is the equalizer of the two maps $a_{0,*}a_0^{-1}\mathcal{F} \to a_{1,*}a_1^{-1}\mathcal{F}$, see Lemma 16.2. By Lemma 15.2

$$a_{0,*}a_0^{-1}\mathcal{F} = \mathcal{H}om(F(K_0)^\#, \mathcal{F}) \quad \text{and} \quad a_{1,*}a_1^{-1}\mathcal{F} = \mathcal{H}om(F(K_1)^\#, \mathcal{F})$$

On the other hand, we know that

$$F(K_1)^\# \rightrightarrows F(K_0)^\# \longrightarrow \text{final object } * \text{ of } Sh(\mathcal{C})$$

is a coequalizer diagram in sheaves of sets by definition of a hypercovering. Thus it suffices to prove that $\mathcal{H}om(-, \mathcal{F})$ transforms coequalizers into equalizers which is immediate from the construction in Sites, Section 26.

Let $\mathcal{G}$ be a cartesian sheaf on $(\mathcal{C}/K)_{total}$. We will show that $\mathcal{G} = a^{-1}\mathcal{F}$ for some sheaf $\mathcal{F}$ on $\mathcal{C}$. This will finish the proof because then $a^{-1}a_*\mathcal{G} = a^{-1}a_*a^{-1}\mathcal{F} = a^{-1}\mathcal{F} = \mathcal{G}$ by the result of the previous paragraph. Set $\mathcal{K}_n = F(K_n)^\#$ for $n \geq 0$. Then we have maps of sheaves

$$\mathcal{K}_2 \,\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\, \mathcal{K}_1 \rightrightarrows \mathcal{K}_0$$

coming from the fact that K is a simplicial semi-representable object. The fact that K is a hypercovering means that

$$\mathcal{K}_1 \to \mathcal{K}_0 \times \mathcal{K}_0 \quad \text{and} \quad \mathcal{K}_2 \to \Big(\text{cosk}_1(\ \mathcal{K}_1 \rightrightarrows \mathcal{K}_0\ )\Big)_2$$

are surjective maps of sheaves. Using the description of cartesian sheaves on $(\mathcal{C}/K)_{total}$ given in Lemma 12.4 and using the description of $Sh(\mathcal{C}/K_n)$ in Lemma 15.3 we find that our problem can be entirely formulated[3] in terms of

(1) the topos $Sh(\mathcal{C})$, and
(2) the simplicial object $\mathcal{K}$ in $Sh(\mathcal{C})$ whose terms are $\mathcal{K}_n$.

Thus, after replacing $\mathcal{C}$ by a different site $\mathcal{C}'$ as in Sites, Lemma 29.5, we may assume $\mathcal{C}$ has all finite limits, the topology on $\mathcal{C}$ is subcanonical, a family $\{V_j \to V\}$ of morphisms of $\mathcal{C}$ is a covering if and only if $\coprod h_{V_j} \to V$ is surjective, and there exists a simplicial object U of $\mathcal{C}$ such that $\mathcal{K}_n = h_{U_n}$ as simplicial sheaves. Working backwards through the equivalences we may assume $K_n = \{U_n\}$ for all n.

Let X be the final object of $\mathcal{C}$. Then $\{U_0 \to X\}$ is a covering, $\{U_1 \to U_0 \times U_0\}$ is a covering, and $\{U_2 \to (\text{cosk}_1\text{sk}_1 U)_2\}$ is a covering. Let us use $d^n_i : U_n \to U_{n-1}$ and $s^n_j : U_n \to U_{n+1}$ the morphisms corresponding to δ^n_i and σ^n_j as in Simplicial, Definition 2.1. By abuse of notation, given a morphism $c : V \to W$ of $\mathcal{C}$ we denote

[3]Even though it does not matter what the precise formulation is, we spell it out: the problem is to show that given an object $\mathcal{G}_0/\mathcal{K}_0$ of $Sh(\mathcal{C})/\mathcal{K}_0$ and an isomorphism

$$\alpha : \mathcal{G}_0 \times_{\mathcal{K}_0, \mathcal{K}(\delta^1_1)} \mathcal{K}_1 \to \mathcal{G}_0 \times_{\mathcal{K}_0, \mathcal{K}(\delta^1_0)} \mathcal{K}_1$$

over $\mathcal{K}_1$ satisfying a cocycle condition in $Sh(\mathcal{C})/\mathcal{K}_2$, there exists $\mathcal{F}$ in $Sh(\mathcal{C})$ and an isomorphism $\mathcal{F} \times \mathcal{K}_0 \to \mathcal{G}_0$ over $\mathcal{K}_0$ compatible with α.

the morphism of topoi $c : Sh(\mathcal{C}/V) \to Sh(\mathcal{C}/W)$ by the same letter. Now $\mathcal{G}$ is given by a sheaf $\mathcal{G}_0$ on $\mathcal{C}/U_0$ and an isomorphism $\alpha : (d^1_1)^{-1}\mathcal{G}_0 \to (d^1_0)^{-1}\mathcal{G}_0$ satisfying the cocycle condition on $\mathcal{C}/U_2$ formulated in Lemma 12.4. Since $\{U_2 \to (\text{cosk}_1\text{sk}_1 U)_2\}$ is a covering, the corresponding pullback functor on sheaves is faithful (small detail omitted). Hence we may replace U by $\text{cosk}_1\text{sk}_1 U$, because this replaces U_2 by $(\text{cosk}_1\text{sk}_1 U)_2$ and leaves U_1 and U_0 unchanged. Then

$$(d^2_0, d^2_1, d^2_2) : U_2 \to U_1 \times U_1 \times U_1$$

is a monomorphism whose its image on T-valued points is described in Simplicial, Lemma 19.6. In particular, there is a morphism c fitting into a commutative diagram

$$\begin{tikzcd}
U_1 \times_{(d^1_1, d^1_0), U_0 \times U_0, (d^1_1, d^1_0)} U_1 \arrow[r, "c"'] \arrow[d] & U_2 \arrow[d] \\
U_1 \times U_1 \arrow[r, "{(\text{pr}_1, \text{pr}_2, s^0_0 \circ d^1_1 \circ \text{pr}_1)}"] & U_1 \times U_1 \times U_1
\end{tikzcd}$$

as going around the other way defines a point of U_2. Pulling back the cocycle condition for α on U_2 translates into the condition that the pullbacks of α via the projections to $U_1 \times_{(d^1_1, d^1_0), U_0 \times U_0, (d^1_1, d^1_0)} U_1$ are the same as the pullback of α via $s^0_0 \circ d^1_1 \circ \text{pr}_1$ is the identity map (namely, the pullback of α by s^0_0 is the identity). By Sites, Lemma 26.1 this means that α comes from an isomorphism

$$\alpha' : \text{pr}_1^{-1}\mathcal{G}_0 \to \text{pr}_2^{-1}\mathcal{G}_0$$

of sheaves on $\mathcal{C}/U_0 \times U_0$. Then finally, the morphism $U_2 \to U_0 \times U_0 \times U_0$ is surjective on associated sheaves as is easily seen using the surjectivity of $U_1 \to U_0 \times U_0$ and the description of U_2 given above. Therefore α' satisfies the cocycle condition on $U_0 \times U_0 \times U_0$. The proof is finished by an application of Sites, Lemma 26.5 to the covering $\{U_0 \to X\}$. $\square$

0D8F **Lemma 17.2.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. The Čech complex of Lemma 9.2 associated to $a^{-1}\mathcal{F}$*

$$a_{0,*}a_0^{-1}\mathcal{F} \to a_{1,*}a_1^{-1}\mathcal{F} \to a_{2,*}a_2^{-1}\mathcal{F} \to \ldots$$

is equal to the complex $\mathcal{H}om(s(\mathbf{Z}^\#_{F(K)}), \mathcal{F})$. Here $s(\mathbf{Z}^\#_{F(K)})$ is as in Hypercoverings, Definition 4.1.

Proof. By Lemma 15.2 we have

$$a_{n,*}a_n^{-1}\mathcal{F} = \mathcal{H}om'(F(K_n)^\#, \mathcal{F})$$

where $\mathcal{H}om'$ is as in Sites, Section 26. The boundary maps in the complex of Lemma 9.2 come from the simplicial structure. Thus the equality of complexes comes from the canonical identifications $\mathcal{H}om'(\mathcal{G}, \mathcal{F}) = \mathcal{H}om(\mathbf{Z}_\mathcal{G}, \mathcal{F})$ for $\mathcal{G}$ in $Sh(\mathcal{C})$. $\square$

0D8G **Lemma 17.3.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. For $E \in D(\mathcal{C})$ the map*

$$E \longrightarrow Ra_*a^{-1}E$$

is an isomorphism.

Proof. First, let $\mathcal{I}$ be an injective abelian sheaf on $\mathcal{C}$. Then the spectral sequence of Lemma 9.3 for the sheaf $a^{-1}\mathcal{I}$ degenerates as $(a^{-1}\mathcal{I})_p = a_p^{-1}\mathcal{I}$ is injective by Lemma 15.4. Thus the complex

$$a_{0,*}a_0^{-1}\mathcal{I} \to a_{1,*}a_1^{-1}\mathcal{I} \to a_{2,*}a_2^{-1}\mathcal{I} \to \ldots$$

computes $Ra_*a^{-1}\mathcal{I}$. By Lemma 17.2 this is equal to the complex $\mathcal{H}om(s(\mathbf{Z}^\#_{F(K)}), \mathcal{I})$. Because K is a hypercovering, we see that $s(\mathbf{Z}^\#_{F(K)})$ is exact in degrees > 0 by Hypercoverings, Lemma 4.4 applied to the simplicial presheaf $F(K)$. Since $\mathcal{I}$ is injective, the functor $\mathcal{H}om(-, \mathcal{I})$ is exact and we conclude that $\mathcal{H}om(s(\mathbf{Z}^\#_{F(K)}), \mathcal{I})$ is exact in positive degrees. We conclude that $R^pa_*a^{-1}\mathcal{I} = 0$ for $p > 0$. On the other hand, we have $\mathcal{I} = a_*a^{-1}\mathcal{I}$ by Lemma 17.1.

Bounded case. Let $E \in D^+(\mathcal{C})$. Choose a bounded below complex $\mathcal{I}^\bullet$ of injectives representing E. By the result of the first paragraph and Leray's acyclicity lemma (Derived Categories, Lemma 16.7) $Ra_*a^{-1}\mathcal{I}^\bullet$ is computed by the complex $a_*a^{-1}\mathcal{I}^\bullet = \mathcal{I}^\bullet$ and we conclude the lemma is true in this case.

Unbounded case. We urge the reader to skip this, since the argument is the same as above, except that we use explicit representation by double complexes to get around convergence issues. Let $E \in D(\mathcal{C})$. To show the map $E \to Ra_*a^{-1}E$ is an isomorphism, it suffices to show for every object U of $\mathcal{C}$ that

$$R\Gamma(U, E) = R\Gamma(U, Ra_*a^{-1}E)$$

We will compute both sides and show the map $E \to Ra_*a^{-1}E$ induces an isomorphism. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. Choose a quasi-isomorphism $a^{-1}\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ for some K-injective complex $\mathcal{J}^\bullet$ on $(\mathcal{C}/K)_{total}$. We have

$$R\Gamma(U, E) = R\operatorname{Hom}(\mathbf{Z}^\#_U, E)$$

and

$$R\Gamma(U, Ra_*a^{-1}E) = R\operatorname{Hom}(\mathbf{Z}^\#_U, Ra_*a^{-1}E) = R\operatorname{Hom}(a^{-1}\mathbf{Z}^\#_U, a^{-1}E)$$

By Lemma 9.1 we have a quasi-isomorphism

$$\Big(\ldots \to g_{2!}(a_2^{-1}\mathbf{Z}^\#_U) \to g_{1!}(a_1^{-1}\mathbf{Z}^\#_U) \to g_{0!}(a_0^{-1}\mathbf{Z}^\#_U)\Big) \longrightarrow a^{-1}\mathbf{Z}^\#_U$$

Hence $R\operatorname{Hom}(a^{-1}\mathbf{Z}^\#_U, a^{-1}E)$ is equal to

$$R\Gamma((\mathcal{C}/K)_{total}, R\mathcal{H}om(\ldots \to g_{2!}(a_2^{-1}\mathbf{Z}^\#_U) \to g_{1!}(a_1^{-1}\mathbf{Z}^\#_U) \to g_{0!}(a_0^{-1}\mathbf{Z}^\#_U), \mathcal{J}^\bullet))$$

By the construction in Cohomology on Sites, Section 35 and since $\mathcal{J}^\bullet$ is K-injective, we see that this is represented by the complex of abelian groups with terms

$$\prod\nolimits_{p+q=n} \operatorname{Hom}(g_{p!}(a_p^{-1}\mathbf{Z}^\#_U), \mathcal{J}^q) = \prod\nolimits_{p+q=n} \operatorname{Hom}(a_p^{-1}\mathbf{Z}^\#_U, g_p^{-1}\mathcal{J}^q)$$

See Cohomology on Sites, Lemmas 34.6 and 35.1 for more information. Thus we find that $R\Gamma(U, Ra_*a^{-1}E)$ is computed by the product total complex $\operatorname{Tot}_\pi(B^{\bullet,\bullet})$ with $B^{p,q} = \operatorname{Hom}(a_p^{-1}\mathbf{Z}^\#_U, g_p^{-1}\mathcal{J}^q)$. For the other side we argue similarly. First we note that

$$s(\mathbf{Z}^\#_{F(K)}) \longrightarrow \mathbf{Z}$$

is a quasi-isomorphism of complexes on $\mathcal{C}$ by Hypercoverings, Lemma 4.4. Since $\mathbf{Z}_U^\#$ is a flat sheaf of $\mathbf{Z}$-modules we see that

$$s(\mathbf{Z}^\#_{F(K)}) \otimes_{\mathbf{Z}} \mathbf{Z}_U^\# \longrightarrow \mathbf{Z}_U^\#$$

is a quasi-isomorphism. Therefore $R\operatorname{Hom}(\mathbf{Z}_U^\#, E)$ is equal to

$$R\Gamma(\mathcal{C}, R\mathcal{H}om(s(\mathbf{Z}^\#_{F(K)}) \otimes_{\mathbf{Z}} \mathbf{Z}_U^\#, \mathcal{I}^\bullet))$$

By the construction of $R\mathcal{H}om$ and since $\mathcal{I}^\bullet$ is K-injective, this is represented by the complex of abelian groups with terms

$$\prod\nolimits_{p+q=n} \operatorname{Hom}(\mathbf{Z}^\#_{K_p} \otimes_{\mathbf{Z}} \mathbf{Z}^\#_U, \mathcal{I}^q) = \prod\nolimits_{p+q=n} \operatorname{Hom}(a_p^{-1}\mathbf{Z}^\#_U, a_p^{-1}\mathcal{I}^q)$$

The equality of terms follows from the fact that $\mathbf{Z}^\#_{K_p} \otimes_{\mathbf{Z}} \mathbf{Z}^\#_U = a_{p!}a_p^{-1}\mathbf{Z}^\#_U$ by Modules on Sites, Remark 27.10. Thus we find that $R\Gamma(U, E)$ is computed by the product total complex $\operatorname{Tot}_\pi(A^{\bullet,\bullet})$ with $A^{p,q} = \operatorname{Hom}(a_p^{-1}\mathbf{Z}^\#_U, a_p^{-1}\mathcal{I}^q)$.

Since $\mathcal{I}^\bullet$ is K-injective we see that $a_p^{-1}\mathcal{I}^\bullet$ is K-injective, see Lemma 15.4. Since $\mathcal{J}^\bullet$ is K-injective we see that $g_p^{-1}\mathcal{J}^\bullet$ is K-injective, see Lemma 3.6. Both represent the object $a_p^{-1}E$. Hence for every $p \geq 0$ the map of complexes

$$A^{p,\bullet} = \operatorname{Hom}(a_p^{-1}\mathbf{Z}^\#_U, a_p^{-1}\mathcal{I}^\bullet) \longrightarrow \operatorname{Hom}(a_p^{-1}\mathbf{Z}^\#_U, g_p^{-1}\mathcal{J}^\bullet) = B^{p,\bullet}$$

induced by g_p^{-1} applied to the given map $a^{-1}\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ is a quasi-isomorphisms as these complexes both compute

$$R\operatorname{Hom}(a_p^{-1}\mathbf{Z}^\#_U, a_p^{-1}E)$$

By More on Algebra, Lemma 103.2 we conclude that the right vertical arrow in the commutative diagram

$$\begin{tikzcd}
R\Gamma(U, E) \arrow[r] \arrow[d] & \operatorname{Tot}_\pi(A^{\bullet,\bullet}) \arrow[d] \\
R\Gamma(U, Ra_*a^{-1}E) \arrow[r] & \operatorname{Tot}_\pi(B^{\bullet,\bullet})
\end{tikzcd}$$

is a quasi-isomorphism. Since we saw above that the horizontal arrows are quasi-isomorphisms, so is the left vertical arrow. ☐

0D8H **Lemma 17.4.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Then we have a canonical isomorphism*

$$R\Gamma(\mathcal{C}, E) = R\Gamma((\mathcal{C}/K)_{total}, a^{-1}E)$$

for $E \in D(\mathcal{C})$.

Proof. This follows from Lemma 17.3 because $R\Gamma((\mathcal{C}/K)_{total}, -) = R\Gamma(\mathcal{C}, -) \circ Ra_*$ by Cohomology on Sites, Remark 14.4. ☐

0D8I **Lemma 17.5.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Let $\mathcal{A} \subset Ab((\mathcal{C}/K)_{total})$ denote the weak Serre subcategory of cartesian abelian sheaves. Then the functor a^{-1} defines an equivalence*

$$D^+(\mathcal{C}) \longrightarrow D^+_{\mathcal{A}}((\mathcal{C}/K)_{total})$$

with quasi-inverse Ra_*.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 12.6. The equivalence is a formal consequence of the results obtained so far. Use Lemmas 17.1 and 17.3 and Cohomology on Sites, Lemma 28.5 $\square$

We urge the reader to skip the following remark.

09X6 **Remark 17.6.** Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets on $\mathcal{C}$. If $\mathcal{C}$ has equalizers and fibre products, then we've defined the notion of a hypercovering of $\mathcal{G}$ in Hypercoverings, Definition 6.1. We claim that all the results in this section have a valid counterpart in this setting. To see this, define the localization $\mathcal{C}/\mathcal{G}$ of $\mathcal{C}$ at $\mathcal{G}$ exactly as in Sites, Lemma 30.3 (which is stated only for sheaves; the topos $Sh(\mathcal{C}/\mathcal{G})$ is equal to the localization of the topos $Sh(\mathcal{C})$ at the sheaf $\mathcal{G}^\#$). Then the reader easily shows that the site $\mathcal{C}/\mathcal{G}$ has fibre products and equalizers and that a hypercovering of $\mathcal{G}$ in $\mathcal{C}$ is the same thing as a hypercovering for the site $\mathcal{C}/\mathcal{G}$. Hence replacing the site $\mathcal{C}$ by $\mathcal{C}/\mathcal{G}$ in the lemmas on hypercoverings above we obtain proofs of the corresponding results for hypercoverings of $\mathcal{G}$. Example: for a hypercovering K of $\mathcal{G}$ we have

$$R\Gamma(\mathcal{C}/\mathcal{G}, E) = R\Gamma((\mathcal{C}/K)_{total}, a^{-1}E)$$

for $E \in D^+(\mathcal{C}/\mathcal{G})$ where $a : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C}/\mathcal{G})$ is the canonical augmentation. This is Lemma 17.4. Let $R\Gamma(\mathcal{G}, -) : D(\mathcal{C}) \to D(Ab)$ be defined as the derived functor of the functor $H^0(\mathcal{G}, -) = H^0(\mathcal{G}^\#, -)$ discussed in Hypercoverings, Section 6 and Cohomology on Sites, Section 13. We have

$$R\Gamma(\mathcal{G}, E) = R\Gamma(\mathcal{C}/\mathcal{G}, j^{-1}E)$$

by the analogue of Cohomology on Sites, Lemma 7.1 for the localization fuctor $j : \mathcal{C}/\mathcal{G} \to \mathcal{C}$. Putting everything together we obtain

$$R\Gamma(\mathcal{G}, E) = R\Gamma((\mathcal{C}/K)_{total}, a^{-1}j^{-1}E) = R\Gamma((\mathcal{C}/K)_{total}, g^{-1}E)$$

for $E \in D^+(\mathcal{C})$ where $g : Sh((\mathcal{C}/K)_{total}) \to Sh(\mathcal{C})$ is the composition of a and j.

18. Cohomological descent for hypercoverings: modules

0D9Y Let $\mathcal{C}$ be a site. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Assume $\mathcal{C}$ has equalizers and fibre products and let K be a hypercovering as defined in Hypercoverings, Definition 6.1. We will study cohomological descent for the augmentation

$$a : (Sh((\mathcal{C}/K)_{total}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$$

of Remark 16.5.

0D9Z **Lemma 18.1.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering. With notation as above*

$$a^* : Mod(\mathcal{O}_\mathcal{C}) \to Mod(\mathcal{O})$$

is fully faithful with essential image the cartesian $\mathcal{O}$-modules. The functor a_* provides the quasi-inverse.

Proof. Since $a^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}$ we have $a^* = a^{-1}$. Hence the lemma follows immediately from Lemma 17.1. $\square$

0DA0 **Lemma 18.2.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering. For $E \in D(\mathcal{O}_\mathcal{C})$ the map*

$$E \longrightarrow Ra_*La^*E$$

is an isomorphism.

Proof. Since $a^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}$ we have $La^* = a^* = a^{-1}$. Moreover Ra_* agrees with Ra_* on abelian sheaves, see Cohomology on Sites, Lemma 20.7. Hence the lemma follows immediately from Lemma 17.3. ☐

0DA1 **Lemma 18.3.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering. Then we have a canonical isomorphism*

$$R\Gamma(\mathcal{C}, E) = R\Gamma((\mathcal{C}/K)_{total}, La^*E)$$

for $E \in D(\mathcal{O}_\mathcal{C})$.

Proof. This follows from Lemma 18.2 because $R\Gamma((\mathcal{C}/K)_{total}, -) = R\Gamma(\mathcal{C}, -) \circ Ra_*$ by Cohomology on Sites, Remark 14.4 or by Cohomology on Sites, Lemma 20.5. ☐

0DA2 **Lemma 18.4.** *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering. Let $\mathcal{A} \subset Mod(\mathcal{O})$ denote the weak Serre subcategory of cartesian $\mathcal{O}$-modules. Then the functor La^* defines an equivalence*

$$D^+(\mathcal{O}_\mathcal{C}) \longrightarrow D^+_\mathcal{A}(\mathcal{O})$$

with quasi-inverse Ra_*.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 12.6 (the required hypotheses hold by the discussion in Remark 16.5). The equivalence is a formal consequence of the results obtained so far. Use Lemmas 18.1 and 18.2 and Cohomology on Sites, Lemma 28.5. ☐

19. Cohomological descent for hypercoverings of an object

0D8J In this section we assume $\mathcal{C}$ has fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. We let K be a hypercovering of X as defined in Hypercoverings, Definition 3.3. We will study the augmentation

$$a : Sh((\mathcal{C}/K)_{total}) \longrightarrow Sh(\mathcal{C}/X)$$

of Remark 16.4. Observe that $\mathcal{C}/X$ is a site which has equalizers and fibre products and that K is a hypercovering for the site $\mathcal{C}/X$[4] by Hypercoverings, Lemma 3.9. This means that every single result proved for hypercoverings in Section 17 has an immediate analogue in the situation in this section.

0D8K **Lemma 19.1.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let K be a hypercovering of X. Then*

1. *$a^{-1} : Sh(\mathcal{C}/X) \to Sh((\mathcal{C}/K)_{total})$ is fully faithful with essential image the cartesian sheaves of sets,*
2. *$a^{-1} : Ab(\mathcal{C}/X) \to Ab((\mathcal{C}/K)_{total})$ is fully faithful with essential image the cartesian sheaves of abelian groups.*

[4] The converse may not be the case, i.e., if K is a simplicial object of $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$ which defines a hypercovering for the site $\mathcal{C}/X$ as in Hypercoverings, Definition 6.1, then it may not be true that K defines a hypercovering of X. For example, if $K_0 = \{U_{0,i}\}_{i \in I_0}$ then the latter condition guarantees $\{U_{0,i} \to X\}$ is a covering of $\mathcal{C}$ whereas the former condition only requires $\coprod h^\#_{U_{0,i}} \to h^\#_X$ to be a surjective map of sheaves.

In both cases a_* provides the quasi-inverse functor.

Proof. Via Remarks 15.5 and 16.4 and the discussion in the introduction to this section this follows from Lemma 17.1. □

0D8L **Lemma 19.2.** *Let $\mathcal{C}$ be a site with fibre product and $X \in \mathrm{Ob}(\mathcal{C})$. Let K be a hypercovering of X. For $E \in D(\mathcal{C}/X)$ the map*

$$E \longrightarrow Ra_*a^{-1}E$$

is an isomorphism.

Proof. Via Remarks 15.5 and 16.4 and the discussion in the introduction to this section this follows from Lemma 17.3. □

09X7 **Lemma 19.3.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let K be a hypercovering of X. Then we have a canonical isomorphism*

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/K)_{total}, a^{-1}E)$$

for $E \in D(\mathcal{C}/X)$.

Proof. Via Remarks 15.5 and 16.4 this follows from Lemma 17.4. □

0D8M **Lemma 19.4.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let K be a hypercovering of X. Let $\mathcal{A} \subset Ab((\mathcal{C}/K)_{total})$ denote the weak Serre subcategory of cartesian abelian sheaves. Then the functor a^{-1} defines an equivalence*

$$D^+(\mathcal{C}/X) \longrightarrow D^+_{\mathcal{A}}((\mathcal{C}/K)_{total})$$

with quasi-inverse Ra_*.

Proof. Via Remarks 15.5 and 16.4 this follows from Lemma 17.5. □

20. Cohomological descent for hypercoverings of an object: modules

0DA3 In this section we assume $\mathcal{C}$ has fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. We let K be a hypercovering of X as defined in Hypercoverings, Definition 3.3. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings on $\mathcal{C}$. Set $\mathcal{O}_X = \mathcal{O}_\mathcal{C}|_{\mathcal{C}/X}$. We will study the augmentation

$$a : (Sh((\mathcal{C}/K)_{total}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}/X), \mathcal{O}_X)$$

of Remark 16.6. Observe that $\mathcal{C}/X$ is a site which has equalizers and fibre products and that K is a hypercovering for the site $\mathcal{C}/X$. Therefore the results in this section are immediate consequences of the corresponding results in Section 18.

0DA4 **Lemma 20.1.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering of X. With notation as above*

$$a^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O})$$

is fully faithful with essential image the cartesian $\mathcal{O}$-modules. The functor a_* provides the quasi-inverse.

Proof. Via Remarks 15.7 and 16.6 and the discussion in the introduction to this section this follows from Lemma 18.1. □

0DA5 **Lemma 20.2.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering of X. For $E \in D(\mathcal{O}_X)$ the map*

$$E \longrightarrow Ra_*La^*E$$

is an isomorphism.

Proof. Via Remarks 15.7 and 16.6 and the discussion in the introduction to this section this follows from Lemma 18.2. ☐

0DA6 **Lemma 20.3.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering of X. Then we have a canonical isomorphism*

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/K)_{total}, La^*E)$$

for $E \in D(\mathcal{O}_\mathcal{C})$.

Proof. Via Remarks 15.7 and 16.6 and the discussion in the introduction to this section this follows from Lemma 18.3. ☐

0DA7 **Lemma 20.4.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let K be a hypercovering of X. Let $\mathcal{A} \subset Mod(\mathcal{O})$ denote the weak Serre subcategory of cartesian $\mathcal{O}$-modules. Then the functor La^* defines an equivalence*

$$D^+(\mathcal{O}_X) \longrightarrow D^+_\mathcal{A}(\mathcal{O})$$

with quasi-inverse Ra_*.

Proof. Via Remarks 15.7 and 16.6 and the discussion in the introduction to this section this follows from Lemma 18.4. ☐

21. Hypercovering by a simplicial object of the site

09X8 Let $\mathcal{C}$ be a site with fibre products and let $X \in \mathrm{Ob}(\mathcal{C})$. In this section we elucidate the results of Section 19 in the case that our hypercovering is given by a simplicial object of the site. Let U be a simplicial object of $\mathcal{C}$. As usual we denote $U_n = U([n])$ and $f_\varphi : U_n \to U_m$ the morphism $f_\varphi = U(\varphi)$ corresponding to $\varphi : [m] \to [n]$. Assume we have an augmentation

$$a : U \to X$$

From this we obtain a simplicial site $(\mathcal{C}/U)_{total}$ and an augmentation morphism

$$a : Sh((\mathcal{C}/U)_{total}) \longrightarrow Sh(\mathcal{C}/X)$$

Namely, from U we obtain a simiplical object K of $\mathrm{SR}(\mathcal{C}, X)$ with degree n part $K_n = \{U_n \to X\}$ and we can apply the constructions in Remark 16.4. More precisely, an object of the site $(\mathcal{C}/U)_{total}$ is given by a V/U_n and a morphism $(\varphi, f) : V/U_n \to W/U_m$ is given by a morphism $\varphi : [m] \to [n]$ in Δ and a morphism $f : V \to W$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & W \arrow[d] \\ U_n \arrow[r, "f_\varphi"] & U_m \end{tikzcd}$$

is commutative. The morphism of topoi a is given by the cocontinuous functor $V/U_n \mapsto V/X$. That's all folks!

In this section we will say the augmentation $a : U \to X$ is a *hypercovering of X in $\mathcal{C}$* if the following hold

(1) $\{U_0 \to X\}$ is a covering of $\mathcal{C}$,
(2) $\{U_1 \to U_0 \times_X U_0\}$ is a covering of $\mathcal{C}$,
(3) $\{U_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n U)_{n+1}\}$ is a covering of $\mathcal{C}$ for $n \geq 1$.

This is equivalent to the condition that K (as above) is a hypercovering of X, see Hypercoverings, Example 3.5.

0DA8 **Lemma 21.1.** *Let $\mathcal{C}$ be a site with fibre product and $X \in \text{Ob}(\mathcal{C})$. Let $a : U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined above. Then*

1. *$a^{-1} : Sh(\mathcal{C}/X) \to Sh((\mathcal{C}/U)_{total})$ is fully faithful with essential image the cartesian sheaves of sets,*
2. *$a^{-1} : Ab(\mathcal{C}/X) \to Ab((\mathcal{C}/U)_{total})$ is fully faithful with essential image the cartesian sheaves of abelian groups.*

In both cases a_* provides the quasi-inverse functor.

Proof. This is a special case of Lemma 19.1. ☐

0D8N **Lemma 21.2.** *Let $\mathcal{C}$ be a site with fibre product and $X \in \text{Ob}(\mathcal{C})$. Let $a : U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined above. For $E \in D(\mathcal{C}/X)$ the map*

$$E \longrightarrow Ra_*a^{-1}E$$

is an isomorphism.

Proof. This is a special case of Lemma 19.2. ☐

09X9 **Lemma 21.3.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \text{Ob}(\mathcal{C})$. Let $a : U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined above. Then we have a canonical isomorphism*

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/U)_{total}, a^{-1}E)$$

for $E \in D(\mathcal{C}/X)$.

Proof. This is a special case of Lemma 19.3. ☐

0DA9 **Lemma 21.4.** *Let $\mathcal{C}$ be a site with fibre product and $X \in \text{Ob}(\mathcal{C})$. Let $a : U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined above. Let $\mathcal{A} \subset Ab((\mathcal{C}/U)_{total})$ denote the weak Serre subcategory of cartesian abelian sheaves. Then the functor a^{-1} defines an equivalence*

$$D^+(\mathcal{C}/X) \longrightarrow D^+_{\mathcal{A}}((\mathcal{C}/U)_{total})$$

with quasi-inverse Ra_*.

Proof. This is a special case of Lemma 19.4 ☐

09WL **Lemma 21.5.** *Let U be a simplicial object of a site $\mathcal{C}$ with fibre products.*

1. *$\mathcal{C}/U$ has the structure of a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites,*
2. *the construction of Lemma 3.1 applied to the structure in (1) reproduces the site $(\mathcal{C}/U)_{total}$ above,*
3. *if $a : U \to X$ is an augmentation, then $a_0 : \mathcal{C}/U_0 \to \mathcal{C}/X$ is an augmentation as in Remark 4.1 part (A) and gives the same morphism of topoi $a : Sh((\mathcal{C}/U)_{total}) \to Sh(\mathcal{C}/X)$ as the one above.*

Proof. Given a morphism of objects $V \to W$ of $\mathcal{C}$ the localization morphism $j : \mathcal{C}/V \to \mathcal{C}/W$ is a left adjoint to the base change functor $\mathcal{C}/W \to \mathcal{C}/V$. The base change functor is continuous and induces the same morphism of topoi as j. See Sites, Lemma 27.3. This proves (1).

Part (2) holds because a morphism $V/U_n \to W/U_m$ of the category constructed in Lemma 3.1 is a morphism $V \to W \times_{U_m, f_\varphi} U_n$ over U_n which is the same thing as a morphism $f : V \to W$ over the morphism $f_\varphi : U_n \to U_m$, i.e., the same thing as a morphism in the category $(\mathcal{C}/U)_{total}$ defined above. Equality of sets of coverings is immediate from the definition.

We omit the proof of (3). $\square$

22. Hypercovering by a simplicial object of the site: modules

0DAA Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings on $\mathcal{C}$. Let $U \to X$ be a hypercovering of X in $\mathcal{C}$ as defined in Section 21. In this section we study the augmentation

$$a : (Sh((\mathcal{C}/U)_{total}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}/X), \mathcal{O}_X)$$

we obtain by thinking of U as a simiplical semi-representable object of $\mathcal{C}/X$ whose degree n part is the singleton element $\{U_n/X\}$ and applying the constructions in Remark 16.6. Thus all the results in this section are immediate consequences of the corresponding results in Section 20.

0DAB **Lemma 22.1.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let U be a hypercovering of X in $\mathcal{C}$. With notation as above*

$$a^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O})$$

is fully faithful with essential image the cartesian $\mathcal{O}$-modules. The functor a_* provides the quasi-inverse.

Proof. This is a special case of Lemma 20.1. $\square$

0DAC **Lemma 22.2.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let U be a hypercovering of X in $\mathcal{C}$. For $E \in D(\mathcal{O}_X)$ the map*

$$E \longrightarrow Ra_*La^*E$$

is an isomorphism.

Proof. This is a special case of Lemma 20.2. $\square$

0DAD **Lemma 22.3.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let U be a hypercovering of X in $\mathcal{C}$. Then we have a canonical isomorphism*

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/U)_{total}, La^*E)$$

for $E \in D(\mathcal{O}_\mathcal{C})$.

Proof. This is a special case of Lemma 20.3. $\square$

0DAE **Lemma 22.4.** *Let $\mathcal{C}$ be a site with fibre products and $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{O}_\mathcal{C}$ be a sheaf of rings. Let U be a hypercovering of X in $\mathcal{C}$. Let $\mathcal{A} \subset Mod(\mathcal{O})$ denote the weak Serre subcategory of cartesian $\mathcal{O}$-modules. Then the functor La^* defines an equivalence*

$$D^+(\mathcal{O}_X) \longrightarrow D^+_\mathcal{A}(\mathcal{O})$$

with quasi-inverse Ra_*.

Proof. This is a special case of Lemma 20.4. $\square$

23. Unbounded cohomological descent for hypercoverings

0DC1 In this section we discuss unbounded cohomological descent. The results themselves will be immediate consequences of our results on bounded cohomological descent in the previous sections and Cohomology on Sites, Lemmas 28.6 and/or 28.7; the real work lies in setting up notation and choosing appropriate assumptions. Our discussion is motivated by the discussion in [LO08] although the details are a good bit different.

Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Assume given for every object U of $\mathcal{C}$ a weak Serre subcategory $\mathcal{A}_U \subset \textit{Mod}(\mathcal{O}_U)$ satisfying the following properties

0DC2 (1) given a morphism $U \to V$ of $\mathcal{C}$ the restriction functor $\textit{Mod}(\mathcal{O}_V) \to \textit{Mod}(\mathcal{O}_U)$ sends $\mathcal{A}_V$ into $\mathcal{A}_U$,

0DC3 (2) given a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ an object $\mathcal{F}$ of $\textit{Mod}(\mathcal{O}_U)$ is in $\mathcal{A}_U$ if and only if the restriction of $\mathcal{F}$ to $\mathcal{C}/U_i$ is in $\mathcal{A}_{U_i}$ for all $i \in I$.

0DC4 (3) there exists a subset $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ such that
   (a) every object of $\mathcal{C}$ has a covering whose members are in $\mathcal{B}$, and
   (b) for every $V \in \mathcal{B}$ there exists an integer d_V and a cofinal system Cov_V of coverings of V such that

$$H^p(V_i, \mathcal{F}) = 0 \text{ for } \{V_i \to V\} \in \text{Cov}_V,\ p > d_V,\ \text{and } \mathcal{F} \in \text{Ob}(\mathcal{A}_V)$$

Note that we require this to be true for $\mathcal{F}$ in $\mathcal{A}_V$ and not just for "global" objects (and thus it is stronger than the condition imposed in Cohomology on Sites, Situation 25.1). In this situation, there is a weak Serre subcategory $\mathcal{A} \subset \textit{Mod}(\mathcal{O}_\mathcal{C})$ consisting of objects whose restriction to $\mathcal{C}/U$ is in $\mathcal{A}_U$ for all $U \in \text{Ob}(\mathcal{C})$. Moreover, there are derived categories $D_\mathcal{A}(\mathcal{O}_\mathcal{C})$ and $D_{\mathcal{A}_U}(\mathcal{O}_U)$ and the restriction functors send these into each other.

0DC5 **Example 23.1.** Let S be a scheme and let X be an algebraic space over S. Let $\mathcal{C} = X_{spaces, étale}$ be the étale site on the category of algebraic spaces étale over X, see Properties of Spaces, Definition 18.2. Denote $\mathcal{O}_\mathcal{C}$ the structure sheaf, i.e., the sheaf given by the rule $U \mapsto \Gamma(U, \mathcal{O}_U)$. Denote $\mathcal{A}_U$ the category of quasi-coherent $\mathcal{O}_U$-modules. Let $\mathcal{B} = \text{Ob}(\mathcal{C})$ and for $V \in \mathcal{B}$ set $d_V = 0$ and let Cov_V denote the coverings $\{V_i \to V\}$ with V_i affine for all i. Then the assumptions (1), (2), (3) are satisfied. See Properties of Spaces, Lemmas 29.2 and 29.7 for properties (1) and (2) and the vanishing in (3) follows from Cohomology of Schemes, Lemma 2.2 and the discussion in Cohomology of Spaces, Section 3.

0DC6 **Example 23.2.** Let S be one of the following types of schemes

1. the spectrum of a finite field,
2. the spectrum of a separably closed field,
3. the spectrum of a strictly henselian Noetherian local ring,
4. the spectrum of a henselian Noetherian local ring with finite residue field,
5. add more here.

Let Λ be a finite ring whose order is invertible on S. Let $\mathcal{C} \subset (Sch/S)_{étale}$ be the full subcategory consisting of schemes locally of finite type over S endowed with the étale topology. Let $\mathcal{O}_\mathcal{C} = \underline{\Lambda}$ be the constant sheaf. Set $\mathcal{A}_U = \textit{Mod}(\mathcal{O}_U)$, in other words, we consider all étale sheaves of Λ-modules. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be the set of quasi-compact objects. For $V \in \mathcal{B}$ set

$$d_V = 1 + 2\dim(S) + \sup\nolimits_{v \in V}(\text{trdeg}_{\kappa(s)}(\kappa(v)) + 2\dim \mathcal{O}_{V,v})$$

and let Cov_V denote the étale coverings $\{V_i \to V\}$ with V_i quasi-compact for all i. Our choice of bound d_V comes from Gabber's theorem on cohomological dimension. To see that condition (3) holds with this choice, use [ILO14, Exposé VIII-A, Corollary 1.2 and Lemma 2.2] plus elementary arguments on cohomological dimensions of fields. We add 1 to the formula because our list contains cases where we allow S to have finite residue field. We will come back to this example later (insert future reference).

Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Assume given weak Serre subcategories $\mathcal{A}_U \subset \mathit{Mod}(\mathcal{O}_U)$ satisfying condition (1). Then

(1) given a semi-representable object $K = \{U_i\}_{i \in I}$ we get a weak Serre subcategory $\mathcal{A}_K \subset \mathit{Mod}(\mathcal{O}_K)$ by taking $\prod \mathcal{A}_{U_i} \subset \prod \mathit{Mod}(\mathcal{O}_{U_i}) = \mathit{Mod}(\mathcal{O}_K)$, and
(2) given a morphism of semi-representable objects $f : K \to L$ the pullback map $f^* : \mathit{Mod}(\mathcal{O}_L) \to \mathit{Mod}(\mathcal{O}_L)$ sends $\mathcal{A}_L$ into $\mathcal{A}_K$.

See Remark 15.6 for notation and explanation. In particular, given a simplicial semi-representable object K it is unambiguous to say what it means for an object $\mathcal{F}$ of $\mathit{Mod}(\mathcal{O})$ as in Remark 16.5 to have restrictions $\mathcal{F}_n$ in $\mathcal{A}_{K_n}$ for all n.

0DC7 **Lemma 23.3.** *Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Assume given weak Serre subcategories $\mathcal{A}_U \subset \mathit{Mod}(\mathcal{O}_U)$ satisfying conditions (1), (2), and (3) above. Assume $\mathcal{C}$ has equalizers and fibre products and let K be a hypercovering. Let $((\mathcal{C}/K)_{total}, \mathcal{O})$ be as in Remark 16.5. Let $\mathcal{A}_{total} \subset \mathit{Mod}(\mathcal{O})$ denote the weak Serre subcategory of cartesian $\mathcal{O}$-modules $\mathcal{F}$ whose restriction $\mathcal{F}_n$ is in $\mathcal{A}_{K_n}$ for all n (as defined above). Then the functor La^* defines an equivalence*

$$D_\mathcal{A}(\mathcal{O}_\mathcal{C}) \longrightarrow D_{\mathcal{A}_{total}}(\mathcal{O})$$

with quasi-inverse Ra_*.

Proof. The cartesian $\mathcal{O}$-modules form a weak Serre subcategory by Lemma 12.6 (the required hypotheses hold by the discussion in Remark 16.5). Since the restriction functor $g_n^* : \mathit{Mod}(\mathcal{O}) \to \mathit{Mod}(\mathcal{O}_n)$ are exact, it follows that $\mathcal{A}_{total}$ is a weak Serre subcategory.

Let us show that $a^* : \mathcal{A} \to \mathcal{A}_{total}$ is an equivalence of categories with inverse given by La_*. We already know that $La_* a^* \mathcal{F} = \mathcal{F}$ by the bounded version (Lemma 18.4). It is clear that $a^*\mathcal{F}$ is in $\mathcal{A}_{total}$ for $\mathcal{F}$ in $\mathcal{A}$. Conversely, assume that $\mathcal{G} \in \mathcal{A}_{total}$. Because $\mathcal{G}$ is cartesian we see that $\mathcal{G} = a^*\mathcal{F}$ for some $\mathcal{O}_\mathcal{C}$-module $\mathcal{F}$ by Lemma 18.1. We want to show that $\mathcal{F}$ is in $\mathcal{A}$. Take $U \in \mathrm{Ob}(\mathcal{C})$. We have to show that the restriction of $\mathcal{F}$ to $\mathcal{C}/U$ is in $\mathcal{A}_U$. As usual, write $K_0 = \{U_{0,i}\}_{i \in I_0}$. Since K is a hypercovering, the map $\coprod_{i \in I_0} h_{U_{0,i}} \to *$ becomes surjective after sheafification. This implies there is a covering $\{U_j \to U\}_{j \in J}$ and a map $\tau : J \to I_0$ and for each $j \in J$ a morphism $\varphi_j : U_j \to U_{0,\tau(j)}$. Since $\mathcal{G}_0 = a_0^*\mathcal{F}$ we find that the restriction of $\mathcal{F}$ to $\mathcal{C}/U_j$ is equal to the restriction of the $\tau(j)$th component of $\mathcal{G}_0$ to $\mathcal{C}/U_j$ via the morphism $\varphi_j : U_j \to U_{0,\tau(i)}$. Hence by (1) we find that $\mathcal{F}|_{\mathcal{C}/U_j}$ is in $\mathcal{A}_{U_j}$ and in turn by (2) we find that $\mathcal{F}|_{\mathcal{C}/U}$ is in $\mathcal{A}_U$.

In particular the statement of the lemma makes sense. The lemma now follows from Cohomology on Sites, Lemma 28.6. Assumption (1) is clear (see Remark 16.5). Assumptions (2) and (3) we proved in the preceding paragraph. Assumption (4) is immediate from (3). For assumption (5) let $\mathcal{B}_{total}$ be the set of objects $U/U_{n,i}$ of the site $(\mathcal{C}/K)_{total}$ such that $U \in \mathcal{B}$ where $\mathcal{B}$ is as in (3). Here we use the description of the site $(\mathcal{C}/K)_{total}$ given in Section 16. Moreover, we set $\text{Cov}_{U/U_{n,i}}$ equal to Cov_U and $d_{U/U_{n,i}}$ equal d_U where Cov_U and d_U are given to us by (3). Then we claim that condition (5) holds with these choices. This follows immediately from Lemma 16.3 and the fact that $\mathcal{F} \in \mathcal{A}_{total}$ implies $\mathcal{F}_n \in \mathcal{A}_{K_n}$ and hence $\mathcal{F}_{n,i} \in \mathcal{A}_{U_{n,i}}$. (The reader who worries about the difference between cohomology of abelian sheaves versus cohomology of sheaves of modules may consult Cohomology on Sites, Lemma 12.4.) □

24. Glueing complexes

0DC8 This section is the continuation of Cohomology, Section 45. The goal is to prove a slight generalization of [BBD82, Theorem 3.2.4]. Our method will be a tiny bit different in that we use the material from Sections 13 and 14. We will also reprove the unbounded version as it is proved in [LO08].

Advice to the reader: We suggest the reader first look at the statement of Lemma 24.5 as well as the second proof of this lemma.

Here is the situation we are interested in.

0DC9 **Situation 24.1.** Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. We are given

1. a category $\mathcal{B}$ and a functor $u : \mathcal{B} \to \mathcal{C}$,
2. an object E_U in $D(\mathcal{O}_{u(U)})$ for $U \in \text{Ob}(\mathcal{B})$,
3. an isomorphism $\rho_a : E_U|_{\mathcal{C}/u(V)} \to E_V$ in $D(\mathcal{O}_{u(V)})$ for $a : V \to U$ in $\mathcal{B}$

such that whenever we have composable arrows $b : W \to V$ and $a : V \to U$ of $\mathcal{B}$, then $\rho_{a \circ b} = \rho_b \circ \rho_a|_{\mathcal{C}/u(W)}$.

We won't be able to prove anything about this without making more assumptions. An interesting case is where $\mathcal{B}$ is a full subcategory such that every object of $\mathcal{C}$ has a covering whose members are objects of $\mathcal{B}$ (this is the case considered in [BBD82]). For us it is important to allow cases where this is not the case; the main alternative case is where we have a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ and $\mathcal{B}$ is a full subcategory of $\mathcal{D}$ such that every object of $\mathcal{D}$ has a covering whose members are objects of $\mathcal{B}$.

In Situation 24.1 a *solution* will be a pair (E, ρ_U) where E is an object of $D(\mathcal{O}_\mathcal{C})$ and $\rho_U : E|_{\mathcal{C}/u(U)} \to E_U$ for $U \in \text{Ob}(\mathcal{B})$ are isomorphisms such that we have $\rho_a \circ \rho_U|_{\mathcal{C}/u(V)} = \rho_V$ for $a : V \to U$ in $\mathcal{B}$.

0DCA **Lemma 24.2.** *In Situation 24.1. Assume negative self-exts of E_U in $D(\mathcal{O}_{u(U)})$ are zero. Let L be a simplicial object of $SR(\mathcal{B})$. Consider the simplicial object $K = u(L)$ of $SR(\mathcal{C})$ and let $((\mathcal{C}/K)_{total}, \mathcal{O})$ be as in Remark 16.5. There exists a cartesian object E of $D(\mathcal{O})$ such that writing $L_n = \{U_{n,i}\}_{i \in I_n}$ the restriction of E to $D(\mathcal{O}_{\mathcal{C}/u(U_{n,i})})$ is $E_{U_{n,i}}$ compatibly (see proof for details). Moreover, E is unique up to unique isomorphism.*

Proof. Recall that $Sh(\mathcal{C}/K_n) = \prod_{i \in I_n} Sh(\mathcal{C}/u(U_{n,i}))$ and similarly for the categories of modules. This product decomposition is also inherited by the derived categories of sheaves of modules. Moreover, this product decomposition is compatible with the morphisms in the simplicial semi-representable object K. See Section 15. Hence we can set $E_n = \prod_{i \in I_n} E_{U_{n,i}}$ ("formal" product) in $D(\mathcal{O}_n)$. Taking (formal) products of the maps ρ_a of Situation 24.1 we obtain isomorphisms $E_\varphi : f_\varphi^* E_n \to E_m$. The assumption about compostions of the maps ρ_a immediately implies that (E_n, E_φ) defines a simplicial system of the derived category of modules as in Definition 14.1. The vanishing of negative exts assumed in the lemma implies that $\mathrm{Hom}(E_n[t], E_n) = 0$ for $n \geq 0$ and $t > 0$. Thus by Lemma 14.7 we obtain E. Uniqueness up to unique isomorphism follows from Lemmas 14.5 and 14.6. □

0DCB **Lemma 24.3** (BBD glueing lemma). *In Situation 24.1. Assume*

1. *$\mathcal{C}$ has equalizers and fibre products,*
2. *there is a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ such that*
   (a) *$\mathcal{D}$ has equalizers and fibre products and u commutes with them,*
   (b) *$\mathcal{B}$ is a full subcategory of $\mathcal{D}$ and $u : \mathcal{B} \to \mathcal{C}$ is the restriction of u,*
   (c) *every object of $\mathcal{D}$ has a covering whose members are objects of $\mathcal{B}$,*
3. *all negative self-exts of E_U in $D(\mathcal{O}_{u(U)})$ are zero, and*
4. *there exists a $t \in \mathbf{Z}$ such that $H^i(E_U) = 0$ for $i < t$ and $U \in \mathrm{Ob}(\mathcal{B})$.*

Then there exists a solution unique up to unique isomorphism.

Proof. By Hypercoverings, Lemma 12.3 there exists a hypercovering L for the site $\mathcal{D}$ such that $L_n = \{U_{n,i}\}_{i \in I_n}$ with $U_{i,n} \in \mathrm{Ob}(\mathcal{B})$. Set $K = u(L)$. Apply Lemma 24.2 to get a cartesian object E of $D(\mathcal{O})$ on the site $(\mathcal{C}/K)_{total}$ restricting to $E_{U_{n,i}}$ on $\mathcal{C}/u(U_{n,i})$ compatibly. The assumption on t implies that $E \in D^+(\mathcal{O})$. By Hypercoverings, Lemma 12.4 we see that K is a hypercovering too. By Lemma 18.4 we find that $E = a^*F$ for some F in $D^+(\mathcal{O}_\mathcal{C})$.

To prove that F is a solution we will use the construction of L_0 and L_1 given in the proof of Hypercoverings, Lemma 12.3. (This is a bit inelegant but there does not seem to be a completely straightforward way around it.)

Namely, we have $I_0 = \mathrm{Ob}(\mathcal{B})$ and so $L_0 = \{U\}_{U \in \mathrm{Ob}(\mathcal{B})}$. Hence the isomorphism $a^*F \to E$ restricted to the components $\mathcal{C}/u(U)$ of $\mathcal{C}/K_0$ defines isomorphisms $\rho_U : F|_{\mathcal{C}/u(U)} \to E_U$ for $U \in \mathrm{Ob}(\mathcal{B})$ by our choice of E.

To prove that ρ_U satisfy the requirement of compatibility with the maps ρ_a of Situation 24.1 we use that I_1 contains the set

$$\Omega = \{(U, V, W, a, b) \mid U, V, W \in \mathcal{B}, a : U \to V, b : U \to W\}$$

and that for $i = (U, V, W, a, b)$ in Ω we have $U_{1,i} = U$. Moreover, the component maps $f_{\delta^1_0, i}$ and $f_{\delta^1_1, i}$ of the two morphisms $K_1 \to K_0$ are the morphisms

$$a : U \to V \quad \text{and} \quad b : U \to V$$

Hence the compatibility mentioned in Lemma 24.2 gives that

$$\rho_a \circ \rho_V|_{\mathcal{C}/u(U)} = \rho_U \quad \text{and} \quad \rho_b \circ \rho_W|_{\mathcal{C}/u(U)} = \rho_U$$

Taking $i = (U, V, U, a, \mathrm{id}_U) \in \Omega$ for example, we find that we have the desired compatibility. The uniqueness of F follows from the uniqueness of E in the previous lemma (small detail omitted). □

0DCC **Lemma 24.4** (Unbounded BBD glueing lemma). *In Situation 24.1. Assume*

1. *$\mathcal{C}$ has equalizers and fibre products,*
2. *there is a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ such that*
   (a) *$\mathcal{D}$ has equalizers and fibre products and u commutes with them,*
   (b) *$\mathcal{B}$ is a full subcategory of $\mathcal{D}$ and $u : \mathcal{B} \to \mathcal{C}$ is the restriction of u,*

(c) *every object of $\mathcal{D}$ has a covering whose members are objects of $\mathcal{B}$,*
(3) *all negative self-exts of E_U in $D(\mathcal{O}_{u(U)})$ are zero, and*
(4) *there exist weak Serre subcategories $\mathcal{A}_U \subset Mod(\mathcal{O}_U)$ for all $U \in \mathrm{Ob}(\mathcal{C})$ satisfying conditions (1), (2), and (3),*
(5) *$E_U \in D_{\mathcal{A}_U}(\mathcal{O}_U)$.*

Then there exists a solution unique up to unique isomorphism.

Proof. The proof is **exactly** the same as the proof of Lemma 24.3. The only change is that E is an object of $D_{\mathcal{A}_{total}}(\mathcal{O})$ and hence we use Lemma 23.3 to obtain F with $E = a^*F$ instead of Lemma 18.4. □

Here is an example application of the general theory above.

0GMG **Lemma 24.5.** *Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Assume $\mathcal{C}$ has fibre products. Let $\{U_i \to X\}_{i \in I}$ be a covering in $\mathcal{C}$. For $i \in I$ let E_i be an object of $D(\mathcal{O}_{U_i})$ and for $i, j \in I$ let* Email of Martin Olsson dated Sep 9, 2021.

$$\rho_{ij} : E_i|_{\mathcal{C}/U_{ij}} \longrightarrow E_j|_{\mathcal{C}/U_{ij}}$$

be an isomorphism in $D(\mathcal{O}_{U_{ij}})$ where $U_{ij} = U_i \times_X U_j$. Assume

(1) *the ρ_{ij} satisfy the cocycle condition on $U_i \times_X U_j \times_X U_k$ for all $i, j, k \in I$,*
(2) *$\mathcal{E}xt^p_{\mathcal{O}_{U_i}}(E_i, E_i) = 0$ for all $p < 0$ and $i \in I$, and*
(3) *there exists a $t \in \mathbf{Z}$ such that $H^p(E_i) = 0$ for $p < t$ and all $i \in I$.*

Then there exists a unique pair (E, ρ_i) where E is an object of $D(\mathcal{O}_X)$ and $\rho_i : E|_{U_i} \to E_i$ are isomorphisms in $D(\mathcal{O}_{U_i})$ compatible with the ρ_{ij}.

First proof. In this proof we deduce the lemma from the very general Lemma 24.3. We urge the reader to look at the second proof in stead.

We may replace $\mathcal{C}$ with $\mathcal{C}/X$. Thus we may and do assume X is the final object of $\mathcal{C}$ and that $\mathcal{C}$ has all finite limits.

Let $\mathcal{B}$ be the full subcategory of $\mathcal{C}$ consisting of $U \in \mathrm{Ob}(\mathcal{C})$ such that there exists an $i(U) \in I$ and a morphism $a_U : U \to U_{i(U)}$. We denote $E_U = a_U^* E_{i(U)}$ in $D(\mathcal{O}_U)$ the pullback (restriction) of E_i via a_U. Given a morphism $a : U \to U'$ of $\mathcal{B}$ we obtain a morphism $(a_{U'} \circ a, a_U) : U \to U_{i(U')} \times_X U_{i(U)} = U_{i(U')i(U)}$ and hence an isomorphism

$$\rho_a : a^* E_{U'} = a^* a_{U'}^* E_{i(U')} \xrightarrow{(a_{U'} \circ a, a_U)^* \rho_{i(U')i(U)}} a_U^* E_{i(U)} = E_U$$

in $D(\mathcal{O}_U)$. The data $\mathcal{B}, E_U, \rho_a$ are as in Situation 24.1; the isomorphisms ρ_a satisfy the cocycle condition exactly because of condition (1) in the statement of the lemma (details omitted).

We are going to apply Lemma 24.3 with $\mathcal{B}$, E_U, ρ_a as above and with $\mathcal{D} = \mathcal{C}$ and $f : \mathcal{C} \to \mathcal{D}$ the identity morphism. Assumptions (1) and (2)(a) of Lemma 24.3 we have seen above. Assumption (2)(b) of Lemma 24.3 is clear. Assumption (2)(c) of Lemma 24.3 holds because $\{U_i \to X\}$ is a covering[5]. Assumption (3) of Lemma 24.3 holds because we have assumed the vanishing of all negative Ext sheaves of E_i which certainly implies that for any object U lying over U_i the negative self-Exts of $E_i|_U$ are zero. Assumption (4) of Lemma 24.3 holds because we have assumed the cohomology sheaves of each E_i are zero to the left of t.

[5]In fact, it would suffice if the map $\coprod_{i \in I} h_{U_i} \to h_X$ becomes surjective on sheafification and the lemma holds in this case with the same proof.

We obtain a unique solution (E, ρ_U). Setting $\rho_i = \rho_{U_i}$ the lemma follows. $\square$

Second proof. We sketch a more direct proof. Denote K the Čech hypercovering of X associated to the covering $\{U_i \to X\}_{i \in I}$, see Hypercoverings, Example 3.4. Thus for example $K_0 = \{U_i \to X\}_{i \in I}$ and $K_1 = \{U_i \times_X U_j \to X\}_{i,j \in I}$ and so on. Let $((\mathcal{C}/K)_{total}, \mathcal{O})$, a, a_n be as in Remark 16.6. The objects E_i determine an object M_0 in $D(\mathcal{O}_0) = \prod D(\mathcal{O}_{U_i})$. Similarly, the isomorphisms ρ_{ij} determine an isomorphism

$$\alpha : L(f_{\delta^1_1})^* M_0 \longrightarrow L(f_{\delta^1_0})^* M_0$$

in $D(\mathcal{O}_1)$ satisfying the cocycle condition. By Lemma 14.3 we obtain a cartesian simplicial system (M_n) of the derived category. By the assumed vanishing of the negative Ext sheaves we see that the objects M_n have vanishing negative self-exts. Thus we find a cartesian object M of $D(\mathcal{O})$ whose associated simplicial system is isomorphic to (M_n) by Lemma 14.7. Since the cohomology sheaves of M are zero in degrees $< t$ we see that by Lemma 20.4 we have $M = La^*E$ for some E in $D(\mathcal{O}_X)$. The isomorphism $La^*E \to M$ restricted to $\mathcal{C}/U_i$ produces the isomorphisms ρ_i. We omit the verification of the compatibility with the isomorphisms ρ_{ij}. $\square$

25. Proper hypercoverings in topology

09XA Let's work in the category LC of Hausdorff and locally quasi-compact topological spaces and continuous maps, see Cohomology on Sites, Section 31. Let X be an object of LC and let U be a simplicial object of LC. Assume we have an augmentation

$$a : U \to X$$

We say that U is a *proper hypercovering* of X if

(1) $U_0 \to X$ is a proper surjective map,
(2) $U_1 \to U_0 \times_X U_0$ is a proper surjective map,
(3) $U_{n+1} \to (\text{cosk}_n \text{sk}_n U)_{n+1}$ is a proper surjective map for $n \geq 1$.

The category LC has all finite limits, hence the coskeleta used in the formulation above exist.

$$\boxed{\text{Principle: Proper hypercoverings can be used to compute cohomology.}}$$

A key idea behind the proof of the principle is to find a topology on LC which is stronger than the usual one such that (a) a surjective proper map defines a covering, and (b) cohomology of usual sheaves with respect to this stronger topology agrees with the usual cohomology. Properties (a) and (b) hold for the qc topology, see Cohomology on Sites, Section 31. Once we have (a) and (b) we deduce the principle via the earlier work done in this chapter.

0DAF **Lemma 25.1.** *Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. There is a commutative diagram*

$$\begin{tikzcd} Sh((LC_{qc}/U)_{total}) \arrow[r, "h"'] \arrow[d, "a_{qc}"'] & Sh(U_{Zar}) \arrow[d, "a"] \\ Sh(LC_{qc}/X) \arrow[r, "h_{-1}"] & Sh(X) \end{tikzcd}$$

where the left vertical arrow is defined in Section 21 and the right vertical arrow is defined in Lemma 2.8.

Proof. Write $Sh(X) = Sh(X_{Zar})$. Observe that both $(LC_{qc}/U)_{total}$ and U_{Zar} fall into case A of Situation 3.3. This is immediate from the construction of U_{Zar} in Section 2 and it follows from Lemma 21.5 for $(LC_{qc}/U)_{total}$. Next, consider the functors $U_{n,Zar} \to LC_{qc}/U_n$, $U \mapsto U/U_n$ and $X_{Zar} \to LC_{qc}/X$, $U \mapsto U/X$. We have seen that these define morphisms of sites in Cohomology on Sites, Section 31. Thus we obtain a morphism of simplicial sites compatible with augmentations as in Remark 5.4 and we may apply Lemma 5.5 to conclude. □

0DAG **Lemma 25.2.** *Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. If $a : U \to X$ gives a proper hypercovering of X, then*

$$a^{-1} : Sh(X) \to Sh(U_{Zar}) \quad \text{and} \quad a^{-1} : Ab(X) \to Ab(U_{Zar})$$

are fully faithful with essential image the cartesian sheaves and quasi-inverse given by a_*. Here $a : Sh(U_{Zar}) \to Sh(X)$ is as in Lemma 2.8.

Proof. We will prove the statement for sheaves of sets. It will be an almost formal consequence of results already established. Consider the diagram of Lemma 25.1. By Cohomology on Sites, Lemma 31.6 the functor $(h_{-1})^{-1}$ is fully faithful with quasi-inverse $h_{-1,*}$. The same holds true for the components h_n of h. By the description of the functors h^{-1} and h_* of Lemma 5.2 we conclude that h^{-1} is fully faithful with quasi-inverse h_*. Observe that U is a hypercovering of X in LC_{qc} (as defined in Section 21) by Cohomology on Sites, Lemma 31.4. By Lemma 21.1 we see that a_{qc}^{-1} is fully faithful with quasi-inverse $a_{qc,*}$ and with essential image the cartesian sheaves on $(LC_{qc}/U)_{total}$. A formal argument (chasing around the diagram) now shows that a^{-1} is fully faithful.

Finally, suppose that $\mathcal{G}$ is a cartesian sheaf on U_{Zar}. Then $h^{-1}\mathcal{G}$ is a cartesian sheaf on LC_{qc}/U. Hence $h^{-1}\mathcal{G} = a_{qc}^{-1}\mathcal{H}$ for some sheaf $\mathcal{H}$ on LC_{qc}/X. We compute

$$\begin{aligned}
(h_{-1})^{-1}(a_*\mathcal{G}) &= (h_{-1})^{-1}\text{Eq}(\ a_{0,*}\mathcal{G}_0 \rightrightarrows a_{1,*}\mathcal{G}_1\ ) \\
&= \text{Eq}(\ (h_{-1})^{-1}a_{0,*}\mathcal{G}_0 \rightrightarrows (h_{-1})^{-1}a_{1,*}\mathcal{G}_1\ ) \\
&= \text{Eq}(\ a_{qc,0,*}h_0^{-1}\mathcal{G}_0 \rightrightarrows a_{qc,1,*}h_1^{-1}\mathcal{G}_1\ ) \\
&= \text{Eq}(\ a_{qc,0,*}a_{qc,0}^{-1}\mathcal{H} \rightrightarrows a_{qc,1,*}a_{qc,1}^{-1}\mathcal{H}\ ) \\
&= a_{qc,*}a_{qc}^{-1}\mathcal{H} \\
&= \mathcal{H}
\end{aligned}$$

Here the first equality follows from Lemma 2.8, the second equality follows as $(h_{-1})^{-1}$ is an exact functor, the third equality follows from Cohomology on Sites, Lemma 31.8 (here we use that $a_0 : U_0 \to X$ and $a_1 : U_1 \to X$ are proper), the fourth follows from $a_{qc}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$, the fifth from Lemma 4.2, and the sixth we've seen above. Since $a_{qc}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$ we deduce that $h^{-1}\mathcal{G} \cong h^{-1}a^{-1}a_*\mathcal{G}$ which ends the proof by fully faithfulness of h^{-1}. □

09XS **Lemma 25.3.** *Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. If $a : U \to X$ gives a proper hypercovering of X, then for $K \in D^+(X)$*

$$K \to Ra_*(a^{-1}K)$$

is an isomorphism where $a : Sh(U_{Zar}) \to Sh(X)$ is as in Lemma 2.8.

Proof. Consider the diagram of Lemma 25.1. Observe that $Rh_{n,*}h_n^{-1}$ is the identity functor on $D^+(U_n)$ by Cohomology on Sites, Lemma 31.11. Hence Rh_*h^{-1} is the identity functor on $D^+(U_{Zar})$ by Lemma 5.3. We have

$$\begin{aligned} Ra_*(a^{-1}K) &= Ra_*Rh_*h^{-1}a^{-1}K \\ &= Rh_{-1,*}Ra_{qc,*}a_{qc}^{-1}(h_{-1})^{-1}K \\ &= Rh_{-1,*}(h_{-1})^{-1}K \\ &= K \end{aligned}$$

The first equality by the discussion above, the second equality because of the commutativity of the diagram in Lemma 25.1, the third equality by Lemma 21.2 (U is a hypercovering of X in LC_{qc} by Cohomology on Sites, Lemma 31.4), and the last equality by the already used Cohomology on Sites, Lemma 31.11. □

09XC **Lemma 25.4.** *Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. If U is a proper hypercovering of X, then*

$$R\Gamma(X, K) = R\Gamma(U_{Zar}, a^{-1}K)$$

for $K \in D^+(X)$ where $a : Sh(U_{Zar}) \to Sh(X)$ is as in Lemma 2.8.

Proof. This follows from Lemma 25.3 because $R\Gamma(U_{Zar}, -) = R\Gamma(X, -) \circ Ra_*$ by Cohomology on Sites, Remark 14.4. □

0DAH **Lemma 25.5.** *Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. Let $\mathcal{A} \subset Ab(U_{Zar})$ denote the weak Serre subcategory of cartesian abelian sheaves. If U is a proper hypercovering of X, then the functor a^{-1} defines an equivalence*

$$D^+(X) \longrightarrow D^+_{\mathcal{A}}(U_{Zar})$$

with quasi-inverse Ra_* where $a : Sh(U_{Zar}) \to Sh(X)$ is as in Lemma 2.8.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 12.6. The equivalence is a formal consequence of the results obtained so far. Use Lemmas 25.2 and 25.3 and Cohomology on Sites, Lemma 28.5. □

09XB **Lemma 25.6.** *Let U be a simplicial object of LC and let $a : U \to X$ be an augmentation. Let $\mathcal{F}$ be an abelian sheaf on X. Let $\mathcal{F}_n$ be the pullback to U_n. If U is a proper hypercovering of X, then there exists a canonical spectral sequence*

$$E_1^{p,q} = H^q(U_p, \mathcal{F}_p)$$

converging to $H^{p+q}(X, \mathcal{F})$.

Proof. Immediate consequence of Lemmas 25.4 and 2.10. □

26. Simplicial schemes

09XT A *simplicial scheme* is a simplicial object in the category of schemes, see Simplicial, Definition 3.1. Recall that a simplicial scheme looks like

$$X_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_0$$

Here there are two morphisms $d^1_0, d^1_1 : X_1 \to X_0$ and a single morphism $s^0_0 : X_0 \to X_1$, etc. These morphisms satisfy some required relations such as $d^1_0 \circ s^0_0 = \mathrm{id}_{X_0} = d^1_1 \circ s^0_0$, see Simplicial, Lemma 3.2. It is useful to think of $d^n_i : X_n \to X_{n-1}$ as the "projection forgetting the ith coordinate" and to think of $s_j^n : X_n \to X_{n+1}$ as the "diagonal map repeating the jth coordinate".

A *morphism of simplicial schemes* $h : X \to Y$ is the same thing as a morphism of simplicial objects in the category of schemes, see Simplicial, Definition 3.1. Thus h consists of morphisms of schemes $h_n : X_n \to Y_n$ such that $h_{n-1} \circ d_j^n = d_j^n \circ h_n$ and $h_{n+1} \circ s_j^n = s_j^n \circ h_n$ whenever this makes sense.

An *augmentation* of a simplicial scheme X is a morphism of schemes $a_0 : X_0 \to S$ such that $a_0 \circ d_0^1 = a_0 \circ d_1^1$. See Simplicial, Section 20.

Let X be a simplicial scheme. The construction of Section 2 applied to the underlying simplicial topological space gives a site X_{Zar}. On the other hand, for every n we have the small Zariski site $X_{n,Zar}$ (Topologies, Definition 3.7) and for every morphism $\varphi : [m] \to [n]$ we have a morphism of sites $f_\varphi = X(\varphi)_{small} : X_{n,Zar} \to X_{m,Zar}$, associated to the morphism of schemes $X(\varphi) : X_n \to X_m$ (Topologies, Lemma 3.17). This gives a simplicial object $\mathcal{C}$ in the category of sites. In Lemma 3.1 we constructed an associated site $\mathcal{C}_{total}$. Assigning to an open immersion its image defines an equivalence $\mathcal{C}_{total} \to X_{Zar}$ which identifies sheaves, i.e., $Sh(\mathcal{C}_{total}) = Sh(X_{Zar})$. The difference between $\mathcal{C}_{total}$ and X_{Zar} is similar to the difference between the small Zariski site S_{Zar} and the underlying topological space of S. We will silently identify these sites in what follows.

Let X_{Zar} be the site associated to a simplicial scheme X. There is a sheaf of rings $\mathcal{O}$ on X_{Zar} whose restriction to X_n is the structure sheaf $\mathcal{O}_{X_n}$. This follows from Lemma 2.2 or from Lemma 3.4. We will say *$\mathcal{O}$ is the structure sheaf of the simplicial scheme X*. At this point all the material developed for simplicial (ringed) sites applies, see Sections 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, and 14.

Let X be a simplicial scheme with structure sheaf $\mathcal{O}$. As on any ringed topos, there is a notion of a *quasi-coherent $\mathcal{O}$-module on X_{Zar}*, see Modules on Sites, Definition 23.1. However, a quasi-coherent $\mathcal{O}$-module on X_{Zar} is just a cartesian $\mathcal{O}$-module $\mathcal{F}$ whose restrictions $\mathcal{F}_n$ are quasi-coherent on X_n, see Lemma 12.10.

Let $h : X \to Y$ be a morphism of simplicial schemes. Either by Lemma 2.3 or by (the proof of) Lemma 5.2 we obtain a morphism of sites $h_{Zar} : X_{Zar} \to Y_{Zar}$. Recall that h_{Zar}^{-1} and $h_{Zar,*}$ have a simple description in terms of the components, see Lemma 2.4 or Lemma 5.2. Let $\mathcal{O}_X$, resp. $\mathcal{O}_Y$ denote the structure sheaf of X, resp. Y. We define $h_{Zar}^\sharp : h_{Zar,*}\mathcal{O}_X \to \mathcal{O}_Y$ to be the map of sheaves of rings on Y_{Zar} given by $h_n^\sharp : h_{n,*}\mathcal{O}_{X_n} \to \mathcal{O}_{Y_n}$ on Y_n. We obtain a morphism of ringed sites

$$h_{Zar} : (X_{Zar}, \mathcal{O}_X) \longrightarrow (Y_{Zar}, \mathcal{O}_Y)$$

Let X be a simplicial scheme with structure sheaf $\mathcal{O}$. Let S be a scheme and let $a_0 : X_0 \to S$ be an augmentation of X. Either by Lemma 2.8 or by Lemma 4.2 we obtain a corresponding morphism of topoi $a : Sh(X_{Zar}) \to Sh(S)$. Observe that $a^{-1}\mathcal{G}$ is the sheaf on X_{Zar} with components $a_n^{-1}\mathcal{G}$. Hence we can use the maps $a_n^\sharp : a_n^{-1}\mathcal{O}_S \to \mathcal{O}_{X_n}$ to define a map $a^\sharp : a^{-1}\mathcal{O}_S \to \mathcal{O}$, or equivalently by adjunction a map $a^\sharp : \mathcal{O}_S \to a_*\mathcal{O}$ (which as usual has the same name). This puts us in the situation discussed in Section 11. Therefore we obtain a morphism of ringed topoi

$$a : (Sh(X_{Zar}), \mathcal{O}) \longrightarrow (Sh(S), \mathcal{O}_S)$$

A final observation is the following. Suppose we are given a morphism $h : X \to Y$ of simplicial schemes X and Y with structure sheaves $\mathcal{O}_X$, $\mathcal{O}_Y$, augmentations $a_0 : X_0 \to X_{-1}$, $b_0 : Y_0 \to Y_{-1}$ and a morphism $h_{-1} : X_{-1} \to Y_{-1}$ such that

$$\begin{tikzcd} X_0 \arrow[r, "h_0"'] \arrow[d, "a_0"'] & Y_0 \arrow[d, "b_0"] \\ X_{-1} \arrow[r, "h_{-1}"] & Y_{-1} \end{tikzcd}$$

commutes. Then from the constructions elucidated above we obtain a commutative diagram of morphisms of ringed topoi as follows

$$\begin{tikzcd} (Sh(X_{Zar}), \mathcal{O}_X) \arrow[r, "h_{Zar}"'] \arrow[d, "a"'] & (Sh(Y_{Zar}), \mathcal{O}_Y) \arrow[d, "b"] \\ (Sh(X_{-1}), \mathcal{O}_{X_{-1}}) \arrow[r, "h_{-1}"] & (Sh(Y_{-1}), \mathcal{O}_{Y_{-1}}) \end{tikzcd}$$

27. Descent in terms of simplicial schemes

0248 Cartesian morphisms are defined as follows.

0249 **Definition 27.1.** Let $a : Y \to X$ be a morphism of simplicial schemes. We say a is *cartesian*, or that Y *is cartesian over* X, if for every morphism $\varphi : [n] \to [m]$ of Δ the corresponding diagram

$$\begin{tikzcd} Y_m \arrow[r, "a"'] \arrow[d, "Y(\varphi)"'] & X_m \arrow[d, "X(\varphi)"] \\ Y_n \arrow[r, "a"] & X_n \end{tikzcd}$$

is a fibre square in the category of schemes.

Cartesian morphisms are related to descent data. First we prove a general lemma describing the category of cartesian simplicial schemes over a fixed simplicial scheme. In this lemma we denote $f^* : Sch/X \to Sch/Y$ the base change functor associated to a morphism of schemes $f : Y \to X$.

07TC **Lemma 27.2.** *Let X be a simplicial scheme. The category of simplicial schemes cartesian over X is equivalent to the category of pairs (V, φ) where V is a scheme over X_0 and*

$$\varphi : V \times_{X_0, d_1^1} X_1 \longrightarrow X_1 \times_{d_0^1, X_0} V$$

is an isomorphism over X_1 such that $(s_0^0)^*\varphi = id_V$ and such that

$$(d_1^2)^*\varphi = (d_0^2)^*\varphi \circ (d_2^2)^*\varphi$$

as morphisms of schemes over X_2.

Proof. The statement of the displayed equality makes sense because $d_1^1 \circ d_2^2 = d_1^1 \circ d_1^2$, $d_1^1 \circ d_0^2 = d_0^1 \circ d_2^2$, and $d_0^1 \circ d_0^2 = d_0^1 \circ d_1^2$ as morphisms $X_2 \to X_0$, see

Simplicial, Remark 3.3 hence we can picture these maps as follows

$$\begin{tikzcd}
 & X_2 \times_{d_1^1 \circ d_0^2, X_0} V \arrow[r, "(d_0^2)^*\varphi"'] \arrow[dl, equal] & X_2 \times_{d_0^1 \circ d_0^2, X_0} V \arrow[dr, equal] & \\
X_2 \times_{d_0^1 \circ d_2^2, X_0} V & & & X_2 \times_{d_0^1 \circ d_1^2, X_0} V \\
 & X_2 \times_{d_1^1 \circ d_2^2, X_0} V \arrow[ul, "(d_2^2)^*\varphi"] \arrow[r, equal] & X_2 \times_{d_1^1 \circ d_1^2, X_0} V \arrow[ur, "(d_1^2)^*\varphi"'] &
\end{tikzcd}$$

and the condition signifies the diagram is commutative. It is clear that given a simplicial scheme Y cartesian over X we can set $V = Y_0$ and φ equal to the composition

$$V \times_{X_0, d_1^1} X_1 = Y_0 \times_{X_0, d_1^1} X_1 = Y_1 = X_1 \times_{X_0, d_0^1} Y_0 = X_1 \times_{X_0, d_0^1} V$$

of identifications given by the cartesian structure. To prove this functor is an equivalence we construct a quasi-inverse. The construction of the quasi-inverse is analogous to the construction discussed in Descent, Section 3 from which we borrow the notation $\tau_i^n : [0] \to [n]$, $0 \mapsto i$ and $\tau_{ij}^n : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Namely, given a pair (V, φ) as in the lemma we set $Y_n = X_n \times_{X(\tau_n^n), X_0} V$. Then given $\beta : [n] \to [m]$ we define $V(\beta) : Y_m \to Y_n$ as the pullback by $X(\tau_{\beta(n)m}^m)$ of the map φ postcomposed by the projection $X_m \times_{X(\beta), X_n} Y_n \to Y_n$. This makes sense because

$$X_m \times_{X(\tau_{\beta(n)m}^m), X_1} X_1 \times_{d_1^1, X_0} V = X_m \times_{X(\tau_m^m), X_0} V = Y_m$$

and

$$X_m \times_{X(\tau_{\beta(n)m}^m), X_1} X_1 \times_{d_0^1, X_0} V = X_m \times_{X(\tau_{\beta(n)}^m), X_0} V = X_m \times_{X(\beta), X_n} Y_n.$$

We omit the verification that the commutativity of the displayed diagram above implies the maps compose correctly. We also omit the verification that the two functors are quasi-inverse to each other. $\square$

024A **Definition 27.3.** Let $f : X \to S$ be a morphism of schemes. The *simplicial scheme associated to* f, denoted $(X/S)_\bullet$, is the functor $\Delta^{opp} \to Sch$, $[n] \mapsto X \times_S \ldots \times_S X$ described in Simplicial, Example 3.5.

Thus $(X/S)_n$ is the $(n+1)$-fold fibre product of X over S. The morphism $d_0^1 : X \times_S X \to X$ is the map $(x_0, x_1) \mapsto x_1$ and the morphism d_1^1 is the other projection. The morphism s_0^0 is the diagonal morphism $X \to X \times_S X$.

024B **Lemma 27.4.** *Let $f : X \to S$ be a morphism of schemes. Let $\pi : Y \to (X/S)_\bullet$ be a cartesian morphism of simplicial schemes. Set $V = Y_0$ considered as a scheme over X. The morphisms $d_0^1, d_1^1 : Y_1 \to Y_0$ and the morphism $\pi_1 : Y_1 \to X \times_S X$ induce isomorphisms*

$$V \times_S X \xleftarrow{(d_1^1, pr_1 \circ \pi_1)} Y_1 \xrightarrow{(pr_0 \circ \pi_1, d_0^1)} X \times_S V.$$

Denote $\varphi : V \times_S X \to X \times_S V$ the resulting isomorphism. Then the pair (V, φ) is a descent datum relative to $X \to S$.

Proof. This is a special case of (part of) Lemma 27.2 as the displayed equation of that lemma is equivalent to the cocycle condition of Descent, Definition 34.1. $\square$

024C **Lemma 27.5.** *Let $f : X \to S$ be a morphism of schemes. The construction*

$$\begin{matrix}\textit{category of cartesian} \\ \textit{schemes over } (X/S)_\bullet\end{matrix} \longrightarrow \begin{matrix}\textit{category of descent data} \\ \textit{relative to } X/S\end{matrix}$$

of Lemma 27.4 is an equivalence of categories.

Proof. The functor from left to right is given in Lemma 27.4. Hence this is a special case of Lemma 27.2. $\square$

We may reinterpret the pullback of Descent, Lemma 34.6 as follows. Suppose given a morphism of simplicial schemes $f : X' \to X$ and a cartesian morphism of simplicial schemes $Y \to X$. Then the fibre product (viewed as a "pullback")

$$f^*Y = Y \times_X X'$$

of simplicial schemes is a simplicial scheme cartesian over X'. Suppose given a commutative diagram of morphisms of schemes

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S. \end{tikzcd}$$

This gives rise to a morphism of simplicial schemes

$$f_\bullet : (X'/S')_\bullet \longrightarrow (X/S)_\bullet.$$

We claim that the "pullback" $f_\bullet^*$ along the morphism $f_\bullet : (X'/S')_\bullet \to (X/S)_\bullet$ corresponds via Lemma 27.5 with the pullback defined in terms of descent data in the aforementioned Descent, Lemma 34.6.

28. Quasi-coherent modules on simplicial schemes

07TE

07TI **Lemma 28.1.** *Let $f : V \to U$ be a morphism of simplicial schemes. Given a quasi-coherent module $\mathcal{F}$ on U_{Zar} the pullback $f^*\mathcal{F}$ is a quasi-coherent module on V_{Zar}.*

Proof. Recall that $\mathcal{F}$ is cartesian with $\mathcal{F}_n$ quasi-coherent, see Lemma 12.10. By Lemma 2.4 we see that $(f^*\mathcal{F})_n = f_n^*\mathcal{F}_n$ (some details omitted). Hence $(f^*\mathcal{F})_n$ is quasi-coherent. The same fact and the cartesian property for $\mathcal{F}$ imply the cartesian property for $f^*\mathcal{F}$. Thus $\mathcal{F}$ is quasi-coherent by Lemma 12.10 again. $\square$

07TJ **Lemma 28.2.** *Let $f : V \to U$ be a cartesian morphism of simplicial schemes. Assume the morphisms $d^n_j : U_n \to U_{n-1}$ are flat and the morphisms $V_n \to U_n$ are quasi-compact and quasi-separated. For a quasi-coherent module $\mathcal{G}$ on V_{Zar} the pushforward $f_*\mathcal{G}$ is a quasi-coherent module on U_{Zar}.*

Proof. If $\mathcal{F} = f_*\mathcal{G}$, then $\mathcal{F}_n = f_{n,*}\mathcal{G}_n$ by Lemma 2.4. The maps $\mathcal{F}(\varphi)$ are defined using the base change maps, see Cohomology, Section 17. The sheaves $\mathcal{F}_n$ are quasi-coherent by Schemes, Lemma 24.1 and the fact that $\mathcal{G}_n$ is quasi-coherent by Lemma 12.10. The base change maps along the degeneracies d^n_j are isomorphisms by Cohomology of Schemes, Lemma 5.2 and the fact that $\mathcal{G}$ is cartesian by Lemma 12.10. Hence $\mathcal{F}$ is cartesian by Lemma 12.2. Thus $\mathcal{F}$ is quasi-coherent by Lemma 12.10. $\square$

07TK **Lemma 28.3.** *Let $f : V \to U$ be a cartesian morphism of simplicial schemes. Assume the morphisms $d^n_j : U_n \to U_{n-1}$ are flat and the morphisms $V_n \to U_n$ are quasi-compact and quasi-separated. Then f^* and f_* form an adjoint pair of functors between the categories of quasi-coherent modules on U_{Zar} and V_{Zar}.*

Proof. We have seen in Lemmas 28.1 and 28.2 that the statement makes sense. The adjointness property follows immediately from the fact that each f_n^* is adjoint to $f_{n,*}$. □

07TL **Lemma 28.4.** *Let $f : X \to S$ be a morphism of schemes which has a section*[6]. *Let $(X/S)_\bullet$ be the simplicial scheme associated to $X \to S$, see Definition 27.3. Then pullback defines an equivalence between the category of quasi-coherent $\mathcal{O}_S$-modules and the category of quasi-coherent modules on $((X/S)_\bullet)_{Zar}$.*

Proof. Let $\sigma : S \to X$ be a section of f. Let $(\mathcal{F}, \alpha)$ be a pair as in Lemma 12.5. Set $\mathcal{G} = \sigma^*\mathcal{F}$. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "(\sigma \circ f, 1)"'] \arrow[d, "f"'] & X \times_S X \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_0"] & X \\ S \arrow[r, "\sigma"] & X & \end{tikzcd}$$

Note that $\mathrm{pr}_0 = d^1_1$ and $\mathrm{pr}_1 = d^1_0$. Hence we see that $(\sigma \circ f, 1)^*\alpha$ defines an isomorphism

$$f^*\mathcal{G} = (\sigma \circ f, 1)^*\mathrm{pr}_0^*\mathcal{F} \longrightarrow (\sigma \circ f, 1)^*\mathrm{pr}_1^*\mathcal{F} = \mathcal{F}$$

We omit the verification that this isomorphism is compatible with α and the canonical isomorphism $\mathrm{pr}_0^* f^*\mathcal{G} \to \mathrm{pr}_1^* f^*\mathcal{G}$. □

29. Groupoids and simplicial schemes

07TM Given a groupoid in schemes we can build a simplicial scheme. It will turn out that the category of quasi-coherent sheaves on a groupoid is equivalent to the category of cartesian quasi-coherent sheaves on the associated simplicial scheme.

07TN **Lemma 29.1.** *Let (U, R, s, t, c, e, i) be a groupoid scheme over S. There exists a simplicial scheme X over S with the following properties*

(1) $X_0 = U$, $X_1 = R$, $X_2 = R \times_{s,U,t} R$,
(2) $s^0_0 = e : X_0 \to X_1$,
(3) $d^1_0 = s : X_1 \to X_0$, $d^1_1 = t : X_1 \to X_0$,
(4) $s^1_0 = (e \circ t, 1) : X_1 \to X_2$, $s^1_1 = (1, e \circ t) : X_1 \to X_2$,
(5) $d^2_0 = pr_1 : X_2 \to X_1$, $d^2_1 = c : X_2 \to X_1$, $d^2_2 = pr_0$, *and*
(6) $X = cosk_2 sk_2 X$.

For all n we have $X_n = R \times_{s,U,t} \ldots \times_{s,U,t} R$ with n factors. The map $d^n_j : X_n \to X_{n-1}$ is given on functors of points by

$$(r_1, \ldots, r_n) \longmapsto (r_1, \ldots, c(r_j, r_{j+1}), \ldots, r_n)$$

for $1 \leq j \leq n-1$ whereas $d^n_0(r_1, \ldots, r_n) = (r_2, \ldots, r_n)$ and $d^n_n(r_1, \ldots, r_n) = (r_1, \ldots, r_{n-1})$.

[6] In fact, it would be enough to assume that f has fpqc locally on S a section, since we have descent of quasi-coherent modules by Descent, Section 5.

Proof. We only have to verify that the rules prescribed in (1), (2), (3), (4), (5) define a 2-truncated simplicial scheme U' over S, since then (6) allows us to set $X = \text{cosk}_2 U'$, see Simplicial, Lemma 19.2. Using the functor of points approach, all we have to verify is that if $(\text{Ob}, \text{Arrows}, s, t, c, e, i)$ is a groupoid, then

$$\begin{tikzcd}
\text{Arrows} \times_{s,\text{Ob},t} \text{Arrows} \arrow[d, shift right=6, "\text{pr}_1"'] \arrow[d, shift right=2, leftarrow, "{1,e}"'] \arrow[d, "c"] \arrow[d, shift left=2, leftarrow, "{e,1}"] \arrow[d, shift left=6, "\text{pr}_0"] \\
\text{Arrows} \arrow[d, shift right=3, "s"'] \arrow[d, leftarrow, "e"] \arrow[d, shift left=3, "t"] \\
\text{Ob}
\end{tikzcd}$$

is a 2-truncated simplicial set. We omit the details.

Finally, the description of X_n for $n > 2$ follows by induction from the description of X_0, X_1, X_2, and Simplicial, Remark 19.9 and Lemma 19.6. Alternately, one shows that cosk_2 applied to the 2-truncated simplicial set displayed above gives a simplicial set whose nth term equals $\text{Arrows} \times_{s,\text{Ob},t} \ldots \times_{s,\text{Ob},t} \text{Arrows}$ with n factors and degeneracy maps as given in the lemma. Some details omitted. $\square$

07TP **Lemma 29.2.** *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let X be the simplicial scheme over S constructed in Lemma 29.1. Then the category of quasi-coherent modules on (U, R, s, t, c) is equivalent to the category of quasi-coherent modules on X_{Zar}.*

Proof. This is clear from Lemmas 12.10 and 12.5 and Groupoids, Definition 14.1. $\square$

In the following lemma we will use the concept of a cartesian morphism $V \to U$ of simplicial schemes as defined in Definition 27.1.

07TQ **Lemma 29.3.** *Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Let X be the simplicial scheme over S constructed in Lemma 29.1. Let $(R/U)_\bullet$ be the simplicial scheme associated to $s : R \to U$, see Definition 27.3. There exists a cartesian morphism $t_\bullet : (R/U)_\bullet \to X$ of simplicial schemes with low degree morphisms given by*

$$\begin{tikzcd}
R \times_{s,U,s} R \times_{s,U,s} R \arrow[r, shift left=3, "pr_{12}"] \arrow[r, "pr_{02}"] \arrow[r, shift right=3, "pr_{01}"'] \arrow[d, "{(r_0,r_1,r_2) \mapsto (r_0 \circ r_1^{-1}, r_1 \circ r_2^{-1})}"'] & R \times_{s,U,s} R \arrow[r, shift left=2, "pr_1"] \arrow[r, shift right=2, "pr_0"'] \arrow[d, "{(r_0,r_1) \mapsto r_0 \circ r_1^{-1}}"'] & R \arrow[d, "t"] \\
R \times_{s,U,t} R \arrow[r, shift left=3, "pr_1"] \arrow[r, "c"] \arrow[r, shift right=3, "pr_0"'] & R \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & U
\end{tikzcd}$$

Proof. For arbitrary n we define $(R/U)_\bullet \to X_n$ by the rule

$$(r_0, \ldots, r_n) \longrightarrow (r_0 \circ r_1^{-1}, \ldots, r_{n-1} \circ r_n^{-1})$$

Compatibility with degeneracy maps is clear from the description of the degeneracies in Lemma 29.1. We omit the verification that the maps respect the morphisms s^n_j. Groupoids, Lemma 13.5 (with the roles of s and t reversed) shows that the two right squares are cartesian. In exactly the same manner one shows all the other squares are cartesian too. Hence the morphism is cartesian. □

30. Descent data give equivalence relations

024D In Section 27 we saw how descent data relative to $X \to S$ can be formulated in terms of cartesian simplicial schemes over $(X/S)_\bullet$. Here we link this to equivalence relations as follows.

024E **Lemma 30.1.** *Let $f : X \to S$ be a morphism of schemes. Let $\pi : Y \to (X/S)_\bullet$ be a cartesian morphism of simplicial schemes, see Definitions 27.1 and 27.3. Then the morphism*

$$j = (d^1_1, d^1_0) : Y_1 \to Y_0 \times_S Y_0$$

defines an equivalence relation on Y_0 over S, see Groupoids, Definition 3.1.

Proof. Note that j is a monomorphism. Namely the composition $Y_1 \to Y_0 \times_S Y_0 \to Y_0 \times_S X$ is an isomorphism as π is cartesian.

Consider the morphism

$$(d^2_2, d^2_0) : Y_2 \to Y_1 \times_{d^1_0, Y_0, d^1_1} Y_1.$$

This works because $d_0 \circ d_2 = d_1 \circ d_0$, see Simplicial, Remark 3.3. Also, it is a morphism over $(X/S)_2$. It is an isomorphism because $Y \to (X/S)_\bullet$ is cartesian. Note for example that the right hand side is isomorphic to $Y_0 \times_{\pi_0, X, \mathrm{pr}_1} (X \times_S X \times_S X) = X \times_S Y_0 \times_S X$ because π is cartesian. Details omitted.

As in Groupoids, Definition 3.1 we denote $t = \mathrm{pr}_0 \circ j = d^1_1$ and $s = \mathrm{pr}_1 \circ j = d^1_0$. The isomorphism above, combined with the morphism $d^2_1 : Y_2 \to Y_1$ give us a composition morphism

$$c : Y_1 \times_{s, Y_0, t} Y_1 \longrightarrow Y_1$$

over $Y_0 \times_S Y_0$. This immediately implies that for any scheme T/S the relation $Y_1(T) \subset Y_0(T) \times Y_0(T)$ is transitive.

Reflexivity follows from the fact that the restriction of the morphism j to the diagonal $\Delta : X \to X \times_S X$ is an isomorphism (again use the cartesian property of π).

To see symmetry we consider the morphism

$$(d^2_2, d^2_1) : Y_2 \to Y_1 \times_{d^1_1, Y_0, d^1_1} Y_1.$$

This works because $d_1 \circ d_2 = d_1 \circ d_1$, see Simplicial, Remark 3.3. It is an isomorphism because $Y \to (X/S)_\bullet$ is cartesian. Note for example that the right hand side is isomorphic to $Y_0 \times_{\pi_0, X, \mathrm{pr}_0} (X \times_S X \times_S X) = Y_0 \times_S X \times_S X$ because π is cartesian. Details omitted.

Let T/S be a scheme. Let $a \sim b$ for $a, b \in Y_0(T)$ be synonymous with $(a, b) \in Y_1(T)$. The isomorphism (d^2_2, d^2_1) above implies that if $a \sim b$ and $a \sim c$, then $b \sim c$. Combined with reflexivity this shows that $\sim$ is an equivalence relation. □

31. An example case

024F In this section we show that disjoint unions of spectra of Artinian rings can be descended along a quasi-compact surjective flat morphism of schemes.

024G **Lemma 31.1.** *Let $X \to S$ be a morphism of schemes. Suppose $Y \to (X/S)_\bullet$ is a cartesian morphism of simplicial schemes. For $y \in Y_0$ a point define*

$$T_y = \{y' \in Y_0 \mid \exists\ y_1 \in Y_1 : d^1_1(y_1) = y, d^1_0(y_1) = y'\}$$

as a subset of Y_0. Then $y \in T_y$ and $T_y \cap T_{y'} \neq \emptyset \Rightarrow T_y = T_{y'}$.

Proof. Combine Lemma 30.1 and Groupoids, Lemma 3.4. $\square$

024H **Lemma 31.2.** *Let $X \to S$ be a morphism of schemes. Suppose $Y \to (X/S)_\bullet$ is a cartesian morphism of simplicial schemes. Let $y \in Y_0$ be a point. If $X \to S$ is quasi-compact, then*

$$T_y = \{y' \in Y_0 \mid \exists\ y_1 \in Y_1 : d^1_1(y_1) = y, d^1_0(y_1) = y'\}$$

is a quasi-compact subset of Y_0.

Proof. Let F_y be the scheme theoretic fibre of $d^1_1 : Y_1 \to Y_0$ at y. Then we see that T_y is the image of the morphism

$$\begin{tikzcd} F_y \arrow[r] \arrow[d] & Y_1 \arrow[r, "d^1_0"] \arrow[d, "d^1_1"] & Y_0 \\ y \arrow[r] & Y_0 & \end{tikzcd}$$

Note that F_y is quasi-compact. This proves the lemma. $\square$

024I **Lemma 31.3.** *Let $X \to S$ be a quasi-compact flat surjective morphism. Let (V, φ) be a descent datum relative to $X \to S$. If V is a disjoint union of spectra of Artinian rings, then (V, φ) is effective.*

Proof. Let $Y \to (X/S)_\bullet$ be the cartesian morphism of simplicial schemes corresponding to (V, φ) by Lemma 27.5. Observe that $Y_0 = V$. Write $V = \coprod_{i \in I} \mathrm{Spec}(A_i)$ with each A_i local Artinian. Moreover, let $v_i \in V$ be the unique closed point of $\mathrm{Spec}(A_i)$ for all $i \in I$. Write $i \sim j$ if and only if $v_i \in T_{v_j}$ with notation as in Lemma 31.1 above. By Lemmas 31.1 and 31.2 this is an equivalence relation with finite equivalence classes. Let $\overline{I} = I/\sim$. Then we can write $V = \coprod_{\overline{i} \in \overline{I}} V_{\overline{i}}$ with $V_{\overline{i}} = \coprod_{i \in \overline{i}} \mathrm{Spec}(A_i)$. By construction we see that $\varphi : V \times_S X \to X \times_S V$ maps the open and closed subspaces $V_{\overline{i}} \times_S X$ into the open and closed subspaces $X \times_S V_{\overline{i}}$. In other words, we get descent data $(V_{\overline{i}}, \varphi_{\overline{i}})$, and (V, φ) is the coproduct of them in the category of descent data. Since each of the $V_{\overline{i}}$ is a finite union of spectra of Artinian local rings the morphism $V_{\overline{i}} \to X$ is affine, see Morphisms, Lemma 11.13. Since $\{X \to S\}$ is an fpqc covering we see that all the descent data $(V_{\overline{i}}, \varphi_{\overline{i}})$ are effective by Descent, Lemma 37.1. $\square$

To be sure, the lemma above has very limited applicability!

32. Simplicial algebraic spaces

0DE7 Let S be a scheme. A *simplicial algebraic space* is a simplicial object in the category of algebraic spaces over S, see Simplicial, Definition 3.1. Recall that a simplicial algebraic space looks like

$$X_2 \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; X_1 \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; X_0$$

Here there are two morphisms $d^1_0, d^1_1 : X_1 \to X_0$ and a single morphism $s^0_0 : X_0 \to X_1$, etc. These morphisms satisfy some required relations such as $d^1_0 \circ s^0_0 = \mathrm{id}_{X_0} = d^1_1 \circ s^0_0$, see Simplicial, Lemma 3.2. It is useful to think of $d^n_i : X_n \to X_{n-1}$ as the "projection forgetting the ith coordinate" and to think of $s^n_j : X_n \to X_{n+1}$ as the "diagonal map repeating the jth coordinate".

A *morphism of simplicial algebraic spaces* $h : X \to Y$ is the same thing as a morphism of simplicial objects in the category of algebraic spaces over S, see Simplicial, Definition 3.1. Thus h consists of morphisms of algebraic spaces $h_n : X_n \to Y_n$ such that $h_{n-1} \circ d^n_j = d^n_j \circ h_n$ and $h_{n+1} \circ s^n_j = s^n_j \circ h_n$ whenever this makes sense.

An *augmentation* $a : X \to X_{-1}$ of a simplicial algebraic space X is given by a morphism of algebraic spaces $a_0 : X_0 \to X_{-1}$ such that $a_0 \circ d^1_0 = a_0 \circ d^1_1$. See Simplicial, Section 20. In this situation we always indicate $a_n : X_n \to X_{-1}$ the induced morphisms for $n \geq 0$.

Let X be a simplicial algebraic space. For every n we have the site $X_{n,spaces,\acute{e}tale}$ (Properties of Spaces, Definition 18.2) and for every morphism $\varphi : [m] \to [n]$ we have a morphism of sites

$$f_\varphi = X(\varphi)_{spaces,\acute{e}tale} : X_{n,spaces,\acute{e}tale} \to X_{m,spaces,\acute{e}tale},$$

associated to the morphism of algebraic spaces $X(\varphi) : X_n \to X_m$ (Properties of Spaces, Lemma 18.8). This gives a simplicial object in the category of sites. In Lemma 3.1 we constructed an associated site which we denote $X_{spaces,\acute{e}tale}$. An object of the site $X_{spaces,\acute{e}tale}$ is a an algebraic space U étale over X_n for some n and a morphism $(\varphi, f) : U/X_n \to V/X_m$ is given by a morphism $\varphi : [m] \to [n]$ in Δ and a morphism $f : U \to V$ of algebraic spaces such that the diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d] & V \arrow[d] \\ X_n \arrow[r, "f_\varphi"] & X_m \end{tikzcd}$$

is commutative. Consider the full subcategories

$$X_{affine,\acute{e}tale} \subset X_{\acute{e}tale} \subset X_{spaces,\acute{e}tale}$$

whose objects are U/X_n with U affine, respectively a scheme. Endowing these categories with their natural topologies (see Properties of Spaces, Lemma 18.6, Definition 18.1, and Lemma 18.3) these inclusion functors define equivalences of topoi

$$Sh(X_{affine,\acute{e}tale}) = Sh(X_{\acute{e}tale}) = Sh(X_{spaces,\acute{e}tale})$$

In the following we will silently identify these topoi. We will say that $X_{\acute{e}tale}$ is the *small étale site of* X and its topos is the *small étale topos of* X.

Let $X_{\acute{e}tale}$ be the small étale site of a simplicial algebraic space X. There is a sheaf of rings $\mathcal{O}$ on $X_{\acute{e}tale}$ whose restriction to X_n is the structure sheaf $\mathcal{O}_{X_n}$. This follows from Lemma 3.4. We will say $\mathcal{O}$ *is the structure sheaf of the simplicial algebraic space* X. At this point all the material developed for simplicial (ringed) sites applies, see Sections 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, and 14.

Let X be a simplicial algebraic space with structure sheaf $\mathcal{O}$. As on any ringed topos, there is a notion of a *quasi-coherent* $\mathcal{O}$*-module on* $X_{étale}$, see Modules on Sites, Definition 23.1. However, a quasi-coherent $\mathcal{O}$-module on $X_{étale}$ is just a cartesian $\mathcal{O}$-module $\mathcal{F}$ whose restrictions $\mathcal{F}_n$ are quasi-coherent on X_n, see Lemma 12.10.

Let $h : X \to Y$ be a morphism of simplicial algebraic spaces over S. By Lemma 5.2 applied to the morphisms of sites $(h_n)_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}$ (Properties of Spaces, Lemma 18.8) we obtain a morphism of small étale topoi $h_{étale} : Sh(X_{étale}) \to Sh(Y_{étale})$. Recall that $h_{étale}^{-1}$ and $h_{étale,*}$ have a simple description in terms of the components, see Lemma 5.2. Let $\mathcal{O}_X$, resp. $\mathcal{O}_Y$ denote the structure sheaf of X, resp. Y. We define $h_{étale}^\sharp : h_{étale,*}\mathcal{O}_X \to \mathcal{O}_Y$ to be the map of sheaves of rings on $Y_{étale}$ given by $h_n^\sharp : h_{n,*}\mathcal{O}_{X_n} \to \mathcal{O}_{Y_n}$ on Y_n. We obtain a morphism of ringed topoi

$$h_{étale} : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

Let X be a simplicial algebraic space with structure sheaf $\mathcal{O}$. Let X_{-1} be an algebraic space over S and let $a_0 : X_0 \to X_{-1}$ be an augmentation of X. By Lemma 4.2 applied to the morphism of sites $(a_0)_{spaces,étale} : X_{0,spaces,étale} \to X_{-1,spaces,étale}$ we obtain a corresponding morphism of topoi $a : Sh(X_{étale}) \to Sh(X_{-1,étale})$. Observe that $a^{-1}\mathcal{G}$ is the sheaf on $X_{étale}$ with components $a_n^{-1}\mathcal{G}$. Hence we can use the maps $a_n^\sharp : a_n^{-1}\mathcal{O}_{X_{-1}} \to \mathcal{O}_{X_n}$ to define a map $a^\sharp : a^{-1}\mathcal{O}_{X_{-1}} \to \mathcal{O}$, or equivalently by adjunction a map $a^\sharp : \mathcal{O}_{X_{-1}} \to a_*\mathcal{O}$ (which as usual has the same name). This puts us in the situation discussed in Section 11. Therefore we obtain a morphism of ringed topoi

$$a : (Sh(X_{étale}), \mathcal{O}) \longrightarrow (Sh(X_{-1}), \mathcal{O}_{X_{-1}})$$

A final observation is the following. Suppose we are given a morphism $h : X \to Y$ of simplicial algebraic spaces X and Y with structure sheaves $\mathcal{O}_X$, $\mathcal{O}_Y$, augmentations $a_0 : X_0 \to X_{-1}$, $b_0 : Y_0 \to Y_{-1}$ and a morphism $h_{-1} : X_{-1} \to Y_{-1}$ such that

$$\begin{tikzcd} X_0 \arrow[r, "h_0"'] \arrow[d, "a_0"'] & Y_0 \arrow[d, "b_0"] \\ X_{-1} \arrow[r, "h_{-1}"] & Y_{-1} \end{tikzcd}$$

commutes. Then from the constructions elucidated above we obtain a commutative diagram of morphisms of ringed topoi as follows

$$\begin{tikzcd} (Sh(X_{étale}), \mathcal{O}_X) \arrow[r, "h_{étale}"'] \arrow[d, "a"'] & (Sh(Y_{étale}), \mathcal{O}_Y) \arrow[d, "b"] \\ (Sh(X_{-1}), \mathcal{O}_{X_{-1}}) \arrow[r, "h_{-1}"] & (Sh(Y_{-1}), \mathcal{O}_{Y_{-1}}) \end{tikzcd}$$

33. Fppf hypercoverings of algebraic spaces

0DH4 This section is the analogue of Section 25 for the case of algebraic spaces and fppf hypercoverings. The reader who wishes to do so, can replace "algebraic space" everywhere with "scheme" and get equally valid results. This has the advantage of replacing the references to More on Cohomology of Spaces, Section 6 with references to Étale Cohomology, Section 100.

We fix a base scheme S. Let X be an algebraic space over S and let U be a simplicial algebraic space over S. Assume we have an augmentation

$$a : U \to X$$

See Section 32. We say that U is an *fppf hypercovering* of X if

(1) $U_0 \to X$ is flat, locally of finite presentation, and surjective,
(2) $U_1 \to U_0 \times_X U_0$ is flat, locally of finite presentation, and surjective,
(3) $U_{n+1} \to (\text{cosk}_n \text{sk}_n U)_{n+1}$ is flat, locally of finite presentation, and surjective for $n \geq 1$.

The category of algebraic spaces over S has all finite limits, hence the coskeleta used in the formulation above exist.

$$\boxed{\text{Principle: Fppf hypercoverings can be used to compute étale cohomology.}}$$

The key idea behind the proof of the principle is to compare the fppf and étale topologies on the category *Spaces*/S. Namely, the fppf topology is stronger than the étale topology and we have (a) a flat, locally finitely presented, surjective map defines an fppf covering, and (b) fppf cohomology of sheaves pulled back from the small étale site agrees with étale cohomology as we have seen in More on Cohomology of Spaces, Section 6.

0DH5 **Lemma 33.1.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. There is a commutative diagram*

$$\begin{tikzcd} Sh((Spaces/U)_{fppf,total}) \arrow[r, "h"'] \arrow[d, "a_{fppf}"'] & Sh(U_{\acute{e}tale}) \arrow[d, "a"] \\ Sh((Spaces/X)_{fppf}) \arrow[r, "h_{-1}"] & Sh(X_{\acute{e}tale}) \end{tikzcd}$$

where the left vertical arrow is defined in Section 21 and the right vertical arrow is defined in Section 32.

Proof. The notation $(Spaces/U)_{fppf,total}$ indicates that we are using the construction of Section 21 for the site $(Spaces/S)_{fppf}$ and the simplicial object U of this site[7]. We will use the sites $X_{spaces,\acute{e}tale}$ and $U_{spaces,\acute{e}tale}$ for the topoi on the right hand side; this is permissible see discussion in Section 32.

Observe that both $(Spaces/U)_{fppf,total}$ and $U_{spaces,\acute{e}tale}$ fall into case A of Situation 3.3. This is immediate from the construction of $U_{\acute{e}tale}$ in Section 32 and it follows from Lemma 21.5 for $(Spaces/U)_{fppf,total}$. Next, consider the functors $U_{n,spaces,\acute{e}tale} \to (Spaces/U_n)_{fppf}$, $U \mapsto U/U_n$ and $X_{spaces,\acute{e}tale} \to (Spaces/X)_{fppf}$, $U \mapsto U/X$. We have seen that these define morphisms of sites in More on Cohomology of Spaces, Section 6 where these were denoted $a_{U_n} = \epsilon_{U_n} \circ \pi_{u_n}$ and

[7]We could also use the étale topology and this would be denoted $(Spaces/U)_{\acute{e}tale,total}$.

$a_X = \epsilon_X \circ \pi_X$. Thus we obtain a morphism of simplicial sites compatible with augmentations as in Remark 5.4 and we may apply Lemma 5.5 to conclude. □

0DH6 **Lemma 33.2.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then*

$$a^{-1} : Sh(X_{\acute{e}tale}) \to Sh(U_{\acute{e}tale}) \quad and \quad a^{-1} : Ab(X_{\acute{e}tale}) \to Ab(U_{\acute{e}tale})$$

are fully faithful with essential image the cartesian sheaves and quasi-inverse given by a_*. Here $a : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is as in Section 32.

Proof. We will prove the statement for sheaves of sets. It will be an almost formal consequence of results already established. Consider the diagram of Lemma 33.1. In the proof of this lemma we have seen that h_{-1} is the morphism a_X of More on Cohomology of Spaces, Section 6. Thus it follows from More on Cohomology of Spaces, Lemma 6.1 that $(h_{-1})^{-1}$ is fully faithful with quasi-inverse $h_{-1,*}$. The same holds true for the components h_n of h. By the description of the functors h^{-1} and h_* of Lemma 5.2 we conclude that h^{-1} is fully faithful with quasi-inverse h_*. Observe that U is a hypercovering of X in $(Spaces/S)_{fppf}$ as defined in Section 21. By Lemma 21.1 we see that a_{fppf}^{-1} is fully faithful with quasi-inverse $a_{fppf,*}$ and with essential image the cartesian sheaves on $(Spaces/U)_{fppf,total}$. A formal argument (chasing around the diagram) now shows that a^{-1} is fully faithful.

Finally, suppose that $\mathcal{G}$ is a cartesian sheaf on $U_{\acute{e}tale}$. Then $h^{-1}\mathcal{G}$ is a cartesian sheaf on $(Spaces/U)_{fppf,total}$. Hence $h^{-1}\mathcal{G} = a_{fppf}^{-1}\mathcal{H}$ for some sheaf $\mathcal{H}$ on $(Spaces/X)_{fppf}$. In particular we find that $h_0^{-1}\mathcal{G}_0 = (a_{0,big,fppf})^{-1}\mathcal{H}$. Recalling that $h_0 = a_{U_0}$ and that $U_0 \to X$ is flat, locally of finite presentation, and surjective, we find from More on Cohomology of Spaces, Lemma 6.7 that there exists a sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ and isomorphism $\mathcal{H} = (h_{-1})^{-1}\mathcal{F}$. Since $a_{fppf}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$ we deduce that $h^{-1}\mathcal{G} \cong h^{-1}a^{-1}\mathcal{F}$. By fully faithfulness of h^{-1} we conclude that $a^{-1}\mathcal{F} \cong \mathcal{G}$.

Fix an isomorphism $\theta : a^{-1}\mathcal{F} \to \mathcal{G}$. To finish the proof we have to show $\mathcal{G} = a^{-1}a_*\mathcal{G}$ (in order to show that the quasi-inverse is given by a_*; everything else has been proven above). Because a^{-1} is fully faithful we have $\text{id} \cong a_*a^{-1}$ by Categories, Lemma 24.4. Thus $\mathcal{F} \cong a_*a^{-1}\mathcal{F}$ and $a_*\theta : a_*a^{-1}\mathcal{F} \to a_*\mathcal{G}$ combine to an isomorphism $\mathcal{F} \to a_*\mathcal{G}$. Pulling back by a and precomposing by θ^{-1} we find the desired isomorphism. □

0DH7 **Lemma 33.3.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then for $K \in D^+(X_{\acute{e}tale})$*

$$K \to Ra_*(a^{-1}K)$$

is an isomorphism. Here $a : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is as in Section 32.

Proof. Consider the diagram of Lemma 33.1. Observe that $Rh_{n,*}h_n^{-1}$ is the identity functor on $D^+(U_{n,\acute{e}tale})$ by More on Cohomology of Spaces, Lemma 6.2. Hence

Rh_*h^{-1} is the identity functor on $D^+(U_{\acute{e}tale})$ by Lemma 5.3. We have

$$\begin{aligned} Ra_*(a^{-1}K) &= Ra_*Rh_*h^{-1}a^{-1}K \\ &= Rh_{-1,*}Ra_{fppf,*}a_{fppf}^{-1}(h_{-1})^{-1}K \\ &= Rh_{-1,*}(h_{-1})^{-1}K \\ &= K \end{aligned}$$

The first equality by the discussion above, the second equality because of the commutativity of the diagram in Lemma 25.1, the third equality by Lemma 21.2 as U is a hypercovering of X in $(Spaces/S)_{fppf}$, and the last equality by the already used More on Cohomology of Spaces, Lemma 6.2. □

0DH8 **Lemma 33.4.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then*

$$R\Gamma(X_{\acute{e}tale}, K) = R\Gamma(U_{\acute{e}tale}, a^{-1}K)$$

for $K \in D^+(X_{\acute{e}tale})$. Here $a : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is as in Section 32.

Proof. This follows from Lemma 33.3 because $R\Gamma(U_{\acute{e}tale}, -) = R\Gamma(X_{\acute{e}tale}, -) \circ Ra_*$ by Cohomology on Sites, Remark 14.4. □

0DH9 **Lemma 33.5.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{A} \subset Ab(U_{\acute{e}tale})$ denote the weak Serre subcategory of cartesian abelian sheaves. If U is an fppf hypercovering of X, then the functor a^{-1} defines an equivalence*

$$D^+(X_{\acute{e}tale}) \longrightarrow D^+_{\mathcal{A}}(U_{\acute{e}tale})$$

with quasi-inverse Ra_*. Here $a : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is as in Section 32.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 12.6. The equivalence is a formal consequence of the results obtained so far. Use Lemmas 33.2 and 33.3 and Cohomology on Sites, Lemma 28.5. □

0DHA **Lemma 33.6.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Let $\mathcal{F}_n$ be the pullback to $U_{n,\acute{e}tale}$. If U is an fppf hypercovering of X, then there exists a canonical spectral sequence*

$$E_1^{p,q} = H^q_{\acute{e}tale}(U_p, \mathcal{F}_p)$$

converging to $H^{p+q}_{\acute{e}tale}(X, \mathcal{F})$.

Proof. Immediate consequence of Lemmas 33.4 and 8.3. □

34. Fppf hypercoverings of algebraic spaces: modules

0DHB We continue the discussion of (cohomological) descent for fppf hypercoverings started in Section 33 but in this section we discuss what happens for sheaves of modules. We mainly discuss quasi-coherent modules and it turns out that we can do unbounded cohomological descent for those.

0DHC **Lemma 34.1.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. There is a commutative diagram*

$$\begin{tikzcd} (Sh((Spaces/U)_{fppf,total}), \mathcal{O}_{big,total}) \arrow[r, "h"'] \arrow[d, "a_{fppf}"'] & (Sh(U_{étale}), \mathcal{O}_U) \arrow[d, "a"] \\ (Sh((Spaces/X)_{fppf}), \mathcal{O}_{big}) \arrow[r, "h_{-1}"] & (Sh(X_{étale}), \mathcal{O}_X) \end{tikzcd}$$

of ringed topoi where the left vertical arrow is defined in Section 22 and the right vertical arrow is defined in Section 32.

Proof. For the underlying diagram of topoi we refer to the discussion in the proof of Lemma 33.1. The sheaf $\mathcal{O}_U$ is the structure sheaf of the simplicial algebraic space U as defined in Section 32. The sheaf $\mathcal{O}_X$ is the usual structure sheaf of the algebraic space X. The sheaves of rings $\mathcal{O}_{big,total}$ and $\mathcal{O}_{big}$ come from the structure sheaf on $(Spaces/S)_{fppf}$ in the manner explained in Section 22 which also constructs a_{fppf} as a morphism of ringed topoi. The component morphisms $h_n = a_{U_n}$ and $h_{-1} = a_X$ are morphisms of ringed topoi by More on Cohomology of Spaces, Section 7. Finally, since the continuous functor $u : U_{spaces,étale} \to (Spaces/U)_{fppf,total}$ used to define h[8] is given by $V/U_n \mapsto V/U_n$ we see that $h_*\mathcal{O}_{big,total} = \mathcal{O}_U$ which is how we endow h with the structure of a morphism of ringed simplicial sites as in Remark 7.1. Then we obtain h as a morphism of ringed topoi by Lemma 7.2. Please observe that the morphisms h_n indeed agree with the morphisms a_{U_n} described above. We omit the verification that the diagram is commutative (as a diagram of ringed topoi – we already know it is commutative as a diagram of topoi). $\square$

0DHD **Lemma 34.2.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then*

$$a^* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$$

is an equivalence fully faithful with quasi-inverse given by a_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 32.

Proof. Consider the diagram of Lemma 34.1. In the proof of this lemma we have seen that h_{-1} is the morphism a_X of More on Cohomology of Spaces, Section 7. Thus it follows from More on Cohomology of Spaces, Lemma 7.1 that

$$(h_{-1})^* : QCoh(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_{big})$$

is an equivalence with quasi-inverse $h_{-1,*}$. The same holds true for the components h_n of h. Recall that $QCoh(\mathcal{O}_U)$ and $QCoh(\mathcal{O}_{big,total})$ consist of cartesian modules whose components are quasi-coherent, see Lemma 12.10. Since the functors h^* and h_* of Lemma 7.2 agree with the functors h_n^* and $h_{n,*}$ on components we conclude that

$$h^* : QCoh(\mathcal{O}_U) \longrightarrow QCoh(\mathcal{O}_{big,total})$$

is an equivalence with quasi-inverse h_*. Observe that U is a hypercovering of X in $(Spaces/S)_{fppf}$ as defined in Section 21. By Lemma 22.1 we see that a_{fppf}^* is fully faithful with quasi-inverse $a_{fppf,*}$ and with essential image the cartesian sheaves of

[8] This happened in the proof of Lemma 33.1 via an application of Lemma 5.5.

$\mathcal{O}_{fppf,total}$-modules. Thus, by the description of $QCoh(\mathcal{O}_{big})$ and $QCoh(\mathcal{O}_{big,total})$ of Lemma 12.10, we get an equivalence

$$a_{fppf}^* : QCoh(\mathcal{O}_{big}) \longrightarrow QCoh(\mathcal{O}_{big,total})$$

with quasi-inverse given by $a_{fppf,*}$. A formal argument (chasing around the diagram) now shows that a^* is fully faithful on $QCoh(\mathcal{O}_X)$ and has image contained in $QCoh(\mathcal{O}_U)$.

Finally, suppose that $\mathcal{G}$ is in $QCoh(\mathcal{O}_U)$. Then $h^*\mathcal{G}$ is in $QCoh(\mathcal{O}_{big,total})$. Hence $h^*\mathcal{G} = a_{fppf}^*\mathcal{H}$ with $\mathcal{H} = a_{fppf,*}h^*\mathcal{G}$ in $QCoh(\mathcal{O}_{big})$ (see above). In turn we see that $\mathcal{H} = (h_{-1})^*\mathcal{F}$ with $\mathcal{F} = h_{-1,*}\mathcal{H}$ in $QCoh(\mathcal{O}_X)$. Going around the diagram we deduce that $h^*\mathcal{G} \cong h^*a^*\mathcal{F}$. By fully faithfulness of h^* we conclude that $a^*\mathcal{F} \cong \mathcal{G}$. Since $\mathcal{F} = h_{-1,*}a_{fppf,*}h^*\mathcal{G} = a_*h_*h^*\mathcal{G} = a_*\mathcal{G}$ we also obtain the statement that the quasi-inverse is given by a_*. □

0DHE **Lemma 34.3.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then for $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module the map*

$$\mathcal{F} \to Ra_*(a^*\mathcal{F})$$

is an isomorphism. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 32.

Proof. Consider the diagram of Lemma 33.1. Let $\mathcal{F}_n = a_n^*\mathcal{F}$ be the nth component of $a^*\mathcal{F}$. This is a quasi-coherent $\mathcal{O}_{U_n}$-module. Then $\mathcal{F}_n = Rh_{n,*}h_n^*\mathcal{F}_n$ by More on Cohomology of Spaces, Lemma 7.2. Hence $a^*\mathcal{F} = Rh_*h^*a^*\mathcal{F}$ by Lemma 7.3. We have

$$\begin{aligned} Ra_*(a^*\mathcal{F}) &= Ra_*Rh_*h^*a^*\mathcal{F} \\ &= Rh_{-1,*}Ra_{fppf,*}a_{fppf}^*(h_{-1})^*\mathcal{F} \\ &= Rh_{-1,*}(h_{-1})^*\mathcal{F} \\ &= \mathcal{F} \end{aligned}$$

The first equality by the discussion above, the second equality because of the commutativity of the diagram in Lemma 25.1, the third equality by Lemma 22.2 as U is a hypercovering of X in $(Spaces/S)_{fppf}$ and $La_{fppf}^* = a_{fppf}^*$ as a_{fppf} is flat (namely $a_{fppf}^{-1}\mathcal{O}_{big} = \mathcal{O}_{big,total}$, see Remark 16.5), and the last equality by the already used More on Cohomology of Spaces, Lemma 7.2. □

0DHF **Lemma 34.4.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Assume $a : U \to X$ is an fppf hypercovering of X. Then $QCoh(\mathcal{O}_U)$ is a weak Serre subcategory of $Mod(\mathcal{O}_U)$ and*

$$a^* : D_{QCoh}(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_U)$$

is an equivalence of categories with quasi-inverse given by Ra_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 32.

Proof. First observe that the maps $a_n : U_n \to X$ and $d_i^n : U_n \to U_{n-1}$ are flat, locally of finite presentation, and surjective by Hypercoverings, Remark 8.4.

Recall that an $\mathcal{O}_U$-module $\mathcal{F}$ is quasi-coherent if and only if it is cartesian and $\mathcal{F}_n$ is quasi-coherent for all n. See Lemma 12.10. By Lemma 12.6 (and flatness of the maps $d_i^n : U_n \to U_{n-1}$ shown above) the cartesian modules for a weak Serre subcategory of $Mod(\mathcal{O}_U)$. On the other hand $QCoh(\mathcal{O}_{U_n}) \subset Mod(\mathcal{O}_{U_n})$ is a weak Serre subcategory for each n (Properties of Spaces, Lemma 29.7). Combined we see that $QCoh(\mathcal{O}_U) \subset Mod(\mathcal{O}_U)$ is a weak Serre subcategory.

To finish the proof we check the conditions (1) – (5) of Cohomology on Sites, Lemma 28.6 one by one.

Ad (1). This holds since a_n flat (seen above) implies a is flat by Lemma 11.1.

Ad (2). This is the content of Lemma 34.2.

Ad (3). This is the content of Lemma 34.3.

Ad (4). Recall that we can use either the site $U_{étale}$ or $U_{spaces,étale}$ to define the small étale topos $Sh(U_{étale})$, see Section 32. The assumption of Cohomology on Sites, Situation 25.1 holds for the triple $(U_{spaces,étale}, \mathcal{O}_U, QCoh(\mathcal{O}_U))$ and by the same reasoning for the triple $(U_{étale}, \mathcal{O}_U, QCoh(\mathcal{O}_U))$. Namely, take

$$\mathcal{B} \subset \text{Ob}(U_{étale}) \subset \text{Ob}(U_{spaces,étale})$$

to be the set of affine objects. For $V/U_n \in \mathcal{B}$ take $d_{V/U_n} = 0$ and take Cov_{V/U_n} to be the set of étale coverings $\{V_i \to V\}$ with V_i affine. Then we get the desired vanishing because for $\mathcal{F} \in QCoh(\mathcal{O}_U)$ and any $V/U_n \in \mathcal{B}$ we have

$$H^p(V/U_n, \mathcal{F}) = H^p(V, \mathcal{F}_n)$$

by Lemma 10.4. Here on the right hand side we have the cohomology of the quasi-coherent sheaf $\mathcal{F}_n$ on U_n over the affine object V of $U_{n,étale}$. This vanishes for $p > 0$ by the discussion in Cohomology of Spaces, Section 3 and Cohomology of Schemes, Lemma 2.2.

Ad (5). Follows by taking $\mathcal{B} \subset \text{Ob}(X_{spaces,étale})$ the set of affine objects and the references given above. □

0DHG **Lemma 34.5.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is an fppf hypercovering of X, then*

$$R\Gamma(X_{étale}, K) = R\Gamma(U_{étale}, a^*K)$$

for $K \in D_{QCoh}(\mathcal{O}_X)$. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 32.

Proof. This follows from Lemma 34.4 because $R\Gamma(U_{étale}, -) = R\Gamma(X_{étale}, -) \circ Ra_*$ by Cohomology on Sites, Remark 14.4. □

0DHH **Lemma 34.6.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{F}$ be quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}_n$ be the pullback to $U_{n,étale}$. If U is an fppf hypercovering of X, then there exists a canonical spectral sequence*

$$E_1^{p,q} = H^q_{étale}(U_p, \mathcal{F}_p)$$

converging to $H^{p+q}_{étale}(X, \mathcal{F})$.

Proof. Immediate consequence of Lemmas 34.5 and 10.3. □

35. Fppf descent of complexes

0DL8 In this section we pull some of the previously shown results together for fppf coverings of algebraic spaces and derived categories of quasi-coherent modules.

0DL9 **Lemma 35.1.** *Let X be an algebraic space over a scheme S. Let $K, E \in D_{QCoh}(\mathcal{O}_X)$. Let $a : U \to X$ be an fppf hypercovering. Assume that for all $n \geq 0$ we have*

$$\mathrm{Ext}^i_{\mathcal{O}_{U_n}}(La_n^*K, La_n^*E) = 0 \text{ for } i < 0$$

Then we have

(1) $\mathrm{Ext}^i_{\mathcal{O}_X}(K, E) = 0$ *for $i < 0$, and*
(2) *there is an exact sequence*

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(K, E) \to \mathrm{Hom}_{\mathcal{O}_{U_0}}(La_0^*K, La_0^*E) \to \mathrm{Hom}_{\mathcal{O}_{U_1}}(La_1^*K, La_1^*E)$$

Proof. Write $K_n = La_n^*K$ and $E_n = La_n^*E$. Then these are the simplicial systems of the derived category of modules (Definition 14.1) associated to La^*K and La^*E (Lemma 14.2) where $a : U_{\acute{e}tale} \to X_{\acute{e}tale}$ is as in Section 32. Let us prove (2) first. By Lemma 34.4 we have

$$\mathrm{Hom}_{\mathcal{O}_X}(K, E) = \mathrm{Hom}_{\mathcal{O}_U}(La^*K, La^*E)$$

Thus the sequence looks like this:

$$0 \to \mathrm{Hom}_{\mathcal{O}_U}(La^*K, La^*E) \to \mathrm{Hom}_{\mathcal{O}_{U_0}}(K_0, E_0) \to \mathrm{Hom}_{\mathcal{O}_{U_1}}(K_1, E_1)$$

The first arrow is injective by Lemma 14.5. The image of this arrow is the kernel of the second by Lemma 14.6. This finishes the proof of (2). Part (1) follows by applying part (2) with $K[i]$ and E for $i > 0$. □

0DLA **Lemma 35.2.** *Let X be an algebraic space over a scheme S. Let $a : U \to X$ be an fppf hypercovering. Suppose given $K_0 \in D_{QCoh}(U_0)$ and an isomorphism*

$$\alpha : L(f_{\delta^1_1})^*K_0 \longrightarrow L(f_{\delta^1_0})^*K_0$$

satisfying the cocycle condition on U_1. Set $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and set $K_n = Lf^*_{\tau^n_n}K_0$. Assume $\mathrm{Ext}^i_{\mathcal{O}_{U_n}}(K_n, K_n) = 0$ for $i < 0$. Then there exists an object $K \in D_{QCoh}(\mathcal{O}_X)$ and an isomorphism $La_0^*K \to K$ compatible with α.

Proof. The objects K_n form the members of a simplicial system of the derived category of modules by Lemma 14.3. Then we obtain an object $K' \in D_{QCoh}(\mathcal{O}_{U_{\acute{e}tale}})$ such that (K_n, K_φ) is the system deduced from K', see Lemma 14.7. Finally, we apply Lemma 34.4 to see that $K' = La^*K$ for some $K \in D_{QCoh}(\mathcal{O}_X)$ as desired. □

36. Proper hypercoverings of algebraic spaces

0DHI This section is the analogue of Section 25 for the case of algebraic spaces. The reader who wishes to do so, can replace "algebraic space" everywhere with "scheme" and get equally valid results. This has the advantage of replacing the references to More on Cohomology of Spaces, Section 8 with references to Étale Cohomology, Section 102.

We fix a base scheme S. Let X be an algebraic space over S and let U be a simplicial algebraic space over S. Assume we have an augmentation

$$a : U \to X$$

See Section 32. We say that U is a *proper hypercovering* of X if

(1) $U_0 \to X$ is proper and surjective,
(2) $U_1 \to U_0 \times_X U_0$ is proper and surjective,
(3) $U_{n+1} \to (\text{cosk}_n \text{sk}_n U)_{n+1}$ is proper and surjective for $n \geq 1$.

The category of algebraic spaces over S has all finite limits, hence the coskeleta used in the formulation above exist.

$\boxed{\text{Principle: Proper hypercoverings can be used to compute étale cohomology.}}$

The key idea behind the proof of the principle is to compare the ph and étale topologies on the category *Spaces*/S. Namely, the ph topology is stronger than the étale topology and we have (a) a proper surjective map defines a ph covering, and (b) ph cohomology of sheaves pulled back from the small étale site agrees with étale cohomology as we have seen in More on Cohomology of Spaces, Section 8.

All results in this section generalize to the case where $U \to X$ is merely a "ph hypercovering", meaning a hypercovering of X in the site $(Spaces/S)_{ph}$ as defined in Section 21. If we ever need this, we will precisely formulate and prove this here.

0DHJ **Lemma 36.1.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. There is a commutative diagram*

$$\begin{tikzcd} Sh((Spaces/U)_{ph,total}) \arrow[r, "h"'] \arrow[d, "a_{ph}"'] & Sh(U_{étale}) \arrow[d, "a"] \\ Sh((Spaces/X)_{ph}) \arrow[r, "h_{-1}"] & Sh(X_{étale}) \end{tikzcd}$$

where the left vertical arrow is defined in Section 21 and the right vertical arrow is defined in Section 32.

Proof. The notation $(Spaces/U)_{ph,total}$ indicates that we are using the construction of Section 21 for the site $(Spaces/S)_{ph}$ and the simplicial object U of this site[9]. We will use the sites $X_{spaces,étale}$ and $U_{spaces,étale}$ for the topoi on the right hand side; this is permissible see discussion in Section 32.

Observe that both $(Spaces/U)_{ph,total}$ and $U_{spaces,étale}$ fall into case A of Situation 3.3. This is immediate from the construction of $U_{étale}$ in Section 32 and it follows from Lemma 21.5 for $(Spaces/U)_{ph,total}$. Next, consider the functors $U_{n,spaces,étale} \to (Spaces/U_n)_{ph}$, $U \mapsto U/U_n$ and $X_{spaces,étale} \to (Spaces/X)_{ph}$, $U \mapsto U/X$. We have seen that these define morphisms of sites in More on Cohomology of Spaces, Section 8 where these were denoted $a_{U_n} = \epsilon_{U_n} \circ \pi_{u_n}$ and $a_X = \epsilon_X \circ \pi_X$. Thus we obtain a morphism of simplicial sites compatible with augmentations as in Remark 5.4 and we may apply Lemma 5.5 to conclude. $\square$

0DHK **Lemma 36.2.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is a proper hypercovering of X, then*

$$a^{-1} : Sh(X_{étale}) \to Sh(U_{étale}) \quad \text{and} \quad a^{-1} : Ab(X_{étale}) \to Ab(U_{étale})$$

are fully faithful with essential image the cartesian sheaves and quasi-inverse given by a_*. Here $a : Sh(U_{étale}) \to Sh(X_{étale})$ is as in Section 32.

[9] To distinguish from $(Spaces/U)_{fppf,total}$ defined using the fppf topology in Section 33.

Proof. We will prove the statement for sheaves of sets. It will be an almost formal consequence of results already established. Consider the diagram of Lemma 36.1. In the proof of this lemma we have seen that h_{-1} is the morphism a_X of More on Cohomology of Spaces, Section 8. Thus it follows from More on Cohomology of Spaces, Lemma 8.1 that $(h_{-1})^{-1}$ is fully faithful with quasi-inverse $h_{-1,*}$. The same holds true for the components h_n of h. By the description of the functors h^{-1} and h_* of Lemma 5.2 we conclude that h^{-1} is fully faithful with quasi-inverse h_*. Observe that U is a hypercovering of X in $(\mathit{Spaces}/S)_{ph}$ as defined in Section 21 since a surjective proper morphism gives a ph covering by Topologies on Spaces, Lemma 8.3. By Lemma 21.1 we see that a_{ph}^{-1} is fully faithful with quasi-inverse $a_{ph,*}$ and with essential image the cartesian sheaves on $(\mathit{Spaces}/U)_{ph,total}$. A formal argument (chasing around the diagram) now shows that a^{-1} is fully faithful.

Finally, suppose that $\mathcal{G}$ is a cartesian sheaf on $U_{\acute{e}tale}$. Then $h^{-1}\mathcal{G}$ is a cartesian sheaf on $(\mathit{Spaces}/U)_{ph,total}$. Hence $h^{-1}\mathcal{G} = a_{ph}^{-1}\mathcal{H}$ for some sheaf $\mathcal{H}$ on $(\mathit{Spaces}/X)_{ph}$. We compute using somewhat pedantic notation

$$\begin{aligned}
(h_{-1})^{-1}(a_*\mathcal{G}) &= (h_{-1})^{-1}\mathrm{Eq}(\ a_{0,small,*}\mathcal{G}_0 \rightrightarrows a_{1,small,*}\mathcal{G}_1\ ) \\
&= \mathrm{Eq}(\ (h_{-1})^{-1}a_{0,small,*}\mathcal{G}_0 \rightrightarrows (h_{-1})^{-1}a_{1,small,*}\mathcal{G}_1\ ) \\
&= \mathrm{Eq}(\ a_{0,big,ph,*}h_0^{-1}\mathcal{G}_0 \rightrightarrows a_{1,big,ph,*}h_1^{-1}\mathcal{G}_1\ ) \\
&= \mathrm{Eq}(\ a_{0,big,ph,*}(a_{0,big,ph})^{-1}\mathcal{H} \rightrightarrows a_{1,big,ph,*}(a_{1,big,ph})^{-1}\mathcal{H}\ ) \\
&= a_{ph,*}a_{ph}^{-1}\mathcal{H} \\
&= \mathcal{H}
\end{aligned}$$

Here the first equality follows from Lemma 4.2, the second equality follows as $(h_{-1})^{-1}$ is an exact functor, the third equality follows from More on Cohomology of Spaces, Lemma 8.5 (here we use that $a_0 : U_0 \to X$ and $a_1 : U_1 \to X$ are proper), the fourth follows from $a_{ph}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$, the fifth from Lemma 4.2, and the sixth we've seen above. Since $a_{ph}^{-1}\mathcal{H} = h^{-1}\mathcal{G}$ we deduce that $h^{-1}\mathcal{G} \cong h^{-1}a^{-1}a_*\mathcal{G}$ which ends the proof by fully faithfulness of h^{-1}. $\square$

0DHL **Lemma 36.3.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is a proper hypercovering of X, then for $K \in D^+(X_{\acute{e}tale})$*

$$K \to Ra_*(a^{-1}K)$$

is an isomorphism. Here $a : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is as in Section 32.

Proof. Consider the diagram of Lemma 36.1. Observe that $Rh_{n,*}h_n^{-1}$ is the identity functor on $D^+(U_{n,\acute{e}tale})$ by More on Cohomology of Spaces, Lemma 8.2. Hence Rh_*h^{-1} is the identity functor on $D^+(U_{\acute{e}tale})$ by Lemma 5.3. We have

$$\begin{aligned}
Ra_*(a^{-1}K) &= Ra_*Rh_*h^{-1}a^{-1}K \\
&= Rh_{-1,*}Ra_{ph,*}a_{ph}^{-1}(h_{-1})^{-1}K \\
&= Rh_{-1,*}(h_{-1})^{-1}K \\
&= K
\end{aligned}$$

The first equality by the discussion above, the second equality because of the commutativity of the diagram in Lemma 25.1, the third equality by Lemma 21.2 as U is a hypercovering of X in $(Spaces/S)_{ph}$ by Topologies on Spaces, Lemma 8.3, and the last equality by the already used More on Cohomology of Spaces, Lemma 8.2. $\square$

0DHM **Lemma 36.4.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. If $a : U \to X$ is a proper hypercovering of X, then*

$$R\Gamma(X_{\acute{e}tale}, K) = R\Gamma(U_{\acute{e}tale}, a^{-1}K)$$

for $K \in D^+(X_{\acute{e}tale})$. Here $a : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is as in Section 32.

Proof. This follows from Lemma 36.3 because $R\Gamma(U_{\acute{e}tale}, -) = R\Gamma(X_{\acute{e}tale}, -) \circ Ra_*$ by Cohomology on Sites, Remark 14.4. $\square$

0DHN **Lemma 36.5.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{A} \subset Ab(U_{\acute{e}tale})$ denote the weak Serre subcategory of cartesian abelian sheaves. If U is a proper hypercovering of X, then the functor a^{-1} defines an equivalence*

$$D^+(X_{\acute{e}tale}) \longrightarrow D^+_{\mathcal{A}}(U_{\acute{e}tale})$$

with quasi-inverse Ra_*. Here $a : Sh(U_{\acute{e}tale}) \to Sh(X_{\acute{e}tale})$ is as in Section 32.

Proof. Observe that $\mathcal{A}$ is a weak Serre subcategory by Lemma 12.6. The equivalence is a formal consequence of the results obtained so far. Use Lemmas 36.2 and 36.3 and Cohomology on Sites, Lemma 28.5. $\square$

0DHP **Lemma 36.6.** *Let S be a scheme. Let X be an algebraic space over S. Let U be a simplicial algebraic space over S. Let $a : U \to X$ be an augmentation. Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}tale}$. Let $\mathcal{F}_n$ be the pullback to $U_{n,\acute{e}tale}$. If U is a ph hypercovering of X, then there exists a canonical spectral sequence*

$$E_1^{p,q} = H^q_{\acute{e}tale}(U_p, \mathcal{F}_p)$$

converging to $H^{p+q}_{\acute{e}tale}(X, \mathcal{F})$.

Proof. Immediate consequence of Lemmas 36.4 and 8.3. $\square$

37. Other chapters

Preliminaries

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